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A DESCRIPTION OF THE TANGENT FUNCTOR CATEGORY

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1. Introduction. Let $\mathscr{M}$ be the category of C^∞-manifolds and let $T\colon \mathscr{M} \to \mathscr{M}$ denote the tangent functor assigning to a manifold M its tangent bundle TM. We shall classify here all the natural transformations $\lambda\colon T^r \to T^n$ between arbitrary iterates of T (r, n are non-negative integers). Such a natural transformation will be viewed as a family of smooth maps $\lambda_{(M)}\colon T^r M \to T^n M$, one for each $M \in \mathscr{M}$ such that for every $f\colon M \to N$ in $\mathscr{M}$ the following diagram commutes:

$$\begin{tikzcd} T^r M \arrow[r, "\lambda_{(M)}"] \arrow[d, "T^r f"'] & T^n M \arrow[d, "T^n f"] \\ T^r N \arrow[r, "\lambda_{(N)}"'] & T^n N \end{tikzcd} \tag{1.1}$$

Let $\mathscr{T}$ be the category whose objects are $I = T^0, T, T^2, T^3, \ldots$ and whose morphisms are the natural transformations $\lambda\colon T^r \to T^n$. One of our results states that to each $r \geqslant 1$ there is associated a sequence of morphisms $i^{(j,r)}\colon T \to T^r$, $j = 1, \ldots, r$, in $\mathscr{T}$ which *makes* T^r *into the sum* (i. e. co-product) *of* r *copies of* T. This means that if n and $\beta^{(j)}\colon T \to T^n$, $j = 1, \ldots, r$, are arbitrarily given, then there exists a unique $\lambda\colon T^r \to T^n$ in $\mathscr{T}$ such that all diagrams

$$\begin{tikzcd} T \arrow[d, "i^{(j,r)}"'] \arrow[dr, "\beta^{(j)}"] & \\ T^r \arrow[r, "\lambda"'] & T^n \end{tikzcd} \qquad (j = 1, \ldots, r) \tag{1.2}$$

commute.

Remark. Neither the differentiability nor even continuity of the maps $\lambda_{(M)}$ need be assumed; their smoothness will be seen to follow from the assumption that all diagrams (1.1) commute.

2. Reduction to Euclidean spaces. Let E, F be Euclidean spaces and take a smooth map $f: E \to F$. The derivative of f at $x_0 \in E$ is a linear map $E \to F$; its value at $x_1 \in E$ will be denoted by $Df(x_0; x_1)$ and it will be called the *derivative at x_0 in the direction x_1*. (This is linear in x_1.) In particular, if f is linear, then $Df(x_0; x_1) = f(x_1)$.

TE will be identified with $E \times E$ so that $\{x_0\} \times E$ is the tangent space at x_0. Consequently, $T^2 E = TE \times TE = E^4$ and, in general, $T^n E = E^{2^n}$. We have $T^n f: E^{2^n} \to F^{2^n}$.

More precisely

$$T^0 f(x_0) = f(x_0),$$
$$T^1 f(x_0, x_1) = \big(f(x_0), Df(x_0; x_1)\big),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T^{n+1} f(x_0, \ldots, x_{2^{n+1}-1})$$
$$= \big(T^n f(x_0, \ldots, x_{2^n-1}), D(T^n f)(x_0, \ldots, x_{2^n-1}; x_{2^n}, \ldots, x_{2^{n+1}-1})\big).$$

(The last derivative is taken at $(x_0, \ldots, x_{2^n-1})$ in the direction $(x_{2^n}, \ldots, x_{2^{n+1}-1})$.)

The following two lemmas give an explicit computation of $T^n f$. If we write $x = (x_0, \ldots, x_{2^n-1})$, then it is obvious that $T^n f(x)$ is a sequence of 2^n terms

$$T^n f(x) = \big(\theta_0 f(x), \theta_1 f(x), \ldots, \theta_{2^n-1} f(x)\big).$$

LEMMA 2.1. *For every i, $\theta_i f(x)$ depends only on $x_0, \ldots, x_{2^r-1}$, where r is the smallest integer such that $i < 2^r$. Moreover, $\theta_0 f(x) = f(x_0)$, and for $i \geqslant 0$, $l \geqslant r$*

$$(\theta_{i+2^l} f)(x_0, \ldots, x_{2^{l+1}-1})$$
$$= D(\theta_i f)(x_0, x_1, \ldots, x_{2^r-1}; x_{2^l}, x_{2^l+1}, \ldots, x_{2^l+2^r-1}).$$

The proof follows directly from the above equalities defining $T^n f$. The Lemma allows to calculate each θ_i. To do this we take the convention of sometimes replacing a suffix $i = 1, 2, 3, \ldots$ by the set $I = \{i_1, \ldots, i_s\}$ of non-negative integers such that $i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_s}$. Thus $\theta_1 = \theta_{\{0\}}$, $x_2 = x_{\{1\}}$, $x_3 = x_{\{0,1\}}$, etc. The above r is then equal to $\max_j (i_j + 1)$.

For $k = 1, 2, \ldots$ and $x_0 \in E$, the k-th derivative $D^k f$ at x_0 is a k-linear map $E^k \to F$. Its value at $(x_1, \ldots, x_k)$ will be denoted by $D^k f(x_0; x_1; \ldots; x_k)$, so that this function is linear in each but the first variable.

