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VARIETIES OF IDEMPOTENT MEDIAL n-QUASIGROUPS

BY

B. CSÁKÁNY AND L. MEGYESI (SZEGED)

Medial n-quasigroups were introduced and studied by Belousov [1], p. 46-52. Varieties of idempotent medial quasigroups (i.e., 2-quasigroups) were investigated by the authors [7]. In this article we prove that varieties of idempotent medial n-quasigroups (shortly, IM-n-quasigroups) are equivalent to varieties of affine modules over some rings. As a consequence we infer that equationally complete varieties of IM-n-quasigroups are equivalent to varieties of affine spaces over finite fields. In particular, for even n, equationally complete varieties of IM-n-quasigroups coincide with equationally complete varieties of idempotent medial quasigroups.

An *n-quasigroup* is an algebra with the n-ary basic operations $f, f_1, \ldots, f_n$, where f_i $(i = 1, \ldots, n)$ denotes the i-th inverse operation for f; i.e., these operations satisfy the identities (1)

$$f(x_1^{i-1}, f_i(x_1^n), x_{i+1}^n) = x_i, \tag{1}$$

$$f_i(x_1^{i-1}, f(x_1^n), x_{i+1}^n) = x_i. \tag{2}$$

Remark that (1), (2) imply the identities

$$f_i(x_1^{i-1}, x_k, x_{i+1}^{k-1}, f_k(x_1^n), x_{k+1}^n) = x_i, \tag{3}$$

$$f_k(x_1^{i-1}, f_i(x_1^n), x_{i+1}^{k-1}, x_i, x_{k+1}^n) = x_k \tag{4}$$

for all i, k such that $(1 \leqslant)\ i < k\ (\leqslant n)$.

Medial and *idempotent n-quasigroups* are defined by the identities

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \ldots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \ldots, f(x_{1n}^{nn})) \tag{5}$$

and

$$f(x, \ldots, x) = x, \tag{6}$$

respectively. Identity (5) is often referred to as the permutability of f with itself.

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(1) Following Belousov, we denote the sequence $a_i, a_{i+1}, \ldots, a_j$ by a_i^j.

Let M be a (unitary right) module over the ring R. Every algebraic (in other terminology: polynomial) operation of M may be written in the form $\sum_{i=1}^{n} x_i \varrho_i$ with $\varrho_i \in R$. Let A_R be the set of all algebraic operations which fulfil

$$\sum_{i=1}^{n} \varrho_i = 1.$$

The algebra $(M; A_R)$ is called an *affine module over R* (see [9] and [6]). The class of all affine modules over a ring R is a variety denoted by $\mathscr{A}(R)$. Concerning affine modules, we need the following two facts:

LEMMA 1. *$(R^k; A_R)$ is a free algebra in $\mathscr{A}(R)$ with the free generating set*

$$\{(0, \dots, 0), (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}.$$

(See, e.g., the proof of Theorem 2 in [6].)

LEMMA 2. *Subvarieties of $\mathscr{A}(R)$ are the same up to equivalence as the varieties $\mathscr{A}(\bar{R})$, where $\bar{R}$ is any homomorphic image of R.* (See the proof of Theorem 4 in [6].)

The equivalence of varieties, a notion going back to A. I. Mal'cev, may be characterized as follows:

LEMMA 3. *Two varieties $\mathscr{F}$ and $\mathscr{G}$ are equivalent if and only if for any $\mathscr{F}$-free F and $\mathscr{G}$-free G with the same free generating set there exist a 1-1 mapping φ of F onto G and a 1-1 mapping ζ of the set of algebraic operations of F onto that of G such that for any m-ary algebraic operation f and elements $a_1, \dots, a_m$ of F the relation*

$$(f(a_1, \dots, a_m))\varphi = (f\zeta)(a_1\varphi, \dots, a_m\varphi)$$

holds (see [4]).

Following Kuroš, a variety is called *Abelian* if every pair of its (basic) operations is permutable (see [3], p. 127, and [8], p. 92).

