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THE CENTER OF AN ALGEBRA OF OPERATORS

BY

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1. Introduction. Let G be a locally compact Abelian group and let $\mathrm{Hom}(L^\infty(G), L^\infty(G))$ denote the algebra of all continuous linear operators on $L^\infty(G)$ which commute with translations by elements of G.

If $\mu \in M(G)$ (the space of all regular complex Borel measures on G), then $\check{\mu}$ denotes the measure of $M(G)$ defined by $\check{\mu}(A) = \mu(-A)$.

Let T_μ be the operator on $L^\infty(G)$ defined by

$$T_\mu(f) = \check{\mu}*f \quad \text{for all } f \in L^\infty(G).$$

Recall that

$$M(G)*L^\infty(G) \subset L^\infty(G) \quad \text{and} \quad \|\mu*f\| \leqslant \|\mu\|\,\|f\|_\infty.$$

It is clear that

$$M(G) \subset \mathrm{Hom}(L^\infty(G), L^\infty(G)).$$

We denote by Z_∞ the center of the algebra $\mathrm{Hom}(L^\infty(G), L^\infty(G))$. In this paper* it is proved that if G is a locally compact, compactly generated, not discrete Abelian group, then

$$M(G) \cap Z_\infty = M_d(G),$$

where $M_d(G)$ is the subspace of $M(G)$ of all discrete measures on G.

If G is a compact Abelian group, then it follows immediately from the previous identity that $Z_\infty = M_d(G)$. A previous result of the author [4] implies that $Z_\infty \subseteq M(G)$ for $G = \boldsymbol{R}$. It follows then that $M_d(\boldsymbol{R}) = Z_\infty(\boldsymbol{R})$.

2. Centrality of discrete measures.

THEOREM 1. *Let G be a locally compact Abelian group. Then every discrete measure $\delta \in M_d(G)$ belongs to the center Z_∞ of the algebra* $\mathrm{Hom}(L^\infty(G), L^\infty(G))$.

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Proof. Note that Z_∞ is a norm closed subspace of $\mathrm{Hom}(L^\infty(G), L^\infty(G))$ and that every discrete measure may be approximated in the $M(G)$ norm (and hence in the operator norm as a convolution operator in $L^\infty(G)$) by finitely supported measures. It suffices, therefore, to prove that every finitely supported measure δ belongs to Z_∞.

Let

$$T_\delta(f) = \sum_{n=1}^{N} a_n \, {}_{a_n}f, \qquad \text{where } {}_{a_n}f(x) = f(x+a_n).$$

If $S \in \mathrm{Hom}(L^\infty(G), L^\infty(G))$ and so $S({}_{a_n}f) = {}_{a_n}S(f)$, then, by linearity, $ST_\delta(f) = T_\delta S(f)$ for every $f \in L^\infty(G)$. This proves that $T_\delta \in Z_\infty$.

We recall that every measure $\mu \in M(G)$ has a unique decomposition $\mu = \mu_d + \mu_c$, where μ_d is discrete and μ_c is continuous. By Theorem 1 and since Z_∞ is a subalgebra of $\mathrm{Hom}(L^\infty(G), L^\infty(G))$, we know that a non-iscrete measure belongs to Z_∞ if and only if its continuous part is in Z_∞.

3. A property of Z_∞. Let χ be a character of the locally compact Abelian group G. For all $T \in \mathrm{Hom}(L^\infty(G), L^\infty(G))$ and for all $f \in L^\infty(G)$, write

$$T_\chi(f) = \chi T(\overline{\chi f}).$$

As it can easily be verified, the operator T_χ still belongs to $\mathrm{Hom}(L^\infty(G), L^\infty(G))$. It is immediate to verify the following properties:

$$(1) \qquad (T_\chi)_\chi = T \quad \text{for all } T \in \mathrm{Hom}(L^\infty(G), L^\infty(G)),$$

$$(2) \qquad (TS)_\chi = T_\chi S_\chi \quad \text{for all } S, T \in \mathrm{Hom}(L^\infty(G), L^\infty(G)).$$

As an immediate consequence of properties (1) and (2) we have the following

LEMMA 1. *If $T \in Z_\infty$, then $T_\chi \in Z_\infty$ for all χ.*

Proof. Let $S \in \mathrm{Hom}(L^\infty(G), L^\infty(G))$; then by properties (1) and (2) we have

$$T_\chi S = T_\chi (S_\chi)_\chi = (TS_\chi)_\chi = (S_\chi T)_\chi = (S_\chi)_\chi T_\chi = ST_\chi.$$

Let $\mu \in M(G)$ and let T_μ be the associated operator. It follows from the definition that $(T_\mu)_\chi = T_{\mu_\chi}$, where $d\mu_\chi = \chi d\bar{\mu}$ if $\bar{\mu} \in M(G)$ is defined by $\bar{\mu}(A) = \overline{\mu(A)}$. In fact, for all $f \in L^\infty(G)$ and for all $x \in G$, we have

$$\begin{aligned}(T_\mu)_\chi(f)(x) &= \chi \overline{T_\mu(\overline{\chi f})(x)} = \chi(x) \overline{\int_G \overline{\chi} f(x+y)\, d\mu(y)} \\ &= \int_G \chi(y) f(x+y)\, d\bar{\mu}(y) = \int_G f(x+y)\, d\mu_\chi(y) \\ &= (\mu_\chi)^\vee * f(x) = T_{\mu_\chi}(f)(x).\end{aligned}$$

The following corollary follows immediately from Lemma 1.

