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PRESENTATIONS OF THE FUNDAMENTAL GROUP OF A MANIFOLD

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The way in which a handle decomposition of a compact manifold M determines a presentation of its fundamental group $\pi_1 M$ is well known (cf. [2]). In the present paper * we show the converse, i.e., that every presentation of the group $\pi_1 M$ can be obtained in that way provided $\dim M > 4$. For smooth manifolds this yields an evaluation of the *first Morse number* $\mu_1 M$ (i.e., the minimum over all Morse functions on M of the number of critical points of index 1) which in the closed case will turn to be the minimal number of generators of $\pi_1 M$. The results, as well as their proofs, are valid both in smooth and PL categories.

We adopt the terminology of handle theory from [6]. In particular, $M \cup_f h^q$ denotes M with a handle h^q of index q added by means of the attaching map f. A presentation of a group with generators $a_1, \ldots, a_i$ and relators $R_1, \ldots, R_t$ will be written as $(a_1, \ldots, a_i; R_1, \ldots, R_t)$. D^n is the disc $\{x \in \boldsymbol{R}^n: \|x\| \leqslant 1\}$, and S^{n-1} is its boundary ∂D^n.

Recall a well-known result of Morse (see [5] and [1]). By an *a-sphere* we mean here, following [6], the left sphere, and by a *b-sphere* — the right sphere in the sense of Milnor [4].

CANCELLATION THEOREM. *Suppose $W' = W \cup h \cup H$. If the a-sphere of H and the b-sphere of h intersect transversally and just at one point, then $W' = W$.*

First we describe a procedure which leads from an arbitrary presentation of the fundamental group of a connected manifold M^n to a presentation of the fundamental group of M^n with attached handles. This procedure will be then applied to handle decompositions of closed manifolds by starting with $M^n = D^n$, and to those of manifolds with boundary by starting with M^n being a collar on a boundary with some 1-handles connecting the collar.

We will consider handles with pairwise disjoint attaching maps and for that purpose we assume a fixed oriented disc D in ∂M containing

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images of attaching maps of 1-handles and having non-empty intersections with the image of attaching maps of 2-handles. With such a disc at hand we are free to use the relative fundamental group $\pi_1(M, D)$.

Let M be a compact connected manifold. Add to M 1-handles $h_1, \ldots, h_k$ with the pairwise disjoint images of attaching maps. Since the group $\pi_1(M \cup h_1 \cup \ldots \cup h_k, D)$ is canonically isomorphic to $\pi_1(M, D) * *Z* \ldots *Z$ (free product of $\pi_1(M, D)$ and of k copies of integers), we may consider $\pi_1(M, D)$ as embedded in $\pi_1(M \cup h_1 \cup \ldots \cup h_k, D)$. The *core* of each handle h_i represents a generator (which we denote by the same letter h_i) of a copy of Z. If we now add to $M \cup h_1 \cup \ldots \cup h_k$ handles $H_1, \ldots, H_v$ of index 2, then the homotopy classes of their attaching maps determine elements of $\pi_1(M \cup h_1 \cup \ldots \cup h_k, D)$, also denoted by the same letters $H_1, \ldots, H_v$. If

$$(a_1, \ldots, a_l;\ R_1, \ldots, R_w) \tag{1}$$

is a presentation of $\pi_1 M$, then the presentation of $\pi_1(M \cup h_1 \cup \ldots \cup h_k, D)$ is

$$(a_1, \ldots, a_l, h_1, \ldots, h_k; R_1, \ldots, R_w). \tag{2}$$

In turn, presentation (2) yields the presentation

$$(a_1, \ldots, a_l, h_1, \ldots, h_k; R_1, \ldots, R_w, H_1, \ldots, H_v) \tag{3}$$

of $\pi_1(M \cup h_1 \cup \ldots \cup h_k \cup H_1 \cup \ldots \cup H_v)$.

The following lemma provides useful operations on handle decompositions of a manifold patterned after those on presentations of a group. By virtue of this lemma each passing from one presentation of $\pi_1 M$ to another is geometrically realizable.

LEMMA. *Let M^n be a compact connected manifold and let the disc D be contained in a connected component A of ∂M so that the inclusion $A \subset M$ induces an epimorphism $\pi_1 A \to \pi_1 M$. Let M be of dimension $n > 4$. Then*

(a) *for each element ω of $\pi_1(M, D)$, there exists a complementary pair h^1, H^2 of handles such that the a-sphere of H determines $h^1\omega \in \pi_1(M \cup h^1, D)$;*

(b) *if the a-sphere of H^2 determines the element $h\omega \in \pi_1(M \cup h, D)$, where $\omega \in \pi_1(M, D) \subset \pi_1(M \cup h, D)$, then h and H are complementary, i.e., they are such as in the Cancellation Theorem;*