LEMMA 2.2. *We have*

$$\theta_I f(x_0, \ldots, x_{2^r-1}) = \sum_{k=1}^{|I|} \sum_{I_j} D^k f(x_0; x_{I_1}; \ldots; x_{I_k}),$$

where $|I|$ is the cardinality of I and the inner sum is over all sequences

$(I_1, \ldots, I_k)$ which partition I into k disjoint subsets, and are lexicographically ordered (i. e. the smallest number in I_1 is smaller than the smallest number in I_2, etc.).

The proof of this lemma is by induction on $|I|$; the inductive step uses the fact (see Lemma 2.1) that if l is greater than each element of I, then

$$\begin{aligned}(\theta_{I\cup\{l\}}f)(x_0, \ldots, x_{2^{l+1}-1}) &= D(\theta_I f)(x_0, \ldots, x_{2^r-1}; x_{2^l}, \ldots, x_{2^l+2^r-1}) \\ &= \sum_{k=1}^{|I|}\sum_{I_j}\Big(D^{k+1}f(x_0; x_{I_1}; \ldots; x_{I_k}; x_{\{l\}}) + \\ &\qquad + \sum_{j=1}^{k} D^k f(x_0; x_{I_1}; \ldots; x_{I_j\cup\{l\}}; \ldots; x_{I_k})\Big).\end{aligned}$$

This is the equality asserted by the Lemma (for $I\cup\{l\}$ at place of I).

Note. For $I = \{i_1, \ldots, i_s\}$, the highest derivative appearing in $\theta_I f$ is $D^{|I|}f(x_0; x_{i_1}; \ldots; x_{i_s})$. It follows that if $\dim E \geqslant 2^n-1$ and $x_1, \ldots, x_{2^n-1}$ are linearly independent in E, then for every $a_1, \ldots, a_{2^n-1} \in F$ there exists an $f\colon E\to F$ (in fact a polynomial) such that $T^n f(x) = \big(f(x_0), a_1, \ldots, a_{2^n-1}\big)$.

LEMMA 2.3. *Let r, n be non-negative integers, let E be a Euclidean space of dimension $\geqslant 2^r-1$ and let $\lambda_{(E)}\colon T^r E\to T^n E$ be a map such that for every smooth $f\colon E\to E$ the diagram*

(2.3.1)
$$\begin{array}{ccc} T^r E & \xrightarrow{\lambda_{(E)}} & T^n E \\ {\scriptstyle T^r f}\downarrow & & \downarrow{\scriptstyle T^n f} \\ T^r E & \xrightarrow[\lambda_{(E)}]{} & T^n E \end{array}$$

commutes. Denote by $\lambda_j\colon T^r E\to E$ the composite of $\lambda_{(E)}$ with the projection $T^n E = E^{2^n}\to E$ onto the j-th summand, so that $\lambda_{(E)} = (\lambda_0, \lambda_1, \ldots, \lambda_{2^n-1})$. Then

(i) $$\lambda_0(x_0, \ldots, x_{2^r-1}) = x_0,$$

(ii) $$\lambda_I(x_0, \ldots, x_{2^r-1}) = \sum_J \lambda(I, J)x_J$$

for every $I\subset\{0, \ldots, n-1\}$, $I\neq\emptyset$, where the sum is over all $J\subset\{0, \ldots, r-1\}$, $J\neq\emptyset$, and $\lambda(I, J)$ are real numbers such that $\lambda(I, J) = 0$ whenever $|J| > |I|$.

Proof. Abbreviate $(x_0, \ldots, x_{2^r-1})$ to x. Then the commutativity of (2.3.1) implies (on the first component of $T^n E$)

(2.3.2) $$\lambda_0\big(f(x_0), \theta_1 f(x), \ldots, \theta_{2^r-1}f(x)\big) = f\big(\lambda_0(x)\big)$$

and (at the remaining components of $T^n E$)

$$(2.3.3)\qquad \lambda_I\big(f(x_0), \theta_1 f(x), \dots, \theta_{2^r-1} f(x)\big)$$
$$= \theta_I f\big(\lambda_0(x), \lambda_1(x), \dots, \lambda_{2^n-1}(x)\big) \quad \text{for } \emptyset \neq I \subset \{0, \dots, n-1\}.$$

Since each $\theta_i f$ in (2.3.2) contains only derivatives of f, a value of f enters on the left-hand side of the equality only at x_0. Since (2.3.2) holds for every f, we must have $\lambda_0(x) = x_0$, whence (i) is proved.

The right-hand side of (2.3.3) contains only derivatives of f at $\lambda_0(x) = x_0$ but the value of f at x_0 does not enter. Hence, by the Note preceding this lemma, it follows that λ_I does not depend on the first variable, and we can rewrite (2.3.3) as

$$(2.3.4)\qquad {}'\lambda_I\big(\theta_1 f(x), \dots, \theta_{2^r-1} f(x)\big) = \theta_I f\big(x_0, \lambda_1(x), \dots, \lambda_{2^n-1}(x)\big).$$

For an arbitrary linear $f\colon E \to E$, we have $\theta_i f(x) = f(x_i)$, by Lemma 2.2. Hence for such f the above becomes

$$\lambda_I\big(f(x_1), \dots, f(x_{2^r-1})\big) = f\big(\lambda_I(x)\big).$$

It is an easy exercise to show that if such identity holds for every linear f, then λ_I must be a linear combination of its variables, so that

$$\lambda_I(x_1, \dots, x_{2^r-1}) = \sum_J \lambda(I, J) x_J,$$

where $J \subset \{0, \dots, r-1\}$, $J \neq \emptyset$.

