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DIFFERENTIATION OF IMPLICIT FUNCTIONS AND STEINHAUS' THEOREM IN TOPOLOGICAL MEASURE SPACES

BY

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Steinhaus' theorem [9], asserting that for any sets $A, B \subset R$ of positive measure the set $A+B$ has interior points, and its group analogue due to Weil [10], chapitre III, § 11, p. 50, admit various generalizations and modifications. For instance, as shown by Erdös and Oxtoby [1], Theorem 3, the operation of addition in Steinhaus' theorem may be replaced by any binary C^1-operation with non-vanishing partial derivatives. The present paper is a continuation of the author's previous papers [4] (joint) and [3], containing extensions of Steinhaus' theorem. (The result of [4] is, in fact, nothing else but the above-mentioned Theorem 3 of [1]; the papers [1] and [4] are independent, and so is also Miller's paper [5] containing the same result.) The proofs given in [4] and [3] are based on a common idea inspired by Weil's convolution proof. The Theorem presented in Section 2 is a joint extension of the results of those papers (to some extent also of Weil's theorem) and its proof follows once more the same pattern. The difficulties that arise are rather of technical nature and require the use of a property corresponding to the rule for the differentiation of an implicit function (Proposition 2).

Weil's idea has also been taken up by Paganoni [6] and carried on by Sander [7]. Their setting the problem is somewhat more general than that presented in this paper (see p. 106). Roughly speaking, in [6] and [7] stronger assertions have been obtained under stronger assumptions, and the method applied differs considerably.

The arrangement of the contents of this paper is quite parallel to that of [3]; propositions and lemmas closely correspond to those in [3]. Facts and arguments contained in [3] are often referred to.

1. Notation and definitions. These are also preserved from [3]. We list briefly some of them, introducing also some other definitions and conventions. For any unexplained non-standard notions or notation we refer the reader to [3].

Let $(X, \mathcal{M})$ be a measurable space (a set X with a σ-field $\mathcal{M}$ of its subsets). In the sequel we shall be concerned with mappings of the form $X \to X$, $X \times X \to X$, and $X \times X \to X \times X$. Speaking of measurability of such mappings we always mean measurability with respect to the σ-fields $\mathcal{M}$ (in X) and $\mathcal{M} \times \mathcal{M}$ (in $X \times X$).

A bijection $g\colon X \to X$ is called *bimeasurable* if both g and g^{-1} are measurable.

Let m be a measure on $\mathcal{M}$. Sets of measure zero are called *nullsets*. A measurable mapping $g\colon X \to X$ is called *non-singular* if $g^{-1}(E)$ is a nullset for any nullset $E \subset X$.

Let m and μ be two measures defined on the same σ-field $\mathcal{M}$. The fact that μ is m-continuous is written symbolically: $\mu \ll m$; the notation $d\mu = \varrho\, dm$ (or $\varrho = d\mu/dm$) means that ϱ is the RN (Radon-Nikodym) derivative (or the density) of μ with respect to m. In the case of μ given by $\mu(E) = m(g(E))$, where $g\colon X \to X$ is some bimeasurable mapping, the function $d\mu/dm$ is also called the RN derivative of the mapping g.

If $\varkappa$ is another measure on $\mathcal{M}$, we have

$$\frac{d\varkappa}{dm} = \frac{d\varkappa}{d\mu} \cdot \frac{d\mu}{dm}, \tag{1}$$

provided the densities on the right-hand side exist. If g and h are bimeasurable mappings, non-singular together with their inverses, and if $\alpha, \beta, \gamma, \delta$ denote the RN derivatives (with respect to the same measure m) of g, h, h^{-1}, $g \circ h^{-1}$, respectively, then

$$\gamma = \frac{1}{\beta \circ h^{-1}} \quad \text{and} \quad \delta = \frac{\alpha \circ h^{-1}}{\beta \circ h^{-1}}. \tag{2}$$

These are rather familiar facts from measure theory (for proofs see, e.g., [2]). Equalities (1) and (2) are asserted to hold m-a.e.; in fact, the functions occurring there are defined up to equivalence.