(c) *for each normal subgroup G of $\pi_1(M, D)$ generated by the a-spheres of $H_1^2, \ldots, H_w^2$, if H_{w+1}^2 determines an element of G, then the attaching map of H_{w+1}^2 can be isotoped in $\partial(M \cup H_1^2 \cup \ldots \cup H_w^2)$ to a trivial embedding (i.e., an embedding extendable to an embedding of 2-disc); if $r \in G$, then there exists a complementary pair H_{w+1}^2, H^3 with H_{w+1}^2 determining r.*

Proof. (a) By general position, every element of $\pi_1(M, D)$ is represented by an embedding of S^1 in ∂M. Add to M a handle h of index 1 in the following way. In $S^0 \times D^{n-1} \subset h$ take the orientation induced by an orientation of $D^1 \times D^{n-1}$. If the normal bundle of the sphere representing $\omega \in \pi_1(M, D)$ is trivial, then we take the attaching map of h which preserves orientations, and if the normal bundle is non-trivial, the attaching map has to preserve orientation on one component of $S^0 \times D^{n-1}$ and change on the second. It is clear that the element $h\omega \in \pi_1(M \cup h, D)$ is represented by a sphere embedded with trivial normal bundle and transversal one-point intersection with the b-sphere of h. By the Cancellation Theorem, a 2-handle H attached by means of that embedding is complementary to h.

(b) We can choose a sphere in the class ω having empty intersection with the b-sphere of h. Therefore, $h\omega$ is represented by a sphere S which intersects the b-sphere of h transversally and at one point only. The sphere S and the a-sphere of H are in the same element of $\pi_1(M \cup h, D)$ and so, by general position, they are isotopic. Now (b) follows from the Cancellation Theorem.

(c) If an embedding represents an element of G, then it is null-homotopic in $\partial(M \cup H_1 \cup \ldots \cup H_w)$. Hence, by general position again, it is isotopic to a trivial embedding.

The complementary pair needed in the second part of (c) can be described as follows. Add to $M \cup H_1 \cup \ldots \cup H_w$ a 2-handle H_{w+1} with the attaching map which is a trivial embedding, and then add a 3-handle H^3 complementary to H_{w+1}. Let an embedding φ represents r. The trivial embedding is isotopic to φ in $\partial(M \cup H_1 \cup \ldots \cup H_w)$. If we change the attaching map of H_{w+1} by that isotopy, we get a pair required in the Lemma.

Consider now handle decompositions of closed manifolds. Without loss of generality we can assume that all considered decompositions are nice (i.e., handles are added in the order of increasing dimensions and attaching maps of handles of the same index are disjoint) and with only one 0-handle. Suppose also that every handle is oriented (but we do not assume orientability of manifolds). For a fixed decomposition of X, denote by X_i submanifolds consisting of all handles of indices not greater than i. Note that $\pi_1 X_2 \cong \pi_1 X$ and that each such decomposition gives a presentation of $\pi_1 X$ in the way described above (where $M^n = D^n$ and the presentation of $\pi_1 D^n$ is trivial). Every 2-handle determines a relator which is a word in the free group generated by 1-handles. If all trivial words are excluded from the set of relators, the resulting presentation is called induced by the handle decomposition.

THEOREM 1. *If X is a connected closed manifold of dimension $n > 4$, then every presentation of $\pi_1 X$ is induced by a handle decomposition.*

Proof. Consider the set of finite presentations of a finitely presented group. A theorem of Tietze (cf. Theorem 2 below or [3], p. 265) says that one can pass from any presentation to another by a finite number of Tietze transformations.

The Tietze transformations consist in the following alterations of presentations:

(A) addition of a new symbol h to the set of generators and a word $h\omega$ (ω is a word in the old set of generators) to the set of relators;

(B) the change reverse to (A): removal of a generator h and of a relator $h\omega$ (h can appear neither in the word ω nor in the remaining relators);

(C) removal or addition of a relator belonging to the normal subgroup generated by the remaining relators.

The Lemma makes Tietze transformations (A)-(C) applicable to the topological situation and the theorem follows.

COROLLARY. *The first Morse number $\mu_1 X$ of a closed connected smooth manifold X of dimension $n > 4$ is equal to the minimal number of generators in presentations of $\pi_1 X$.*

In the case of manifolds with boundary, presentations depend not only on handle decompositions, but also on presentations of the fundamental group of the boundary. Suppose that X is compact and connected and consider nice oriented handle decompositions of X on ∂X having no 0-handles. If the boundary ∂X is connected, then every presentation $(a_1, \ldots, a_\alpha; R_1, \ldots, R_t)$ of $\pi_1 \partial X$ can be extended to a presentation

$$(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta; R_1, \ldots, R_t, Q_1, \ldots, Q_u) \quad \text{of } \pi_1 X$$

in the same way as in the closed case. Namely,

$$\pi_1 X_1 \cong \pi_1 \partial X * Z * \ldots * Z$$

has the presentation

$$(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta; R_1, \ldots, R_t),$$

where 1-handles are in 1-1 correspondence with elements of the set $(b_1, \ldots, b_\beta)$ of generators. An attaching map of the i-th 2-handle carries the generator of $\pi_1(S^1 \times D^{n-2})$ determined by orientation of the handle to an element of $\pi_1 X_1$. This element can be represented by a word in generators $a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta$ and the word can be taken as the relator Q_i (thus the relators are determined modulo the normal subgroup of $(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta)$ generated by $\{R_1, \ldots, R_t\}$).