We claim that λ_I does not depend on those x_J for which $|J| > |I|$. Indeed, suppose λ_I depends on x_J, where $J = \{j_1, \dots, j_l\}$ and $l > |I|$. Then in (2.3.4) it would be possible to have an f and x such that the value of the left-hand side could be altered by suitably altering $\theta_J f(x)$ and this could be attained by varying f so that only $D^{|J|} f(x_0; x_{j_1}; \dots; x_{j_l})$ undergoes a change (see the Note preceding this lemma). However, the right-hand side of (2.3.4) contains no derivatives of order higher than $|I|$. Hence the right-hand side would remain unchanged. This contradiction proves that we must have in this case $\lambda(I, J) = 0$.

Lemma 2.4. *Let $\lambda_{(E)}'\colon T^r E \to T^n E$ be the map in the preceding Lemma. Suppose further that F is a Euclidean space and $\beta_{(F)}\colon T^r F \to T^n F$ a map such that for every linear $f\colon F \to E$*

$$(2.4.1)\qquad \begin{array}{ccc} T^r F & \xrightarrow{\beta_{(F)}} & T^n F \\ {\scriptstyle T^r f}\big\downarrow & & \big\downarrow{\scriptstyle T^n f} \\ T^r E & \xrightarrow[\lambda_{(E)}]{} & T^n E \end{array}$$

commutes. Then, writing $\beta_{(F)} = (\beta_0, \ldots, \beta_{2^n-1})$, *we have*

(a) $$\beta_0(x_0, \ldots, x_{2^r-1}) = x_0,$$

(b) $$\beta_I(x_0, \ldots, x_{2^r-1}) = \sum_J \lambda(I, J)x_J,$$

where $\lambda(I, J)$ *are exactly the same numbers which appear in part* (ii) *of the previous Lemma.*

Proof. Since f is linear, $T^r f(x) = \big(f(x_0), f(x_1), \ldots, f(x_{2^r-1})\big)$. Thus (2.4.1) implies, at the i-th component of $T^n E$ that

$$f\big(\beta_i(x_0, \ldots, x_{2^r-1})\big) = \lambda_i\big(f(x_0), \ldots, f(x_{2^r-1})\big).$$

Taking $i = 0$, we obtain (a), due to the arbitrariness of f, and for $0 < i = 2^{i_1} + \ldots + 2^{i_s}$, $I = \{i_1, \ldots, i_s\}$, the above implies

$$f\big(\beta_I(x_0, \ldots, x_{2^r-1})\big) = \sum_J \lambda(I, J)f(x_J) = f\Big(\sum_J \lambda(I, J)x_J\Big)$$

by the structure of $\lambda_{(E)}$. Thus (b) follows.

COROLLARY 1. *If* λ: $T^r \to T^n$ *is a natural transformation, then its action on a Euclidean space* E *is a map* $\lambda_{(E)}$: $T^r E \to T^n E$ *satisfying the assertions of Lemma* 2.3. *Moreover, the constants* $\lambda(I, J)$ *do not depend on* E.

Indeed, Lemma 2.4 with $\lambda_{(F)}$ at place of $\beta_{(F)}$ shows that the $\lambda(I, J)$ do not depend on the Euclidean space.

COROLLARY 2. *A natural transformation* λ: $T^r \to T^n$ *is completely determined by its effect* $\lambda_{(R)}$: $T^r R \to T^n R$ *on the real line.*

Proof. Indeed, if $\lambda_{(R)}$ is known, then, by the above, $\lambda_{(E)}$ is known for each Euclidean space E. Thus for any open subset $U \subset E$, denoting by f: $U \to E$ the inclusion map, we have

$$\begin{array}{ccc} T^r U & \xrightarrow{\lambda_{(U)}} & T^n U \\ {\scriptstyle T^r f}\downarrow & & \downarrow{\scriptstyle T^n f} \\ T^r E & \xrightarrow[\lambda_{(E)}]{} & T^n E \end{array}.$$

Since $T^n f$ is injective, there is at most one $\lambda_{(U)}$ satisfying the diagram. Now, if $M \in \mathcal{M}$, then there are open injections φ: $U \to M$ (U open in a Euclidean space) such that M is covered by the sets $\varphi(U)$. For each of these,

$$\begin{array}{ccc} T^r U & \xrightarrow{\lambda_{(U)}} & T^n U \\ {\scriptstyle T^r \varphi}\downarrow & & \downarrow{\scriptstyle T^n \varphi} \\ T^r M & \xrightarrow[\lambda_{(M)}]{} & T^n M \end{array}.$$

Here $T^n\varphi, T^r\varphi$ are open injections, whence the restriction of $\lambda_{(M)}$ to $T^r\varphi$ $(T^r U)$ is completely determined by $\lambda_{(U)}$. Hence $\lambda_{(M)}$ is determined.

3. The Structure Theorem. For a set S let P_*S denote the set of all non-empty subsets of S. Assume that

$$\lambda\colon P_*\{0,\ldots,n-1\}\times P_*\{0,\ldots,r-1\}\to R \tag{3.1}$$

is a given map and define for every Euclidean space E a mapping $\lambda_{(E)}\colon T^rE\to T^nE$ by letting

$$\lambda_{(E)}(x) = \big(x_0, \lambda_1(x), \ldots, \lambda_{2^n-1}(x)\big) \quad \text{for } x = (x_0, x_1, \ldots, x_{2^r-1}) \tag{3.2}$$

and

$$\lambda_I(x) = \sum_J \lambda(I, J)x_J \quad \text{for } I\in P_*\{0,\ldots,n-1\}, \tag{3.3}$$

the sum being over all $J\in P_*\{0,\ldots,r-1\}$.