If X is a topological space, we denote by $\mathcal{B}(X)$ the σ-field of Borel subsets of X. A measure m defined on $\mathcal{B}(X)$ is *regular* if it is finite on compact sets and satisfies

$$\begin{aligned} m(E) &= \sup\{m(K)\colon K \text{ compact},\ K \subset E\} \\ &= \inf\{m(U)\colon U \text{ open},\ U \supset E\} \end{aligned} \tag{3}$$

for any set $E \in \mathcal{B}(X)$. If $g\colon X \to X$ is a homeomorphism, then the regularity of m implies the regularity of the measure $E \mapsto m(g(E))$. If ϱ is a continuous non-negative function on X, then the measure $d\mu = \varrho\, dm$ is regular, provided m is regular.

Let $f\colon X\times X\to X$. The following solvability conditions have been introduced by the author in [3]. The first of them makes sense if X is any set, the second and the fourth are meaningful if X is a topological space; in the third one, X has to be a measurable space:

(S) The equation $f(x, y) = z$ has a unique solution y, respectively x, for any fixed values of x, z, respectively y, z; this solution will be denoted by $\varphi(x, z)$, respectively $\psi(z, y)$.

Condition (S) defines thus two mappings $\varphi, \psi\colon X\times X\to X$.

(CS) f satisfies (S), f, φ, and ψ are continuous.

(MS) f satisfies (S), f, φ, and ψ are measurable.

(BS) f satisfies (MS) with respect to the σ-field $\mathscr{B}(X)$.

Condition (S) means that, for $x, y, z\in X$ fixed, the mappings

$$f(\cdot, y),\ f(x, \cdot), \tag{4}$$

hence also

$$\varphi(\cdot, z),\ \psi(z, \cdot) \tag{5}$$

are bijections. Clearly, for $z\in X$ fixed, mappings (5) are mutually inverse.

Condition (CS) implies that mappings (4) and (5) are homeomorphisms. Similarly, condition (MS) implies that they are bimeasurable. In the case of a topological space with a countable base, (BS) is a consequence of (CS).

2. Extension of Steinhaus' theorem. We now formulate the main result:

THEOREM. *Let X be a topological space and let m be a σ-finite regular measure on $\mathscr{B}(X)$. Let $f\colon X\times X\to X$ be a transformation satisfying conditions* (CS) *and* (BS) *and such that, for any $x, y\in X$ fixed, homeomorphisms* (4) *are non-singular together with their inverses. Further, suppose that there exist positive-valued continuous measurable functions α and β on $X\times X$ such that, for $x, y\in X$ fixed, the functions $\alpha(\cdot, y)$ and $\beta(x, \cdot)$ are the RN derivatives of mappings* (4). *Then for any sets $A, B\in\mathscr{B}(X)$ of positive measure the set $f(A\times B)$ has interior points.*

Clearly, for $X = R$, m being Lebesgue measure, and $f(x, y) = x+y$ (then $\alpha = \beta\equiv 1$), this is precisely Steinhaus' theorem. If X is a locally compact σ-compact group, m is Haar measure, and f is the group operation (then either $\alpha\equiv 1$ or $\beta\equiv 1$, according as m is left or right invariant), we obtain Weil's theorem.

In the case of f being coordinatewise measure-preserving (i.e., in the case where $\alpha = \beta\equiv 1$), this result has been established in [3]. In the present situation the assumptions concerning f are somewhat less restrictive. A result of this type for $X = R$ has been obtained in three independent papers [1], [4], and [5].

The proof of the Theorem is split into several steps, similarly to the proof of the Theorem in [3]. Propositions 1 and 2 are direct analogues of Propositions 1 and 2 of [3]; Lemma 1 corresponds to Lemma 3 of [3].

In this section we only give Proposition 1 sketching its proof. The proof of the Theorem will be completed in the next section.