LEMMA 4. *Any variety of medial n-quasigroups is Abelian.*

In fact, f is permutable with itself by (5). Further, the permutability of pairs (f, f_1), (f_1, f_1), (f_1, f_2) can be established by making use of (1)-(5):

$$(\alpha) \quad f\big(f_1(a_{11}^{n1}), \dots, f_1(a_{1n}^{nn})\big)$$

$$\overset{(2)}{=} f_1\Big(f\Big(f\big(f_1(a_{11}^{n1}), \dots, f_1(a_{1n}^{nn})\big), f(a_{21}^{2n}), \dots, f(a_{n1}^{nn})\Big), f(a_{21}^{2n}), \dots, f(a_{n1}^{nn})\Big)$$

$$\overset{(5)}{=} f_1\Big(f\Big(f\big(f_1(a_{11}^{n1}), a_{21}^{n1}\big), \dots, f\big(f_1(a_{1n}^{nn}), a_{2n}^{nn}\big)\Big), f(a_{21}^{2n}), \dots, f(a_{n1}^{nn})\Big)$$

$$\overset{(1)}{=} f_1\big(f(a_{11}^{1n}), \dots, f(a_{n1}^{nn})\big),$$

$$(\beta) \quad f_1\big(f_1(a_{11}^{1n}), \dots, f_1(a_{n1}^{nn})\big)$$

$$\overset{(2)}{=} f_1\Big(f\Big(f_1\big(f_1(a_{11}^{1n}), \dots, f_1(a_{n1}^{nn})\big), f_1(a_{12}^{n2}), \dots, f_1(a_{1n}^{nn})\Big), f_1(a_{12}^{n2}), \dots, f_1(a_{1n}^{nn})\Big)$$

$$\overset{(\alpha)}{=} f_1\Big(f_1\big(f(f_1(a_{11}^{1n}), a_{12}^{1n}), \ldots, f(f_1(a_{n1}^{nn}), a_{n2}^{nn})\big), f_1(a_{12}^{n2}), \ldots, f_1(a_{1n}^{nn})\Big)$$

$$\overset{(1)}{=} f_1(f_1(a_{11}^{n1}), \ldots, f_1(a_{1n}^{nn})),$$

$$(\gamma) \quad f_1(f_2(a_{11}^{1n}), \ldots, f_2(a_{n1}^{nn}))$$

$$\overset{(4)}{=} f_2\Big(f_1\big(f_1(a_{12}^{n2}), f_1(f_2(a_{11}^{1n}), \ldots, f_2(a_{n1}^{nn})), f_1(a_{13}^{n3}), \ldots, f_1(a_{1n}^{nn})\big), f_1(a_{12}^{n2}), \ldots, f_1(a_{1n}^{nn})\Big)$$

$$\overset{(\beta)}{=} f_2\Big(f_1\big(f_1(a_{12}, f_2(a_{11}^{1n}), a_{13}^{1n}), \ldots, f_1(a_{n2}, f_2(a_{n1}^{nn}), a_{n3}^{nn})\big), f_1(a_{12}^{n2}), \ldots, f_1(a_{1n}^{nn})\Big)$$

$$\overset{(3)}{=} f_2(f_1(a_{11}^{n1}), \ldots, f_1(a_{1n}^{nn})).$$

Analogously we get the permutability of each other pair of operations.

Let P_n denote the ring of all fractions of the form

$$\frac{g(x_1, \ldots, x_{n-1})}{x_1^{k_1} \ldots x_{n-1}^{k_{n-1}}(1-x_1- \ldots -x_{n-1})^l},$$

where g is a polynomial in variables $x_1, \ldots, x_{n-1}$ with integer coefficients, and $k_1, \ldots, k_{n-1}, l$ are non-negative integers.

THEOREM 1. *The variety of all* IM-*n-quasigroups is equivalent to* $\mathscr{A}(P_n)$. *Any variety of* IM-*n-quasigroups is equivalent to* $\mathscr{A}(\bar{P}_n)$ *for some homomorphic image* $\bar{P}_n$ *of* P_n.

Proof. It is proved in [6] that if a variety is Abelian, Hamiltonian (i.e., in any algebra every subalgebra is a class of some congruence), idempotent (i.e., in any algebra every one-element set is a subalgebra), and regular (i.e., in any algebra two congruences coincide provided they have a class in common), then it is equivalent to the variety of all affine modules over some commutative ring. Next we show that any variety $\mathfrak{Q}$ of IM-n-quasigroups fulfils the four conditions listed before.

By Lemma 4, $\mathfrak{Q}$ is Abelian. In view of (2) we have

$$f_i(x) = f_i(x, \ldots, x, f(x, \ldots, x), x, \ldots, x) = x,$$

showing that $\mathfrak{Q}$ is idempotent. Further, for

$$t(x, y, z) = f_2(f_1(x, z, \ldots, z), y, z, \ldots, z),$$

the identity

$$t(x, x, z) = z$$

and the identical implication

$$(t(x, y, z) = z) \Rightarrow (x = y)$$

hold in $\mathfrak{Q}$. In accordance with the result of [5] this means that $\mathfrak{Q}$ is regular.