STRUCTURE THEOREM. *All diagrams*

$$\begin{tikzcd} T^rE \arrow[r, "\lambda_{(E)}"] \arrow[d, "T^rf"'] & T^nE \arrow[d, "T^nf"] \\ T^rF \arrow[r, "\lambda_{(F)}"'] & T^nF \end{tikzcd}, \tag{3.4}$$

where $f\colon E\to F$ is an arbitrary smooth map, commute if and only if

(1) $\lambda(I,J) = 0$ *whenever* $|J|>|I|$;

(2) *for every* $I\subset\{0,\ldots,n-1\}$, $|I|\geqslant 2$ *and* $j\in\{0,\ldots,r-1\}$,

$$\sum_{I_1,I_2}^{I}\lambda(I_1,\{j\})\lambda(I_2,\{j\}) = 0,$$

where the sum is over all partitions $I = I_1\cup I_2$;

(3) *for every* $|I|\geqslant|J|$, $J = \{j_1,\ldots,j_l\}$, $|J|\geqslant 2$,

$$\lambda(I,J) = \sum_{\varphi\in\Phi}\lambda\big(\varphi^{-1}(j_1),\{j_2\}\big)\ldots\lambda\big(\varphi^{-1}(j_l),\{j_l\}\big),$$

where the sum is over the set $\Phi = \Phi(I,J)$ *of all surjections* $\varphi\colon I\to J$.

Proof. Assume that (3.4) commutes. Then (1) follows from Lemma 2.3. Combining (3.3) with Lemma 2.2, we get

$$\lambda_I\big(T^rf(x)\big) = \sum_J\lambda(I,J)\sum_{k=1}^{|J|}\sum_{J_j}D^kf(x_0;\, x_{J_1};\, \ldots;\, x_{J_k}). \tag{3.5}$$

Since (3.4) commutes, this must be equal to

$$\theta_I f(\lambda_1(x), \ldots, \lambda_{2^n-1}(x)) = \sum_{s=1}^{|I|} \sum_{I_j} D^s f(x_0;\ \lambda_{I_1}(x);\ \ldots;\ \lambda_{I_s}(x)). \tag{3.6}$$

Consider one of the sequences $(J_1, \ldots, J_k)$ appearing in (3.5). Choosing E of $\dim \geqslant 2^r - 1$, $x_1, \ldots, x_{2^r-1}$ independent, and taking a suitable f, we can assume that $D^k f(x_0;\ x_{J_1};\ \ldots;\ x_{J_k}) = 1$ and all other derivatives in (3.5) are zero. Hence (3.5) is $\lambda(I, J)$. But then the only non-zero terms in (3.6) are those where $s = k$ and where there is a permutation $(l_1, \ldots, l_k)$ of $(1, \ldots, k)$ such that $|J_{l_1}| \leqslant |I_1|, \ldots, |J_{l_k}| \leqslant |I_k|$ (see (3.3) and (1) above). The sum of these terms, which must be equal to $\lambda(I, J)$, can be written as

$$\lambda(I, J) = \sum_{I_j}^{*} \lambda(I_1, J_1)\lambda(I_2, J_2) \ldots \lambda(I_k, J_k), \tag{3.7}$$

where * denotes summation over all sequences $(I_1, \ldots, I_k)$ giving a partition of I (not necessarily in increasing lexicographical order as in (3.6)) and such that $|J_1| \leqslant |I_1|, \ldots, |J_k| \leqslant |I_k|$.

Suppose now that $J_1, \ldots, J_k$ is a sequence of subsets of $\{0, \ldots, r-1\}$ which are not pairwise disjoint, so that $J_{m_1} \cap J_{m_2} \neq \emptyset$ for some $m_1 \neq m_2$. Then the coefficient of $D^k f(x_0;\ x_{J_1};\ \ldots;\ x_{J_k})$ in (3.6) is precisely the sum in (3.7) but in the present case the sum must be zero, since a derivative of this kind does not appear at all in (3.5). Thus, with the summation as in (3.7), for non-disjoint $J_1, \ldots, J_k$, we get

$$0 = \sum_{I_j}^{*} \lambda(I_1, J_1)\lambda(I_2, J_2) \ldots \lambda(I_k, J_k). \tag{3.8}$$

It is clear from the way the equations (1), (3.7) and (3.8) were obtained that they are also sufficient for (3.4) to commute.

It remains then to be shown that (3.7) and (3.8) are equivalent to assertions (2) and (3) of our Theorem. Indeed, (3) is the special case of (3.7) in which $k = |J| = l$ and $J_1, \ldots, J_k$ are one-element sets. Further, (2) is the special case of (3.8) in which $k = 2$ and $J_1 = J_2 = \{j\}$.

Assume now that (2) and (3) hold. To prove (3.7) we have to show that

$$\sum_{\varphi \in \Phi} \lambda(\varphi^{-1}(j_1), \{j\}) \ldots \lambda(\varphi^{-1}(j_l), \{j_l\}) = \sum_{I_j}^{*} \lambda(I_1, J_1) \ldots \lambda(I_k, J_k). \tag{3.9}$$

To see this, replace each of the $\lambda(I_1, J_1), \ldots, \lambda(I_k, J_k)$ by the corresponding sum in (3). Let $\Phi(I_1, \ldots, I_k)$ be the set of all surjections

$\varphi\colon I \to J$ such that $\varphi^{-1}(J_1) = I_1, \ldots, \varphi^{-1}(J_k) = I_k$. Then, rather evidently'

$$\lambda(I_1, J_1) \ldots \lambda(I_k, J_k) = \sum_{\varphi \in \Phi(I_1, \ldots, I_k)} \lambda\big(\varphi^{-1}(j_1), \{j_1\}\big) \ldots \lambda\big(\varphi^{-1}(j_l), \{j_l\}\big).$$

Applying $\sum\limits_{I_j}^{*}$, as in (3.7), to these equalities, we get (3.9).