PROPOSITION 1. *Let X and m be as in the Theorem and let $f: X \times X \to X$ be a transformation with the following properties:*

(a) *f satisfies condition* (S), *φ is measurable, and ψ is continuous;*

(b) *$f(x, \cdot)$ and $\psi(z, \cdot)$ are measurable for any $x, z \in X$;*

(c) *$\varphi(x, \cdot)$ is non-singular for any $x \in X$;*

(d) *for any compact $B \subset X$ the function $m(\psi(\cdot, B))$ is lower semicontinuous.*

Then for any two sets $A, B \in \mathscr{B}(X)$ of positive measure the set $f(A \times B)$ has interior points.

Outline of the proof. This proposition differs from Proposition 1 of [3] very slightly; namely, in [3], instead of (c) and (d), it is required that $f(x, \cdot)$ and $\psi(z, \cdot)$ preserve measure. However, the proof given in [3] retains its validity. Just as there, we fix compact sets A and B with $m(A) > 0$ and $m(B) > 0$ (by regularity, it suffices to prove the assertion for compact sets) and we introduce the function ω: $X \to R$ given by

$$\omega(z) = m\big(A \cap \psi(z, B)\big).$$

Applying assumption (d) we can verify that ω is lower semicontinuous; using (c) it is not difficult to show that

$$\int\limits_X \omega(z)\,dm(z) > 0;$$

for details see [3]. Hence the set $\Omega = \{z \in X: \ \omega(z) > 0\}$ is open and nonempty. This completes the proof, since $\Omega \subset f(A \times B)$, as is easy to see.

3. An abstract analogue of the rule for the differentiation of an implicit function and the proof of the main result. The proof of Theorem 2 of [4] is essentially based on the formula expressing the dervative of an implicit function. The following proposition can be regarded as an analogue of that formula.

PROPOSITION 2. *Let X, m, f, α, and β satisfy the assumptions of the Theorem. Then for every $z \in X$ mappings* (5) *are non-singular and the functions*

$$x \mapsto \frac{\alpha\big(x, \varphi(x, z)\big)}{\beta\big(x, \varphi(x, z)\big)} \quad \text{and} \quad y \mapsto \frac{\beta\big(\psi(z, y), y\big)}{\alpha\big(\psi(z, y), y\big)} \tag{6}$$

are the RN derivatives of mappings (5).

(Obviously, α and β play the role of the partial derivatives in the classical theorem.)

For the proof we shall need two lemmas. The first of them is, in fact, a version of Proposition 2 of purely measure-theoretic character.

LEMMA 1. *Let* $(X, \mathcal{M}, m)$ *be a measure space with* m σ*-finite and let* $f: X\times X\to X$ *be a transformation satisfying condition* (MS). *Suppose that there exist positive-valued measurable functions* α *and* β *on* $X\times X$ *such that, for almost every* $x\in X$ *and almost every* $y\in X$, *mappings* (4) *are non-singular together with their inverses and the functions* $\alpha(\cdot, y)$ *and* $\beta(x, \cdot)$ *are the RN derivatives of mappings* (4). *Then for every set* $A\in\mathcal{M}$ *and almost every* $z\in X$ *we have*

$$m\big(\varphi(A, z)\big) = \int_A \frac{\alpha\big(x, \varphi(x, z)\big)}{\beta\big(x, \varphi(x, z)\big)}\, dm(x) \tag{7}$$

and, similarly, for every set $B\in\mathcal{M}$ *and almost every* $z\in X$ *we have*

$$m\big(\psi(z, B)\big) = \int_B \frac{\beta\big(\psi(z, y), y\big)}{\alpha\big(\psi(z, y), y\big)}\, dm(y). \tag{8}$$

Proof. Let $F, G: X\times X\to X\times X$ be defined by

$$F(x, y) = \big(x, f(x, y)\big) \quad \text{and} \quad G(x, y) = \big(f(x, y), y\big).$$