In order to prove that $\mathfrak{Q}$ is Hamiltonian consider a $K \in \mathfrak{Q}$. From the description of medial n-quasigroups by Belousov in [1] it follows that there exists an Abelian group $(K; +)$ having pairwise permutable automorphisms $a_1, \dots, a_n$ whose sum is the identical map such that

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i a_i \tag{7}$$

for any $x_1, \dots, x_n \in K$. Consequently,

$$f_i(x_1, \dots, x_n) = \sum_{k \neq i} x_k(-a_k a_i^{-1}) + x_i a_i^{-1} \tag{8}$$

for $i = 1, \dots, n$. Now, (7) and (8) imply

$$\begin{aligned}(9)\quad & f\big(f_1(x_1, x_2, x_4, \dots, x_{2n-2}), f_2(x_{2n-2}, x_3, x_2, \dots, x_{2n-4}),\\ & f_3(x_{2n-4}, x_{2n-2}, x_5, x_2, \dots, x_{2n-6}), \dots, f_n(x_2, x_4, \dots, x_{2n-2}, x_{2n-1})\big)\\ & \qquad = x_1 - x_2 + x_3 - x_4 + \dots + x_{2n-1}.\end{aligned}$$

Thus, any subalgebra L of K is closed with respect to $(2n-1)$-ary alternating sums in the group $(K; +)$, whence L is a coset relative to the subgroup

$$L_0(\subseteq K) = \{l - l_0 \mid l \in L\}$$

with fixed $l_0 \in L$.

It remains to check that the congruence of $(K; +)$, determined by L_0, is also a congruence of the n-quasigroup K. First we show that each of the automorphisms a_i, a_i^{-1} $(i = 1, \dots, n)$ maps L_0 into itself. Indeed,

$$(l - l_0)a_1 = la_1 - l_0 a_1 + f(l_0, \dots, l_0) - l_0 = f(l, l_0, \dots, l_0) - l_0 \in L, \tag{10}$$

$$(l - l_0)a_1^{-1} = la_1^{-1} - l_0 a_1^{-1} + f_1(l_0, \dots, l_0) - l_0 = f_1(l, l_0, \dots, l_0) - l_0 \in L, \tag{11}$$

and analogously for $i > 1$. Now, if

$$a_i = l_i - l_0 + b_i \quad (a_i, b_i \in K;\ l_i \in L;\ i = 1, \dots, n),$$

then (10) gives

$$f(a_1^n) = \sum_{i=1}^{n} a_i a_i = \sum_{i=1}^{n} (l_i - l) a_i + \sum_{i=1}^{n} b_i a_i \in L_0 + f(b_1^n).$$

In a similar way from (11) we obtain

$$f_1(a_1^n) \in L_0 + f_1(b_1^n).$$

Thus, $\mathfrak{Q}$ is equivalent to $\mathscr{A}(R)$ for some commutative ring R. We can apply Lemma 3 to $\mathfrak{Q}$ and $\mathscr{A}(R)$. Let R_n be free in $\mathscr{A}(R)$ with the free generating set

$$e_0 = (0, \dots, 0), \quad e_1 = (1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1)$$

and let Q_n be a free IM-n-quasigroup with the same free generating set.

Using the representation of R_n in Lemma 1 we get

$$f\zeta = \sum_{i=1}^{n} x_i a_i,$$

where $a_i \in R$ and

$$\sum_{i=1}^{n} a_i = 1. \tag{12}$$

If

$$f_j\zeta = \sum_{i=1}^{n} x_i \beta_{ji} \quad \text{for } 1 \leqslant j \leqslant n,$$

then

$$\begin{aligned} e_j = e_j\varphi &= \big(f(e_0, \ldots, e_0, f_j(e_0, \ldots, e_0, e_j, e_0, \ldots, e_0), e_0, \ldots, e_0)\big)\varphi \\ &= e_0 a_i + \ldots + (e_0\beta_{j1} + \ldots + e_j\beta_{jj} + \ldots + e_0\beta_{jn}) a_j + \ldots + e_0 a_n \\ &= (0, \ldots, 0, \beta_{jj}a_j, 0, \ldots, 0), \end{aligned}$$