It remains to prove (3.8). Without loss of generality we may assume that $J_1 \cap J_2 \neq \emptyset$. Consider any sequence $(I_1, \ldots, I_k)$ appearing in the summation of (3.8), and denote this sequence henceforth by $(\bar{I}_1, \ldots, \bar{I}_k)$. Let $\bar{I} = I - (\bar{I}_3 \cup \ldots \cup \bar{I}_k)$ and call (I_1, I_2) an *admissible pair* if $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \bar{I}$ and $|J_1| \leqslant |I_1|$, $|J_2| \leqslant |I_2|$. Then it is clear that $(I_1, I_2, \bar{I}_3, \ldots, \bar{I}_k)$ appears in the summation of (3.8) iff (I_1, I_2) is admissible. Therefore (3.8) will follow if we prove that

$$\sum_{(I_1, I_2)} \lambda(I_1, J_1)\lambda(I_2, J_2) = 0,$$

the sum being over all admissible pairs. But this equality is the same as (3.8) for $k = 2$, and $\bar{I}$ instead of I. Thus we have reduced the proof of (3.8) to the case $k = 2$.

Thus suppose that $k = 2$ and $J_1 = \{a, r_2, \ldots, r_u\}$, $J_2 = \{a, s_1, \ldots, s_v\}$. Denoting by $\Phi(I_i)$ the set of all surjections $I_i \to J_i$ $(i = 1, 2)$, we have

$$\lambda(I_1, J_1) = \sum_{\varphi_1 \in \Phi(I_1)} \lambda\big(\varphi_1^{-1}(a), \{a\}\big)\lambda\big(\varphi_1^{-1}(r_1), \{r_1\}\big) \ldots \lambda\big(\varphi_1^{-1}(r_u), \{r_u\}\big)$$

and

$$\lambda(I_2, J_2) = \sum_{\varphi_2 \in \Phi(I_2)} \lambda\big(\varphi_2^{-1}(a), \{a\}\big)\lambda\big(\varphi_2^{-1}(s_1), \{s_1\}\big) \ldots \lambda\big(\varphi_2^{-1}(s_v), \{s_v\}\big).$$

We have to show that

$$(3.10)\quad \sum_{(I_1, I_2)} \sum_{\varphi_1 \in \Phi(I_1), \varphi_2 \in \Phi(I_2)} \lambda\big(\varphi_1^{-1}(a), \{a\}\big)\lambda\big(\varphi_2^{-1}(a), \{a\}\big)\lambda\big(\varphi_1^{-1}(r_1), \{r_1\}\big) \ldots$$
$$\ldots \lambda\big(\varphi_2^{-1}(s_v), \{s_v\}\big) = 0,$$

where the first sum is over all partitions $I_1 \cup I_2 = I$. Given such a partition, and another one $(\tilde{I}_1, \tilde{I}_2)$, and also $\varphi_i \in \Phi(I_i)$, $\tilde{\varphi}_i \in \Phi(\tilde{I}_i)$, let us call the pairs $(\tilde{\varphi}_1, \tilde{\varphi}_2)$, (φ_1, φ_2) *associated* if

1° $I_i - \varphi_i^{-1}(a) = \tilde{I}_i - \tilde{\varphi}_i^{-1}(a)$, $i = 1, 2$, and

2° $\varphi_i | \big(I_i - \varphi_i^{-1}(a)\big) = \tilde{\varphi}_i | \big(\tilde{I}_i - \tilde{\varphi}_i^{-1}(a)\big)$, $i = 1, 2$ (equal restrictions).

Writing $(\varphi_1, \varphi_2) \simeq (\tilde{\varphi}_1, \tilde{\varphi}_2)$ in that case, we get an equivalence relation. It is clear that if (φ_1, φ_2) remains in one equivalence class, then the set $I^0 = \varphi_1^{-1}(a) \cup \varphi_2^{-1}(a)$ is constant. Furthermore, if the summation of products (3.10) is performed over just one equivalence class, then in each of the products only the first two terms vary, and thus we get a multiple of a sum of type (2), with I and j in (2) replaced by I^0 and a. Such sum vanishes by assumption, whence (3.10).

4. Classification theorems.

THEOREM 1. *There is a bijective correspondence between the set of all natural transformations* $T^r \to T^n$ *and the set of all maps* (3.1) *satisfying conditions* (1), (2) *and* (3) *in Section* 3. *Given a map* (3.1), *the corresponding natural transformation acts on Euclidean spaces as described by* (3.2) *and* (3.3).

Proof. By Corollary 1 in Section 2 and the Structure Theorem, the proof reduces to showing that if we have for every Euclidean space E a map $\lambda_{(E)}\colon T^r E \to T^n E$ such that (3.2), (3.3) and (1), (2), (3) in Section 3 hold, then there is a unique natural transformation λ whose actions on Euclidean spaces are the $\lambda_{(E)}$. The uniqueness is by Corollary 2, Section 2, and the existence is obtained as follows.