It is not difficult to show (see [3], Lemma 1) that, in view of condition (MS), F and G are bimeasurable. Put $\Phi = G\circ F^{-1}$, i.e.,

$$\Phi(x, z) = \big(z, \varphi(x, z)\big). \tag{9}$$

For $A, B\in\mathcal{M}$ we have

$$F(A\times B) = \{(x, z):\ x\in A,\ z\in f(x, B)\}.$$

Hence, applying the fact that $\alpha(\cdot, y)$ and $\beta(x, \cdot)$ are the RN derivatives of mappings (4), by the Fubini theorem we obtain

$$M\big(F(A\times B)\big) = \int_A m\big(f(x, B)\big)\, dm(x) = \int_A\int_B \beta(x, y)\, dm(y)\, dm(x) = \int_{A\times B} \beta\, dM,$$

where M denotes the product measure $m\times m$. This shows that β is the RN derivative of F, and so F and F^{-1} are M-non-singular. Similarly, α is the RN derivative of G, and so G and G^{-1} are M-non-singular. Consequently, according to (2), the RN derivative of Φ is the function

$$(x, z)\mapsto \frac{\alpha\big(F^{-1}(x, z)\big)}{\beta\big(F^{-1}(x, z)\big)}.$$

But $F^{-1}(x, z) = (x, \varphi(x, z))$, as can easily be verified. It follows that the RN derivative of Φ is just the integrand in (7). Thus

$$(10)\qquad M(\Phi(A\times C)) = \int_{A\times C} \frac{a(x, \varphi(x, z))}{\beta(x, \varphi(x, z))}\, dM(x, z) \quad \text{for } A, C \in \mathcal{M}.$$

Now, let $A \in \mathcal{M}$ be fixed and take any set $C \in \mathcal{M}$. In view of (9) we have

$$\Phi(A\times C) = \{(z, y)\colon z \in C, y \in \varphi(A, z)\},$$

whence

$$(11)\qquad M(\Phi(A\times C)) = \int_C m(\varphi(A, z))\, dm(z).$$

The set C being arbitrary, the equality of the right-hand sides of (10) and (11) implies, by the Fubini theorem, that (7) holds for almost every $z \in X$. Assertion (8) follows by symmetry.

Remark. In the assertions of Lemma 1, "for almost every $z \in X$" means "for all $z \in X$ off a nullset which depends on A, respectively on B". In the case where this nullset can be chosen independently of A and B the lemma states that functions (6) are the RN derivatives of mappings (5) for almost every $z \in X$. In order to pass from almost all to all $z \in X$ we need the topological assumptions of the Theorem.

LEMMA 2. *Let X and m be as in the Theorem and let $f\colon X\times X \to X$ be a transformation satisfying condition* (CS). *Then for every $z_0 \in X$ and any compact sets $A, B \subset X$ we have*

$$(12)\qquad m(\varphi(A, z_0)) \geqslant \limsup_{z\to z_0} m(\varphi(A, z)),$$

$$(13)\qquad m(\psi(z_0, B)) \geqslant \limsup_{z\to z_0} m(\psi(z, B))$$

(*i.e., the functions $m(\varphi(A, \cdot))$ and $m(\psi(\cdot, B))$ are upper semicontinuous*).

Proof. Fix an $\varepsilon > 0$ and take an open set $V \supset \varphi(A, z_0)$ with

$$m(V) \leqslant m(\varphi(A, z_0)) + \varepsilon$$

(which exists by the regularity of m). By assumption, φ is continuous; consequently, A being compact, it is easy to find a neighbourhood W of z_0 such that $\varphi(A\times W) \subset V$. Then

$$m(\varphi(A, z)) \leqslant m(\varphi(A, z_0)) + \varepsilon \quad \text{for } z \in W$$

and (12) follows. (13) is proved analogously.

Proof of Proposition 2. Let $A, B \subset X$ be fixed compact sets. According to Lemma 1, equalities (7) and (8) are satisfied for $x \in X\setminus Z$, where Z is a nullset. It is not difficult to prove (see [3], Lemma 4) that the set $X \setminus Z$ is dense (unless $m \equiv 0$, but in this case the proposition is obvious). Thus every point $z_0 \in X$ can be approached by points $z \in X \setminus Z$, and so we can pass to the limit in (7) and (8) as $z \to z_0$, $z \in X \setminus Z$. The convergence under the integral sign is uniform by the compactness of A and B and by the continuity of α and β. Thus, on account of (12) and (13), we obtain, after passing to the limit,

$$m\big(\varphi(A, z_0)\big) \geqslant \int_A \frac{\alpha\big(x, \varphi(x, z_0)\big)}{\beta\big(x, \varphi(x, z_0)\big)}\, dm(x), \tag{14}$$

$$m\big(\psi(z_0, B)\big) \geqslant \int_B \frac{\beta\big(\psi(z_0, y), y\big)}{\alpha\big(\psi(z_0, y), y\big)}\, dm(y). \tag{15}$$

This holds for any $z_0 \in X$ and any compact sets $A, B \subset X$.