whence

$$\beta_{jj}a_j = 1 \quad (j = 1, \ldots, n). \tag{13}$$

By a similar computation we get

$$\beta_{jk}a_j + a_k = 0 \quad (j = 1, \ldots, n;\ k \neq j), \tag{14}$$

$$\sum_{k=1}^{n} \beta_{jk} = 1 \quad (j = 1, \ldots, n). \tag{15}$$

The ring R contains all elements of the form

$$g(a_1, \ldots, a_{n-1})\beta_{11}^{k_1} \ldots \beta_{nn}^{k_n}, \tag{16}$$

where g is a polynomial with integer coefficients and k_j $(j = 1, \ldots, n)$ are non-negative integers. On the other hand, all elements of R can be represented in form (16). Indeed, take an arbitrary $\varrho \in R$ and consider the 1-1 mapping ζ (of Lemma 3) between algebraic operations of Q_2 and R_2. For a suitable binary p we have $p\zeta = x\varrho + y(1-\varrho)$. Let

$$p(x, y) = f(p_1(x, y), \ldots, p_n(x, y)), \quad p_i\zeta = x\varrho_i + y(1-\varrho_i) \quad (i = 1, \ldots, n)$$

and suppose that

$$\varrho_i = g_i(a_1, \ldots, a_{n-1})\beta_{11}^{k_{i1}} \ldots \beta_{nn}^{k_{in}}.$$

Then, applying (12) and (13), we obtain

$$\begin{aligned} \varrho &= \sum_{i=1}^{n} a_i g_i(a_1, \ldots, a_{n-1})\beta_{11}^{k_{i1}} \ldots \beta_{nn}^{k_{in}} \\ &= \Big(\sum_{i=1}^{n} a_i g_i(a_1, \ldots, a_{n-1}) a_1^{\mu_1} \ldots (1 - a_1 - \ldots - a_{n-1})^{\mu_n}\Big) \beta_{11}^{\nu_1} \ldots \beta_{nn}^{\nu_n}, \end{aligned}$$

where $\mu_s = (\max_j k_{j_s}) - k_{i_s}$, $\nu_r = \max_j k_{j_r}$ $(s, r = 1, \ldots, n)$, and the outer parentheses enclose a polynomial of $a_1, \ldots, a_{n-1}$ with integer coefficients. The case of $p(x, y) = f_k(p_1(x, y), \ldots, p_n(x, y))$ can be considered analogously, by using (14). Thus the mapping

$$\frac{g(x_1, \ldots, x_{n-1})}{x^{k_1} \ldots x_{n-1}^{k_{n-1}}(1-x_1-\ldots-x_{n-1})^l} \to g(a_1, \ldots, a_{n-1})\beta_{11}^{k_1} \ldots \beta_{n-1}^{k_{n-1}} \beta_{nn}^l$$

turns out to be onto; it is also homomorphic as we can easily verify by making use of (12)-(15). The second part of the theorem is proved.

In order to prove the first part, we show that $\mathscr{A}(P_n)$ is also equivalent to some variety of IM-n-quasigroups. For this aim it is enough to find, in any affine module over P_n, n-ary algebraic operations $f, f_1, \ldots, f_n$ satisfying identities (1)-(6) and such that all operations of P_n are algebraic over the system $\{f, f_1, \ldots, f_n\}$. The desired operations are the following:

$$f(t_1, \ldots, t_n) = t_1x_1 + \ldots + t_{n-1}x_{n-1} + t_n(1-x_1-\ldots-x_{n-1}),$$

$$f_1(t_1, \ldots, t_n) = t_1\frac{1}{x_1} + t_2\left(-\frac{x_2}{x_1}\right) + \ldots + t_{n-1}\left(-\frac{x_{n-1}}{x_1}\right) + t_n\frac{x_1+\ldots+x_{n-1}-1}{x_1}, \ldots,$$

$$f_n(t_1, \ldots, t_n) = t_1\frac{1}{1-x_1-\ldots-x_{n-1}} + t_2\frac{-x_2}{1-x_1-\ldots-x_{n-1}} + \ldots + t_{n-1}\frac{-x_{n-1}}{1-x_1-\ldots-x_{n-1}} + t_n\frac{-x_1}{1-x_1-\ldots-x_{n-1}}.$$

Indeed, identities (1)-(6) may be verified by computation. Further, any binary operation of an affine module over P_n is algebraic over $\{f, f_1, \ldots, f_n\}$. This is clear for the operations

$$t_1x_i + t_2(1-x_i), \quad t_1\frac{1}{x_i} + t_2\left(1-\frac{1}{x_i}\right),$$

$$t_1\frac{1}{1-x_1-\ldots-x_{n-1}} + t_2\left(1-\frac{1}{1-x_1-\ldots-x_{n-2}}\right).$$