For U open in E, we have the obvious inclusions $T^r U \subset T^r E$ and $T^n U \subset T^n E$. Moreover, (3.2) and (3.3) imply that $\lambda_{(E)}(T^r U) \subset T^n U$. Thus we may define $\lambda_{(U)}\colon T^r U \to T^n U$ by $\lambda_{(U)} = \lambda_{(E)} | T^r U$. It follows that if $f\colon U \to W$ is a smooth map between open subsets of Euclidean spaces which can be extended to the corresponding Euclidean spaces, then, by (3.4),

$$
\begin{array}{ccc}
T^r U & \xrightarrow{\lambda_{(U)}} & T^n U \\
{\scriptstyle T^r f}\downarrow & & \downarrow{\scriptstyle T^n f} \\
T^r W & \xrightarrow[\lambda_{(W)}]{} & T^n W
\end{array}
\tag{4.1}
$$

commutes. From this it can be seen that the assumption of the extendability of f beyond U is not essential, for indeed U can be covered by smaller subsets such that from each of these f is extendable to the whole space, so the restriction of (4.1) to any of these smaller subsets commutes, and therefore (4.1) commutes.

If $M \in \mathcal{M}$, then M can be covered by neighbourhoods Q which admit diffeomorphisms $\psi\colon U \to Q$, where $U \subset E$ are as above. Thus, for each such Q, there is a unique $\lambda_{(Q)}$ such that

$$
\begin{array}{ccc}
T^r U & \xrightarrow{\lambda_{(U)}} & T^n U \\
{\scriptstyle T^r \psi}\downarrow & & \downarrow{\scriptstyle T^n \psi} \\
T^r Q & \xrightarrow[\lambda_{(Q)}]{} & T^n Q
\end{array}
\tag{4.2}
$$

commutes. Combining (4.1) and (4.2), one sees that $\lambda(Q)$ does not depend on the choice of ψ or U, and also that $\lambda_{(Q_1)} = \lambda_{(Q_2)}$ on $Q_1 \cap Q_2$ whenever the latter set is non-empty. Thus there is a unique map $\lambda_{(M)}\colon T^r M \to T^n M$ compatible with all the $\lambda_{(Q)}$. Again by (4.1) and (4.2) one sees that the collection of all $\lambda_{(M)}$, $M \in \mathscr{M}$, is a natural transformation.

THEOREM 2. *There is a bijective correspondence between the set of all natural transformations $T^r \to T^n$ and the set of all maps*

$$\lambda\colon P_*\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, r-1\} \to R \tag{4.3}$$

satisfying condition (2) *of Section* 3.

Indeed, a map λ as above admits a unique extension to a map (3.1) which satisfies conditions (1), (2) and (3) of the Structure Theorem; it suffices to define $\lambda(I, J)$ by conditions (1) and (3).

5. The Co-product Theorem. Let $\eta^{(j,r)}$ be $T^j \to T^{j+1} \to \ldots \to T^r$ $(0 \leqslant j < r)$, where each arrow is the natural transformation corresponding to the zero-section of a manifold into its tangent bundle. Define, for $j = 1, \ldots, r$, $i^{(j,r)}_{(R)}\colon TR \to T^r R$ by

$$i^{(j,r)}_{(R)}(x_0, x_1) = (x_0, 0, \ldots, 0, x_1, 0, \ldots, 0), \tag{5.1}$$

where x_1 is at the 2^{j-1} place. Then, by Corollary 2, Section 2, it follows that (5.1) is the action on R of the natural transformation

$$i^{(j,r)} = \begin{cases} \eta^{(1,r)} & \text{if } j = 1, \\ \eta^{(j,r)} \circ T\eta^{(0,j-1)} & \text{if } j = 2, 3, \ldots, r. \end{cases}$$

The following theorem states that in the tangent functor category $\mathscr{T}$ the morphisms $i^{(j,r)}\colon T \to T^r$, $j = 1, \ldots, r$, make T^r into the sum (co-product) of r copies of T.

CO-PRODUCT THEOREM. *Let r, $n > 0$ be given. Then for any natural transformations $\beta^{(j)}\colon T \to T^n$, $j = 1, \ldots, r$, there is a unique natural transformation $\lambda\colon T^r \to T^n$ such that for every $j = 1, \ldots, r$ diagram* (1.2)′ *commutes.*

Proof. We show first that there is at most one λ satisfying $\beta^{(j)} = \lambda \circ i^{(j,r)}$ for every $j = 1, \ldots, r$. Indeed, if such λ exists, then, for every $(x_0, x_1) \in TR$ and $\emptyset \neq I \subset \{0, \ldots, n-1\}$ we have, by (5.1)

$$\beta^{(j)}_I(x_0, x_1) = \beta^{(j)}(I, \{0\})x_1 = \lambda(I, \{j-1\})x_1. \tag{5.2}$$

Thus function (4.3) is determined by the $\beta^{(j)}$, $j = 1, \ldots, r$, whence, by Theorem 2 in Section 4, the natural transformation $\lambda\colon T^r \to T^n$ is unique. The existence of λ follows again by Theorem 2 in Section 4, taking (5.2) as the definition of λ. Indeed, condition (2) in Section 3 must then hold for λ because it must hold for each of the natural transformations $\beta^{(j)}$.