Now let z_0 be fixed and write

$$\varrho(x) = \frac{\alpha\big(x, \varphi(x, z_0)\big)}{\beta\big(x, \varphi(x, z_0)\big)} \quad \text{and} \quad \tau(y) = \frac{\beta\big(\psi(z_0, y), y\big)}{\alpha\big(\psi(z_0, y), y\big)};$$

we have

$$\tau(y) = \frac{1}{\varrho(x)} \quad \text{for } x, y \text{ connected by } f(x, y) = z_0 \tag{16}$$

(i.e., for $y = \varphi(x, z_0)$ or $x = \psi(z_0, y)$). Consider the measures μ, ν and m_φ, m_ψ on $\mathscr{B}(X)$ given by

$$d\mu = \varrho\, dm, \qquad d\nu = \tau\, dm,$$

$$m_\varphi(E) = m\big(\varphi(E, z_0)\big), \qquad m_\psi(E) = m\big(\psi(z_0, E)\big).$$

Then (14) and (15) can be rewritten as

$$m_\varphi(A) \geqslant \mu(A) \quad \text{and} \quad m_\psi(B) \geqslant \nu(B)$$

for A and B compact. According to the remarks after formula (3), all these measures are regular, and so

$$m_\varphi \geqslant \mu \quad \text{and} \quad m_\psi \geqslant \nu. \tag{17}$$

Since, by assumption, $\varrho > 0$ and $\tau > 0$, the measures m, μ, and ν are mutually continuous. Hence, by (17),

$$m \ll m_\varphi \quad \text{and} \quad m \ll m_\psi. \tag{18}$$

We shall show that the converse relations also hold. Suppose that $m(E) = 0$. Writing $F = \varphi(E, z_0)$ (equivalently, $E = \psi(z_0, F)$), we have $m_\psi(F) = 0$. Hence, by (18), $m(F) = 0$; consequently, $m_\varphi(E) = 0$ for any E with $m(E) = 0$, i.e., $m_\varphi \ll m$.

By symmetry also $m_\psi \ll m$. Thus all the five occurring measures are equivalent (mutually continuous), and so their relative RN derivatives exist; the equivalence between m, m_φ and m_ψ is a part of the assertion of the proposition.

Write

$$\xi = \frac{dm_\varphi}{dm} \quad \text{and} \quad \eta = \frac{dm_\psi}{dm}.$$

Mappings (5) being mutually inverse, by (2) we have

$$\eta(y) = \frac{1}{\xi(x)} \quad \text{for } f(x, y) = z_0. \tag{19}$$

From (17) we get, in view of (1),

$$\xi = \frac{dm_\varphi}{d\mu}\cdot\frac{d\mu}{dm} = \frac{dm_\varphi}{d\mu}\cdot\varrho \geqslant \varrho,$$

$$\eta = \frac{dm_\psi}{d\nu}\cdot\frac{d\nu}{dm} = \frac{dm_\psi}{d\nu}\cdot\tau \geqslant \tau.$$

(These equalities and inequalities hold a.e. relative to any of the equivalent measures in question). On account of (16) and (19), we infer that the last inequalities are in fact equalities

$$dm_\varphi = \varrho\, dm \quad \text{and} \quad dm_\psi = \tau\, dm.$$

This, by the definitions of ϱ, τ and m_φ, m_ψ, in view of the fact that z_0 was chosen arbitrarily in X, completes the proof of Proposition 2.

The Theorem formulated in the preceding section can now be derived easily from Propositions 1 and 2.

