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ON MIKUSIŃSKI-PEXIDER FUNCTIONAL EQUATION

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1. The authors of [3] deal with Mikusiński's functional equation

$$f(x+y)\big(f(x+y)-f(x)-f(y)\big) = 0 \tag{1}$$

and they find its general solution. In connexion therewith the following question presents itself in a natural way: find the general solution of the functional equation

$$f(x+y)\big(g(x+y)-h(x)-k(y)\big) = 0. \tag{2}$$

Similarly as in [3], we observe that it is worth-while to consider (2) in an equivalent conditional form

$$f(x+y) \neq 0 \text{ implies } g(x+y) = h(x)+k(y) \tag{3}$$

which allows us to eliminate the sign of multiplication. Thus we may consider our equation for structures endowed with one operation "+" only.

Evidently, the character of equation (3) is slightly pathological. As a matter of fact, since f does not occur on the right-hand side of (3), the left-hand side expresses simply the fact that $x+y$ is not a member of an arbitrarily given set Z which, of course, may always be treated as a counter-image of $\{0\}$ for a certain function f. Thus it seems more natural to consider the functional equation (3) in the form

$$x+y \notin Z \text{ implies } g(x+y) = h(x)+k(y), \tag{4}$$

as well as,

$$f(x+y) \neq 0 \text{ implies } f(x+y) = h(x)+k(y). \tag{5}$$

Equations (4) and (5) may be referred to as *Mikusiński-Pexider equations* in view of their connexion with (1) and with the Pexider functional equation (see [1])

$$g(x+y) = h(x)+k(y).$$

In the present paper we are going to investigate (4) and (5) in a slightly more general form suggested by the results obtained in [2].

2. Suppose that we are given two abelian groups $\mathscr{G}$ and $\mathscr{H}$ (both written additively) and functions

$$\Phi: X \xrightarrow{\text{onto}} \mathscr{G}, \qquad \Psi: Y \xrightarrow{\text{onto}} \mathscr{G},$$

where X and Y are non-empty sets of an arbitrary nature. We are going to study the conditional equation

$$\Phi(x)+\Psi(y)\notin Z \text{ implies } g\big(\Phi(x)+\Psi(y)\big) = h(x)+k(y) \tag{6}$$

with unknown functions

$$g: \mathscr{G}\to\mathscr{H}, \qquad h: X\to\mathscr{H}, \qquad k: Y\to\mathscr{H}. \tag{7}$$

Z denotes here an arbitrarily fixed subset of $\mathscr{G}$.

In order to simplify the statements, in the sequel we exclude the trivial case $Z = \mathscr{G}$ (then, of course, every triple (7) yields a solution of (6)). Moreover, we shall use the following notation:

$$Z(\varphi) = \{s\in\mathscr{G}: \varphi(s) = 0\}, \qquad \varphi\in\mathscr{H}^{\mathscr{G}};$$

$$A-a_0 = \{a-a_0: a\in A\}, \qquad A\subset\mathscr{G},\ a_0\in\mathscr{G};$$

$$A' = \mathscr{G}\setminus A, \qquad A\subset\mathscr{G}.$$

Finally, we shall permanently assume that one of the given functions, say Φ, is invertible.

LEMMA 1. *Equation* (6) *is equivalent to the following system*:

$$\Phi(x)+\Psi(y)\notin Z \textit{ implies } g\big(\Phi(x)+\Psi(y)\big) = h(x)-h\big(\Phi^{-1}(s_0-\Psi(y))\big)+g(s_0), \tag{8}$$

$$k(y) = -h\big(\Phi^{-1}(s_0-\Psi(y))\big)+g(s_0), \qquad y\in Y; \tag{9}$$

s_0 being an arbitrarily fixed point of Z'.

Proof. Indeed, for fixed $y\in Y$ it suffices to put $x = \Phi^{-1}\big(s_0-\Psi(y)\big)$ in (6).