Let $p, g \in P_n$ and suppose that $t_1p + t_2(1-p)$ and $t_1q + t_2(1-q)$ are algebraic over $f, f_1, \ldots, f_n$ in some affine module over P_n. Then

$$t_1(p+q) + t_2(1-(p+q)) = f_2\big(f_1(t_2, t_1p+t_2(1-p), t_2, \ldots, t_2),$$
$$f(t_2, t_1q+t_2(1-q), t_2, \ldots, t_2), t_2, \ldots, t_2\big),$$

$$t_1(-p)+t_2(1+p)$$
$$= f_1\big(f\big(z, f_2(t_1p+t_2(1-p)),\ t_2, \ldots, t_2\big),\ t_2, \ldots, t_2\big),\ t_2, \ldots, t_2\big),$$
$$t_1(pq)+t_2(1-pq) = \big(t_1p+t_2(1-p)\big)q+t_2(1-q).$$

Since P_n is generated by

$$\left\{x_1, \ldots, x_{n-1}, \frac{1}{x_1}, \ldots, \frac{1}{x_{n-1}}, \frac{1}{1-x_1-\ldots-x_{n-1}}\right\},$$

the operation $t_1r+t_2(1-r)$ is algebraic over $\{f, f_1, \ldots, f_n\}$ for any $r \in P_n$. Finally, if all operations of arity less than l are algebraic over $\{f, f_1, \ldots, f_n\}$, then this is valid also for the l-ary operations, since for arbitrary $r_1, \ldots, r_l \in P_n$ we have

$$t_1r_1+\ldots+t_lr_l = f\left(t_1\frac{r_1}{x_1}+\ldots+t_{l-2}\frac{r_{l-2}}{x_1}+t_{l-1}\frac{r_{l-1}+r_l+x_1-1}{x_1},\right.$$
$$\left. t_{l-1}\frac{x_2-r_l}{x_2}+t_l\frac{r_l}{x_2},\ t_{l-1}, \ldots, t_{l-1}\right).$$

Now consider the variety $\mathscr{Q}_0$ of all IM-n-quasigroups. By the second part of our theorem there exists a homomorphic image R_0 of P_n such that $\mathscr{Q}_0$ is equivalent to $\mathscr{A}(R_0)$. However, in view of the equivalence of $\mathscr{A}(P_n)$ to some variety of IM-n-quasigroups, P_n is also a homomorphic image of R_0 by Lemma 2. For the ideals of P_n the ascending chain condition holds, whence $R_0 \cong P_n$, which completes the proof.

THEOREM 2. *Equationally complete varieties of IM-n-quasigroups are the same up to equivalence as varieties of affine spaces over finite fields, except for $GF(2)$ in the case of even n.*

Proof. By Lemma 2, any equationally complete variety of IM-n-quasigroups is equivalent to the variety of all affine modules over some simple homomorphic image, i.e., over some factor-field of P_n. We show that every factor-field of P_n is finite. It is known ([2], p. 68) that the polynomial ring $Z[x_1, \ldots, x_{2n-1}]$ has only finite factor-fields. $Z[x_1, \ldots, x_{2n-1}]$ is free with the free generating set $\{x_1, \ldots, x_{2n-1}\}$ in the variety of all commutative rings with 1. Hence any factor-field of P_n is finite.

Conversely, let $GF(q)$ be a finite field whose multiplicative group is generated by $a_1 \in GF(q)$. Further, let $a_2, \ldots, a_{n-1}$ be non-zero elements of $GF(q)$ with sum different from 1. Such elements do not exist only in the case where $q = 2$ and n is even. The mapping

$$\frac{g(x_1, \ldots, x_{n-1})}{x_1^{k_1}\ldots x_{n-1}^{k_{n-1}}(1-x_1-\ldots-x_{n-1})^l} \to$$
$$\to g(a_1, \ldots, a_{n-1})a_1^{-k_1}\ldots a_{n-1}^{-k_{n-1}}(1-a_1-\ldots-a_n)^l$$

is a homomorphism of P_n onto $GF(q)$. This proves the theorem.

COROLLARY 1. *There exist countably many varieties (as well as equationally complete varieties) of IM-n-quasigroups for arbitrary $n \geqslant 2$.*

COROLLARY 2. *The set of all equationally complete varieties of IM-n-quasigroups is uniquely determined up to equivalence by the parity of n.*

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