Proof of the Theorem. It suffices to verify that the assumptions of Proposition 1 are fulfilled. Now, (a), (b), and (c) are contained in the assumptions of the Theorem: (a) and (b) hold since f satisfies conditions (CS) and (BS); (c) follows from the fact that the mapping $\varphi(x, \cdot)$ is inverse to $f(x, \cdot)$. To obtain (d), observe that for any $B \in \mathscr{B}(X)$ and $z \in X$ we have, in virtue of Proposition 2,

$$m\big(\psi(z, B)\big) = \int_B \frac{\beta\big(\psi(z, y), y\big)}{\alpha\big(\psi(z, y), y\big)}\, dm(y);$$

for compact B this quantity depends continuously on z, since the integrand is a continuous function. Thus (d) is also satisfied and the Theorem follows from Proposition 1.

4. The case of $X = R^n$. As a consequence of our Theorem we obtain

COROLLARY. *Let $A, B \subset R^n$ be sets of positive n-dimensional Lebesgue measure, let $\Delta \subset R^{2n}$ be an open set containing $A \times B$, and let $f\colon \Delta \to R^n$*

be a C^1-mapping such that the Jacobians Df/Dx and Df/Dy are different from zero in Δ. Then the set $f(A \times B)$ has interior points.

(For $n = 1$ this has been proved in [1], [4], and [5].)

Apparently, this fact is a particular case of the Theorem if Δ is the entire R^{2n} and for every $x, y \in R^n$ the mappings $f(x, \cdot)$ and $f(\cdot, y)$ are diffeomorphisms of R^n onto R^n: it suffices to put $\alpha = |Df/Dx|$ and $\beta = |Df/Dy|$ (cf., e. g., [8], Chapter VIII, 5.3). The general case can be reduced to that just described by means of the following lemma:

LEMMA 3. *Let f satisfy the conditions of the Corollary. Then for every point $p_0 = (x_0, y_0) \in \Delta$ there exist a neighbourhood $U \times V$ of p_0 with $U \times V \subset \Delta$ and a C^1-mapping $\tilde{f}$: $R^{2n} \to R^n$ such that $\tilde{f} = f$ in $U \times V$, and for any fixed $x, y \in R^n$ the mappings $\tilde{f}(x, \cdot)$ and $\tilde{f}(\cdot, y)$ are diffeomorphisms of R^n onto R^n.*

The assertion of the Corollary is then clear: we may assume that A and B are compact; each point in Δ has a neighbourhood $U \times V$ with the properties above; finitely many such neighbourhoods cover $A \times B$; at least one of them is such that $U \cap A$ and $V \cap B$ have positive measure, and it suffices to apply the Theorem to the mapping $\tilde{f}$ whose existence is asserted by Lemma 3. Thus it remains to prove the lemma.

Proof of Lemma 3. We may assume that $x_0 = y_0 = f(x_0, y_0) = 0$. We have

$$f = L + h \quad \text{in } \Delta, \tag{20}$$

where $L = f'(0)$ denotes the differential of f at the point $0 \in R^{2n}$ and h: $\Delta \to R^n$ is a C^1-mapping with

$$h(0) = 0 \quad \text{and} \quad h'(0) = 0. \tag{21}$$

Write

$$L_1 x = L(x, 0) \quad \text{and} \quad L_2 y = L(0, y).$$

By assumption, $f(\cdot, 0)$ and $f(0, \cdot)$ have non-zero Jacobians, and so L_1 and L_2 are linear automorphisms of R^n. Let

$$\delta = \frac{1}{2} \min \left(\frac{1}{\|L_1^{-1}\|}, \frac{1}{\|L_2^{-1}\|} \right) \tag{22}$$

($\|\cdot\|$ denotes the operator norm). In view of (21), h being of class C^1, there is $r > 0$ such that

$$\Delta_r = \{(x, y): |x| < r, |y| < r\} \subset \bar{\Delta}_r \subset \Delta \tag{23}$$

and

$$\|h'(p)\| \leqslant \tfrac{1}{6} \delta \quad \text{for } p \in \bar{\Delta}_r. \tag{24}$$