For $s_0\in Z'$, henceforth fixed, we put

$$G(s) = g(s+s_0)-g(s_0), \qquad H(s) = h\big(\Phi^{-1}(s+s_0)\big)-g(s_0), \qquad s\in\mathscr{G}, \tag{10}$$

and

$$T = Z'-s_0. \tag{11}$$

LEMMA 2. *Equation* (8) *is equivalent to the following one*:

$$t\in T \textit{ implies } G(t) = H(t+s)-H(s), \qquad s\in\mathscr{G}; \tag{12}$$

G, H and T are defined by (10) *and* (11), *respectively.*

Proof. Applying (8), (10) and (11) we obtain

$$\Phi(x)+\Psi(y)-s_0\in T \text{ implies } G\big(\Phi(x)+\Psi(y)-s_0\big)+g(s_0) = H\big(\Phi(x)-s_0\big)+g(s_0)-H\big(-\Psi(y)\big),$$

whence (12) results by setting

$$t = \Phi(x)+\Psi(y)-s_0, \quad s = -\Psi(y).$$

Conversely, it follows from (12), (10) and (11) that

$$t\in T \text{ implies } g(t+s_0)-g(s_0) = h\big(\Phi^{-1}(t+s+s_0)\big)-h\big(\Phi^{-1}(s+s_0)\big).$$

Making the same substitutions as above we obtain:

$$\Phi(x)+\Psi(y)\notin Z \text{ implies } g\big(\Phi(x)+\Psi(y)\big) = h(x)-h\big(\Phi^{-1}(s_0-\Psi(y))\big)+g(s_0).$$

Thus the problem of finding the general solution of (6) may be reduced to that for equation (12).

LEMMA 3. *The general solution of equation* (12) *is given by*

$$G(s) = \begin{cases} \varphi(s) & \text{for } s\in T,\\ \text{arbitrary} & \text{for } s\notin T,\end{cases}$$

$$H(s) = \varphi(s)+c_0, \quad s\in \mathcal{G};$$

φ is an arbitrary element of $\mathcal{H}^{\mathcal{G}}$ which satisfies the condition

$$(13) \qquad \varphi(s+t) = \varphi(s)+\varphi(t), \quad (s,t)\in\mathcal{G}\times T,$$

whereas c_0 is an arbitrary constant from $\mathcal{H}$ and T is defined by (11).

Proof. Putting in (12) $s = 0$ and $c_0 = H(0)$, we get $G(t) = H(t)-c_0$ for $t\in T$. Thus

$$H(t)-c_0 = H(t+s)-c_0-\big(H(s)-c_0\big), \quad (s,t)\in\mathcal{G}\times T,$$

whence (13) is obtained by setting

$$\varphi(s) = H(s)-c_0, \quad s\in\mathcal{G}.$$

Conversely, every pair of functions G and H, defined in the statement of this lemma, satisfies (12).

Lemmas 1,2 and 3 imply the following

THEOREM 1. *The general solution of equation* (6) *is given by*

$$g(s) = \begin{cases} \varphi(s-s_0)+c & \text{for } s\notin Z,\\ \text{arbitrary} & \text{for } s\in Z,\end{cases}$$

$$h(x) = \varphi\big(\Phi(x)-s_0\big)+c-c_0, \quad x\in X,$$

$$k(y) = -\varphi\big(-\Psi(y)\big)+c_0, \quad y\in Y,$$

where φ is an arbitrary element of $\mathscr{H}^{\mathscr{G}}$ fulfilling (13) *with T defined by* (11), *whereas s_0 is an arbitrarily fixed point of Z', and c_0, c are arbitrary constants from $\mathscr{H}$.*

Remark. A slight modification allows us to get a similar result for the functional equation

$$\sum_{i=1}^{n} \Phi_i(x_i) \notin Z \text{ implies } g\Big(\sum_{i=1}^{n} \Phi_i(x_i)\Big) = \sum_{i=1}^{n} h_i(x_i)$$

with unknown functions

$$g\colon \mathscr{G} \to \mathscr{H}, \quad h_i\colon X_i \to \mathscr{H}, \quad i = 1, \ldots, n,$$

where the functions

$$\Phi_i\colon X_i \xrightarrow{\text{onto}} \mathscr{G}, \quad i = 1, \ldots, n,$$

are given, Φ_1 is invertible, whereas X_i, $i = 1, \ldots, n$, are any non-empty sets and Z denotes an arbitrarily fixed subset of $\mathscr{G}$.

The question about the general solution of (13) becomes now obtruding itself. In general, one cannot expect φ to be additive, since the behaviour of solutions depends obviously on Z. Take, for instance, $\mathscr{G} = \mathscr{H} = R$, the additive group of all real numbers, and $Z = R \setminus \{a\}$, $a \in R$. Thus an arbitrary member φ of R^R, fulfilling $\varphi(0) = 0$, satisfies (13).