Let now ∂X have k connected components $N_1, \ldots, N_k$. A 1-handle joins N_i to N_j if its attaching map carries one component of $S^0 \times D^{n-1}$ into N_i and the second into N_j. By the connectivity of X it follows that any two components are joined by a sequence of handles. By appropriate shifting of attaching maps of 1-handles one can get a decomposition with all 1-handles attached to N_1 except for $k-1$ handles $h_2, \ldots, h_k$ such that h_j joins N_1 to N_j for $j = 2, \ldots, k$. In the rest of the paper we will assume that each considered decomposition has that property.

We say that

$$(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta; R_1, \ldots, R_t, Q_1, \ldots, Q_u)$$

is a *presentation induced by the handle decomposition and the presentation* $(a_1, \ldots, a_\alpha; R_1, \ldots, R_t)$ *of* $\pi_1 N_1 * \ldots * \pi_1 N_k$ if

(i) $(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta; R_1, \ldots, R_t)$ is the presentation of

$$\pi_1 X_1 \cong \pi_1 N_1 * \ldots * \pi_1 N_k * Z * \ldots * Z$$

(Z taken β times) and each 1-handle, except for joining handles, determines a generator of a copy of Z;

(ii) $Q_1, \ldots, Q_u$ are non-trivial elements of $\pi_1 X_1$, each determined by a 2-handle.

For the proof of a relative version of Theorem 1 we need the following mild strengthening of the Tietze theorem:

THEOREM 2. *Suppose that*

(1) $(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta; R_1, \ldots, R_t, Q_1, \ldots, Q_u)$,

(2) $(a_1, \ldots, a_\alpha, c_1, \ldots, c_\gamma; R_1, \ldots, R_t, T_1, \ldots, T_v)$

are presentations of a group G. If $R_1, \ldots, R_t$ are words in $a_1, \ldots, a_\alpha$ only, then one can pass from (1) *to* (2) *by a finite number of Tietze transformations preserving $a_1, \ldots, a_\alpha$ and $R_1, \ldots, R_t$.*

Proof. Denote by $\boldsymbol{a}$ the sequence $a_1, \ldots, a_\alpha$, by $\boldsymbol{b}$ — the sequence $b_1, \ldots, b_\beta$, etc. There exist

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\gamma) \quad \text{and} \quad \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\beta)$$

such that

$$\boldsymbol{c} = \boldsymbol{\omega}(\boldsymbol{a}, \boldsymbol{b}) \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{\varphi}(\boldsymbol{a}, \boldsymbol{c}).$$

Using Tietze transformations of type (A), we change the presentation (1) to

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{R}, \boldsymbol{Q}, \boldsymbol{c}^{-1}\boldsymbol{\omega})$$

and further to

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{R}, \boldsymbol{Q}, \boldsymbol{c}^{-1}\boldsymbol{\omega}, \boldsymbol{T}, \boldsymbol{b}^{-1}\boldsymbol{\varphi}).$$

The latter presentation can be obtained from (2) as well. Since Tietze transformations are reversible, the theorem follows.

For any compact connected manifold X of dimension $n > 4$ and any handle decomposition of X, the manifold $M^n = X_1$ together with the component $A = \partial X_1 \setminus \partial X$ satisfy assumptions of the Lemma. Thus the class of presentations induced by handle decompositions is invariant under Tietze transformations. Hence, by Theorem 2, we get the following generalization of Theorem 1.

THEOREM 3. *Let X be a compact connected manifold of dimension $n > 4$, with the boundary $\partial X = N_1 \cup \ldots \cup N_k$, where $N_1, \ldots, N_k$ are disjoint* *and connected. If* $(a_1, \ldots, a_a; R_1, \ldots, R_t)$ *is a presentation of* $\pi_1 N_1 * \ldots * \pi_1 N_k$, *then each presentation of* $\pi_1 X$ *of the form*

$$(a_1, \ldots, a_a, b_1, \ldots, b_\beta; R_1, \ldots, R_t, Q_1, \ldots, Q_u)$$

is induced by a handle decomposition of X.

Theorem 3 yields the following evaluation of the first Morse number of a smooth and compact, not necessarily closed manifold:

COROLLARY. *Under the assumptions of Theorem* 3, $\beta = \mu_1 X^n - k + 1$ *is the minimal number with the following property: there exists a presentation*

$$(a_1, \ldots, a_a, b_1, \ldots, b_\beta; R_1, \ldots, R_t, Q_1, \ldots, Q_u) \quad \textit{of } \pi_1 X$$

such that $(a_1, \ldots, a_a; R_1, \ldots, R_t)$ *is a presentation of* $\pi_1 N_1 * \ldots * \pi_1 N_k$.

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