We now construct a C^1-mapping $\tilde{h}\colon R^{2n} \to R^n$ with the following properties:

$$\tilde{h}(p) = h(p) \quad \text{for } p \in \Delta_{r/2} \tag{25}$$

and

$$\|\tilde{h}'(p)\| \leqslant \delta \quad \text{for all } p \in R^{2n}. \tag{26}$$

The construction is standard: we take a C^1-function $\eta\colon [0, +\infty) \to [0,1]$ with

$$\eta(t) = 1 \text{ for } 0 \leqslant t \leqslant \tfrac{1}{2}, \quad \eta(t) = 0 \text{ for } t \geqslant 1,$$
$$|\eta'(t)| \leqslant \tfrac{5}{2} \quad \text{for all } t \geqslant 0$$

and we put

$$\tilde{h}(p) = \begin{cases} \eta\left(\dfrac{|x|}{r}\right)\eta\left(\dfrac{|y|}{r}\right)h(p) & \text{for } p = (x, y) \in \Delta, \\ 0 & \text{for } p \notin \Delta. \end{cases}$$

It is not difficult to verify by an elementary calculation that $\tilde{h}$ is of class C^1 on R^{2n} and satisfies (25) and (26).

Now we can define the desired neighbourhoods U, V and the mapping $\tilde{f}$. Namely, we put

$$U = \{x \in R^n\colon |x| < \tfrac{1}{2}r\}, \qquad V = \{y \in R^n\colon |y| < \tfrac{1}{2}r\},$$
$$\tilde{f} = L + \tilde{h}. \tag{27}$$

Then $\tilde{f}$ is a C^1-mapping of R^{2n} into R^n. For $p \in U \times V$ we have $\tilde{f}(p) = f(p)$ by (20) and (25). It remains to verify that for x, y fixed the mappings $\tilde{f}(x, \cdot)$ and $\tilde{f}(\cdot, y)$ are diffeomorphisms of R^n onto R^n.

Thus fix $y \in R^n$ and write

$$q(z) = \tilde{f}(L_1^{-1}z, y) \quad \text{for } z \in R^n.$$

Then $q\colon R^n \to R^n$ is a C^1-mapping and

$$\tilde{f}(x, y) = q(L_1 x) \quad \text{for } x \in R^n. \tag{28}$$

Writing (27) in the form

$$\tilde{f}(x, y) = L_1 x + L_2 y + \tilde{h}(x, y)$$

and putting $z = L_1 x$ we obtain

$$q(z) = z + L_2 y + \tilde{h}(L_1^{-1}z, y) \quad \text{for } z \in R^n. \tag{29}$$

The last summand is a contraction in z: indeed, for any $z_1, z_2 \in R^n$ we have, by the Lagrange inequality, in view of (26) and (22),

$$\begin{aligned} |\tilde{h}(L_1^{-1}z_1, y) - \tilde{h}(L_1^{-1}z_2, y)| &\leqslant \delta\,|(L_1^{-1}z_1, y) - (L_1^{-1}z_2, y)| \\ &= \delta\,|L_1^{-1}z_1 - L_1^{-1}z_2| \leqslant \delta\|L_1^{-1}\|\cdot|z_1 - z_2| \leqslant \tfrac{1}{2}\,|z_1 - z_2|. \end{aligned}$$

Thus (29) can be rewritten as

$$q = \text{identity} + \text{const} + (\tfrac{1}{2}\text{-Lipschitz map}).$$

Consequently, q is a bijection of R^n onto R^n. The same decomposition shows that $\|q'\| \geqslant \frac{1}{2}$ at each point, and so q is a diffeomorphism. Hence, by (28), also $\tilde{f}(\cdot, y)$ is a diffeomorphism of R^n onto R^n.

The assertion concerning $\tilde{f}(x, \cdot)$ for x fixed follows from symmetry.

5. An example and problems. The smoothness assumption in the Corollary is not nice, as the remaining conditions and the assertion involve pure measure and topology. However, the following example shows that this assumption cannot be entirely omitted: it does not suffice to require that f be continuous and coordinatewise monotonic (for $n = 1$).