However, φ must be additive whenever the set Z' is not too small. Namely, we have the following

THEOREM 2. *If $T \cap (T-q) \neq \emptyset$ for every $q \in \mathscr{G}$, then every $\varphi \in \mathscr{H}^{\mathscr{G}}$ which satisfies* (13) *must be an additive function.*

Proof. According to (13), one can write

$$\varphi(s_1+t) - \varphi(s_1) = \varphi(s_2+t) - \varphi(s_2), \quad (s_1, s_2, t) \in \mathscr{G} \times \mathscr{G} \times T.$$

Taking p, q arbitrary from $\mathscr{G}$ and putting $s_1 = p+q$, $s_2 = q$, we have

$$\varphi(p+q+t) - \varphi(p+q) = \varphi(q+t) - \varphi(q), \quad (p, q, t) \in \mathscr{G} \times \mathscr{G} \times T. \tag{14}$$

On the other hand (cf. also (13)),

$$\varphi(p+q+t) = \varphi(p) + \varphi(q+t), \quad (p, q, t) \in \mathscr{G} \times \mathscr{G} \times (T-q). \tag{15}$$

Choosing a $t \in T \cap (T-q)$, we infer from (14) and (15) that

$$\varphi(p+q) = \varphi(p) + \varphi(q), \quad (p, q) \in \mathscr{G} \times \mathscr{G},$$

which was to be proved.

3. Now we apply these results to the functional equation

$$f\big(\Phi(x) + \Psi(y)\big) \neq 0 \text{ implies } f\big(\Phi(x) + \Psi(y)\big) = h(x) + k(y), \tag{16}$$

where the functions occurring are subjected to the same general conditions as in the preceding section.

THEOREM 3. *The general solution of equation* (16) *is given by*

$$f(s) = \begin{cases} \varphi(s-s_0)+c & \text{for } s \notin Z, \\ 0 & \text{for } s \in Z, \end{cases}$$

$$h(x) = \varphi\big(\Phi(x)-s_0\big)+c-c_0, \qquad x \in X,$$

$$k(y) = -\varphi\big(-\Psi(y)\big)+c_0, \qquad y \in Y,$$

where Z is an arbitrarily fixed subset of $\mathcal{G}$, whereas φ, s_0, c, c_0 have the same meaning as in theorem 1.

In particular, φ must be additive whenever there does not exist a $q \in \mathcal{G}$ such that $Z \cup (Z-q) = \mathcal{G}$.

Proof. Fix arbitrarily a set Z contained in $\mathcal{G}$ and consider equation (6) with the symbol f instead of g. It suffices to choose from the family of all solutions of this equation those for which the condition $Z \subset Z(f)$ is satisfied.

In the case where $X = Y = \mathcal{G}$, $\Phi = \Psi$ are identities on $\mathcal{G}$, and $f = h = k$, (16) assumes the form

$$f(s+t) \neq 0 \text{ implies } f(s+t) = f(s)+f(t), \tag{17}$$

which was investigated in [3]. The main result obtained there reads as follows: f must be additive if $\mathcal{G}$ has no subgroups of index 2; if $\mathcal{G}$ has a subgroup K of index 2, then either f is additive or $f = \text{const}\ (\neq 0)$ on K' and $f = 0$ on K.

This result can easily be deduced from our theorem 3 ([1]).

In fact, the condition $f = h = k$ implies

$$f(s) = \begin{cases} \varphi(s-s_0)+c, & s \notin Z, \\ \varphi(s-s_0)+c-c_0, & s \in \mathcal{G}, \\ -\varphi(-s)+c_0, & s \in \mathcal{G}, \end{cases}$$

whence

$$f(s) = -\varphi(-s), \qquad s \in \mathcal{G}.$$

Now, if $\mathcal{G}$ has no subgroup of index 2, then $Z(f) \cup \big(Z(f)-q\big) = \mathcal{G}$ is not possible for any $q \in \mathcal{G}$ ([2]) and, consequently, f must be additive. If $\mathcal{G}$ has a subgroup K of index 2, then, besides the additive solutions, equation (17) may also have non-additive solutions f such that $Z(f) = K$. For

([1]) However, only in the case of commutative $\mathcal{G}$ and $\mathcal{H}$. In [3] $\mathcal{G}$ and $\mathcal{H}$ are not assumed commutative.

([2]) (17) implies that $Z = Z(f)$ is a group (cf. [3]).

every $s_0 \notin K$, we have $K' - s_0 = K$ and thus f satisfies

$$f(s+t) = f(s), \qquad (s, t) \in \mathcal{G} \times K. \tag{18}$$

Fix an $s_0 \notin K$ and take an arbitrary $s \notin K$. Then $t = s_0 - s \in K$, whence by (18) we have $f(s) = f(s_0)$, i.e. $f = \text{const}$ ($\neq 0$) on K', whereas $f = 0$ on K by the condition $Z(f) = K$.

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