In view of the above theorem, a description of all the natural transformations $T \to T^n$ may be of interest. In this case we may by-pass Corollary 2 of Section 2. Indeed, if $\lambda_{(R)}\colon TR \to T^n R$ is known for a natural transformation $\lambda\colon T \to T^n$, then from the commutativity of

(5.3)
$$\begin{tikzcd} TR \arrow[r, "\lambda_{(R)}"] \arrow[d, "Tf"'] & T^n R \arrow[d, "T^n f"] \\ TM \arrow[r, "\lambda_{(M)}"'] & T^n M \end{tikzcd}$$

for every curve $f\colon R \to M$, where $M \in \mathcal{M}$, we obtain $\lambda_{(M)}$ on every tangent vector to M.

Given integers $i, j > 0$, write $i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$, $j = 2^{j_1} + 2^{j_2} + + \ldots + 2^{j_l}$, where $0 \leqslant i_1 < \ldots < i_k$, $0 \leqslant j_1 < \ldots < j_l$, and set

$$a_{ij} = \begin{cases} 1 & \text{if } \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition. *A map $\lambda_{(R)}\colon TR \to T^n R$ describes in the above way* (*i. e. by requiring that all diagrams* (5.3) *commute*) *a natural transformation $\lambda\colon T \to T^n$ if and only if*

$$\lambda_{(R)}(x_0, x_1) = (x_0, \lambda_1 x_1, \lambda_2 x_1, \ldots, \lambda_{2^n-1} x_1)$$

for all $(x_0, x_1) \in TR$, where $\lambda_1, \ldots, \lambda_{2^n-1}$ are real numbers satisfying the equations

$$\sum_{i+j=k} \lambda_i \lambda_j a_{ij} = 0, \qquad k = 2, 3, 4, \ldots, 2^n.$$

From this it follows in particular that for $n > 1$, $\lambda_1, \ldots, \lambda_{2^n-1}$ belong in the above sense to a natural transformation $T \to T^n$ iff $\lambda_1, \ldots, \lambda_{2^{n-1}-1}$ belong to a natural transformation $T \to T^{n-1}$. Moreover, for given $\lambda_1, \ldots \ldots, \lambda_{2^{n-1}-1}$, all the possible continuations $\lambda_{2^{n-1}}, \ldots \lambda_{2^n-1}$ are obtained by solving the above equations for $k = 2^{n-1}+1, \ldots, 2^n$ which are *linear* with respect to the variables $\lambda_{2^{n-1}}, \ldots, \lambda_{2^n-1}$.

All this can be easily deduced from the preceding results.

6. Invertible transformations.

Lemma. *Suppose $\lambda\colon T^r \to T^n$ $(r, n \geqslant 1)$ is a natural transformation such that $\lambda(\{i\}, \{j\}) \neq 0$ for some $0 \leqslant i \leqslant r-1$, $0 \leqslant j \leqslant n-1$. Then $\lambda(I, \{j\}) = 0$ whenever $i \notin I \subset \{0, \ldots, r-1\}$.*

Proof. We use induction on $m = |I|$. Thus let $m = 1$ and let $i \notin I$, that is $I = \{k\}$ and $k \neq i$. Condition (2) of the Structure Theorem (Section 3) applied to $\{k, i\}$ and j yields $\lambda(\{k\}, \{j\})\lambda(\{i\}, \{u\}) = 0$, whence $\lambda(\{k\}, \{j\}) = 0$.

Assuming that the Lemma is valid for some $m \geqslant 1$, take any I with $i \notin I$ and $|I| = m+1$, and put $I_0 = I \cup \{i\}$. Then in (condition (2) again)

$$\sum_{I_1, I_2}^{I_0} \lambda(I_1, \{j\})\lambda(I_2, \{j\}) = 0$$

one of the summands is $\lambda(I, \{j\})\lambda(\{i\}, \{j\})$. For each other summand we have that either I_1 or I_2 contains i, and something else. But if, say, $i \in I_1$ and $|I_1| \geqslant 2$, then $i \notin I_2$ and $|I_2| \leqslant m$, whence by the inductive assumption that summand vanishes. Thus the above sum reduces to the term first mentioned, and as this must vanish, we have $\lambda(I, \{j\}) = 0$.

THEOREM. *Let $Gp(T_n)$ denote the group of all invertible natural transformations $T^n \to T^n$. Then $Gp(T^n)$ is a Lie group of dimension $n \cdot 2^{n-1}$ and its connected component of identity $\mathring{Gp}(T^n)$ is solvable.*

Proof. By Theorem 1 in Section 4, we may identify $\lambda\colon T^n \to T^n$ with the matrix $\bar{\lambda} \in Gl(2^n-1, R)$ whose coefficients are $\lambda(I, J)$, $\emptyset \neq I, J \subset \{0, 1, \ldots, n-1\}$. Composition of natural transformations corresponds to multiplication of their matrices, i.e. if λ has an inverse, then so does $\bar{\lambda}$. Conversely, suppose $(\bar{\lambda})^{-1}$ exists. It is clear that then for every Euclidean space $\lambda_{(E)}\colon T^n E \to T^n E$ has an inverse $\lambda_{(E)}^{-1}$. Moreover, all diagrams (3.4) commute if $\lambda_{(E)}, \lambda_{(F)}$ are replaced by $\lambda_{(E)}^{-1}, \lambda_{(F)}^{-1}$ (and $r = n$). Thus, by the Structure Theorem, the matrix $(\bar{\lambda})^{-1}$ satisfies conditions (1), (2) and (3). By Theorem 1, Section 4, the matrix $(\bar{\lambda})^{-1}$ defines a natural transformation, and the latter is evidently λ^{-1}.