Example. There exist a compact set $A \subset R$ of positive Lebesgue measure and a continuous function $f\colon R^2 \to R$ such that for any fixed $x, y \in R$ mappings (4) are increasing homeomorphisms of R onto R and the set $f(A \times A)$ has no interior points.

Construction. Consider the set

$$K = \Big\{\sum_{n=1}^{\infty} 4\varepsilon_n \cdot 5^{-n}\colon\ \varepsilon_n = 0 \text{ or } \varepsilon_n = 1\Big\}.$$

K is a subset of $[0, 1]$, homeomorphic to the Cantor set. Let $g\colon [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function constructed with respect to the set K, i.e., the function determined uniquely by the conditions

$$g(x) = \sum_{n=1}^{\infty} \varepsilon_n \cdot 2^{-n} \quad \text{for } x = \sum_{n=1}^{\infty} 4\varepsilon_n \cdot 5^{-n} \in K,$$

and

$$g \text{ is non-decreasing on } [0, 1].$$

Then the function $h\colon [0, 1] \to [0, 1]$ given by

$$h(x) = \tfrac{1}{2}x + \tfrac{1}{2}g(x)$$

is continuous and strictly increasing. We extend h to a homeomorphism of R onto R by putting $h(x) = x$ for $x \notin [0, 1]$ (and preserving the symbol h) and we write

$$f(x, y) = h^{-1}(x) + h^{-1}(y).$$

Now let $A = h(K)$. Then A has measure $\frac{1}{2}$ and we obtain $f(A \times A) = K + K$.

By definition, K consists precisely of those numbers $x \in [0, 1]$ which can be written in the quinary numeration system with the use only of the digits 0 and 4. The sum of two such numbers admits a quinary expansion in which the digit 2 does not occur, and so the set $f(A \times A) = K + K$ has no interior.

Note that the above argument involves essentially the fact that g is singular. Therefore, it is natural to ask:

PROBLEM 1. In the situation of the Corollary for $n = 1$, can the assumption of class C^1 of f be replaced by the requirement that f is continuous and, for any $x, y \in R$, functions (4) are strictly increasing and absolutely continuous together with their inverses? (**P 1029**)

The smoothness of f in the Corollary corresponds to the continuity of the functions α and β in the Theorem (and in Proposition 2). We are thus led to the following question, an extension of Problem 1:

PROBLEM 2. Is the assumption of continuity of α and β necessary in Proposition 2 and in the Theorem ? (**P 1030**)

(The mere existence of these functions is a consequence of the remaining assumptions and the Radon-Nikodym theorem.)

To conclude the paper, we indicate some possible directions of further research.

First, we can ask whether Proposition 2 and the Theorem admit a local version. That means: do their assertions remain true if f is defined (as in the Corollary) not necessarily on the whole of $X \times X$ and other conditions are satisfied locally (in some sense which requires a precise definition)?

Secondly, we can try to replace one space X by three topological measure spaces X, Y, Z and to let f be defined on a subset of $X \times Y$ and have values in Z. Actually, in all considerations of this paper we could have assumed that f is of type $X \times Y \to Z$ and all the proofs would have gone through. However, this would not do any good, since the "global" conditions (CS) and (BS) imposed on mappings (4) (in a formulation involving the triple X, Y, Z) imply that X, Y, and Z are, in fact, the same space. Therefore, this generalization is interesting in the "local version" only.

We can go a step further: since "Steinhaus type" theorems involve measure in X and Y, and topology in Z only, one might assume that X and Y are measure spaces and Z is a topological space. But, of course, these structures have to be related in some way (by f).

Certain positive results relevant to these topics have been obtained by Paganoni [6] and Sander [7] under additional assumptions of very technical character.

As it has already been remarked, Proposition 2 is an analogue of the rule for the differentiation of an implicit function. It would be nice to have an abstract version of the implicit function theorem itself, i.e., to have a theorem asserting the local solvability of the equation $f(x, y) = z$ under assumptions formulated in terms of the topological and measure-theoretic properties of mappings (4). The author does not know whether any such generalizations of the implicit function theorem do exist.

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