It follows that $Gp(T^n) \subset Gl(2^n-1, R)$ is composed of precisely those matrices which are invertible and whose coefficients $\lambda(I, J)$ are subject to the polynomial identities (1), (2) and (3) of the Structure Theorem. So $Gp(T^n)$ is a Lie group.

Let $k\colon \mathring{Gp}(T^n) \to Gl(n, R)$ be the map assigning to λ the matrix $\big(k(\lambda)\big)_{i,j} = \lambda(\{i\}, \{j\})$, where $i, j = 0, \ldots, n-1$. The fact that $\lambda(\{i\}, J) = 0$, whenever $|J| > 1$, implies that k is a homomorphism. If $\lambda(\{i\}, \{j\}) \neq 0$ for some i, j, then, by the Lemma above, $\lambda(\{k\}, \{j\}) = 0$ for all $k \neq j$. Thus the matrix $k(\lambda)$ has exactly one non-zero coefficient in each column. Since $\lambda(\{j\}, \{j\}) = 1$ holds for the identity transformation $\lambda \in \mathring{Gp}(T^n)$, we conclude that the connected group $k\big(\mathring{Gp}(T^n)\big)$ is composed of diagonal matrices with positive entries on the diagonal.

From $\lambda(\{j\}, \{j\}) > 0$ it follows, by the above Lemma, that $\lambda(I, \{j\}) = 0$ whenever $j \notin I$. By indentity (3) of the Structure Theorem, we conclude that $\lambda(I, J) = 0$ unless $J \subset I$, which means that $\mathring{Gp}(T^n)$ is composed of triangular matrices, and is therefore solvable.

By Theorem 2, Section 4, $\lambda\colon T^n \to T^n$ is completely determined by the real numbers $\lambda(I, \{j\})$, $\emptyset \neq I, \{j\} \subset \{0, 1, \ldots, n-1\}$, and any choice of such numbers will yield a natural transformation, provided (2) in Section 3 is satisfied. But for $\lambda \in \mathring{Gp}(T^n)$ we must have $\lambda(I, \{j\}) = 0$ whenever $j \notin I$ and these equalities imply condition (2) of Section 3. Therefore, to obtain an element $\lambda \in \mathring{Gp}(T^n)$ we may choose quite arbitrarily the numbers $\lambda(\{j\}, \{j\}) > 0$ and the $\lambda(I, \{j\})$, where $j \in I$. As there are $n \cdot 2^{n-1}$ of these, $\dim \mathring{Gp}(T^n) = n \cdot 2^{n-1}$.

Remark. A slight refinement of the above proof yields that $Gp(T^n)/\mathring{Gp}(T^n)$ is isomorphic to the full symmetric group on n symbols.

7. Monads. A triple (T, λ, η) is called a *monad* if $\lambda\colon T^2 \to T$ and $\eta\colon I \to T$ are natural transformations satisfying the diagrams

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, "\lambda"] & T \arrow[l, "T_\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \tag{7.1}$$

and

$$\begin{tikzcd} T^3 \arrow[r, "T_\lambda"] \arrow[d, "\lambda_T"'] & T^2 \arrow[d, "\lambda"] \\ T^2 \arrow[r, "\lambda"'] & T \end{tikzcd} .$$

As well known from category theory, the existence of a monad (T, λ, η) is equivalent to the existence of a pair of adjoint functors $F\colon \mathcal{M} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{M}$, where $\mathcal{C}$ is some category and F is the left adjoint of G, such that $T = G \circ F$.

Suppose $\lambda\colon T^2 \to T$ is a natural transformation. By the Structure Theorem,

$$\lambda_{(R)}(x_0, x_1, x_2, x_3) = \big(x_0, \lambda(\{0\}, \{0\})x_1 + \lambda(\{0\}, \{1\})x_2\big),$$

where the numbers $a = \lambda(\{0\}, \{0\})$ and $b = \lambda(\{0\}, \{1\})$ are arbitrary. Let $\tau_{(M)}\colon TM \to M$ denote the "base point projection". Then the natural transformation $\tau\colon T \to I$ satisfies, for $x = (x_0, x_1, x_2, x_3) \in T^2R$,

$$(\tau_T)_{(R)}(x) = (x_0, x_1) \quad \text{and} \quad (T\tau)_{(R)}(x) = (x_0, x_2).$$

Thus $\lambda = a\tau_T + bT\tau$ is the general formula for λ.

If $\eta\colon I \to T$ is a natural transformation, then $\eta_{(R)}(x_0) = (x_0, \eta_1(x_0))$ for all $x_0 \in R$, where $\eta_1\colon R \to R$. The requirement that

$$\begin{tikzcd} R \arrow[r, "\eta(R)"] \arrow[d, "f"'] & TR \arrow[d, "Tf"] \\ R \arrow[r, "\eta(R)"'] & TR \end{tikzcd}$$

should commute for every smooth f leads quickly to the conclusion that η_1 is identically 0, and therefore η is the natural transformation such that $\eta_{(M)}\colon M \to TM$ is the zero-section of a manifold into its tangent bundle.

THEOREM. *$(T, \tau_T + T\tau, \eta)$ is the only monad for the functor T.*

Proof. Indeed, the first diagram (7.1) will commute for $\lambda = a\tau_T + bT\tau$ if and only if $a = b = 1$. It so happens that in this case also the second (square) diagram commutes.

Using the description of the natural transformations $T \to T^2$ given in Section 4, one can verify that the dual object, a co-monad for the functor T, does not exist.

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