COROLLARY 1. *Let $\mu \in M(G)$ belong to the center Z_∞. Then $\mu_\chi \in Z_\infty$ for all characters χ of G, where $d\mu_\chi = \chi d\bar{\mu}$.*

Let us conclude this section with the following lemma:

LEMMA 2. *If there is a continuous non-zero measure μ in the center Z_∞, then there is also a continuous non-zero measure ν with the property $\nu(G) \neq 0$.*

Proof. Let $\mu \in M_c(G)$. For any choice of χ, also $\mu_\chi \in M_c(G)$; moreover, it is possible to choose χ so that $\mu_\chi(G) \neq 0$. If $\mu \in Z_\infty$, then the lemma follows from Corollary 1.

4. A property of continuous positive measures.

LEMMA 3. *Let G be a locally compact not discrete Abelian group and let $\mu \in M^+(G)$. Let $H \subset G$ be a compact set such that*

$$\mu(H+x) = 0 \quad \textit{for all } x \in G.$$

Then for every $\varepsilon > 0$ there exists a neighborhood V of the identity e such that

$$\mu(V+H+x) < \varepsilon \quad \textit{for all } x \in G.$$

Proof. Suppose that there exists $\varepsilon_0 > 0$ such that, for every neighborhood V of e and for at least one $x \in G$, $\mu(V+H+x) \geqslant \varepsilon_0$. Let $\mathscr{V}$ be a basic system of compact symmetric neighborhoods of e, partially ordered by inclusion. For every $V \in \mathscr{V}$, let x_V be an element of G such that

$$\mu(V+H+x_V) \geqslant \varepsilon_0.$$

We prove that there exists a compact set $K_0 \subset G$ and $V_0 \in \mathscr{V}$ such that $x_V \in K_0$ for all $V \in \mathscr{V}$, $V \subset V_0$. If not, then it would be possible to choose a subnet $\{x_a\}_{a \in \mathscr{A}}$ with the property that for any compact $K \subset G$ there exists $a_0 \in \mathscr{A}$ such that $x_a \notin K$ for $a \geqslant a_0$. This implies that it is possible to choose a sequence $\{x_n\} \subset \{x_a\}_{a \in \mathscr{A}}$ such that the open sets $V_n + H + x_n$, where $V_n = V_{a(n)}$, $x_n = x_{V_{a(n)}}$, are a disjoint collection. But then, as μ is positive, we would have

$$\mu\Big(\bigcup_1^\infty (V_n+H+x_n)\Big) = \sum_{n=1}^\infty \mu(V_n+H+x_n) \geqslant \sum_{n=1}^\infty \varepsilon_0 = +\infty.$$

This is impossible, as μ is a bounded measure. Therefore, there exists a subnet $\{x_\gamma\}_{\gamma \in \Gamma}$ which converges to an element $x_0 \in G$.

Let U be a neighborhood of e. We prove that there is $\gamma_0 \in \Gamma$ such that

$$V_\gamma + H + x_\gamma \subset U + H + x_0 \quad \text{for all } \gamma \geqslant \gamma_0.$$

In fact, if W denotes a neighborhood of e for which $W+W \subset U$, then there exists $\gamma_0 \in \Gamma$ such that $x_\gamma - x_{\gamma_0} \in W$ for all $\gamma \geqslant \gamma_0$, and $V_\gamma \subset W$

(because $\{V_\gamma\}$ is basic). As μ is a positive measure, this inclusion implies

$$\mu(U+H+x_0) \geqslant \mu(V_\gamma+H+x_\gamma) \quad \text{for } \gamma \geqslant \gamma_0.$$

Then $\mu(H+x_0) \geqslant \varepsilon_0$, since U is an arbitrary neighborhood of e, and μ is regular. This contradicts the hypothesis.

COROLLARY 2. *Let G be a locally compact not discrete Abelian group and let $\mu \in M_c^+(G)$. Then for every $\varepsilon > 0$ there exists a neighborhood V of the identity e such that*

$$\mu(V+x) < \varepsilon \quad \textit{for all } x \in G.$$

Proof. Since μ is continuous, $\mu(\{x\}) = 0$ for all $x \in G$.

THEOREM 2. *Let G be a locally compact, compactly generated, not discrete Abelian group and let $\mu \in M_c^+(G)$. For every $\varepsilon > 0$, G contains a dense open set A with*

$$\mu(A+x) < \varepsilon \quad \textit{for all } x \in G.$$

Proof. By Corollary 2, given any positive integer n, there exists a neighborhood V_n of e, with compact closure, such that

$$\mu(V_n+x) \leqslant \frac{1}{n} \quad \text{for all } x \in G.$$

It follows from [2], Theorem (8.7), that G has a compact subgroup

$$N \subset \bigcap_1^\infty V_n$$

such that G/N is metrizable and has a countable basis for its open sets. Let $\{N+x_n\}$ denote a sequence of cosets of N whose union is dense in G. Since, for any n and for all $x \in G$,

$$N+x_n+x \subset \bigcap_{m=1}^\infty (V_m+x_n+x) \subset (V_m+x_n+x) \quad \text{for all } m,$$

we have $\mu(N+x_n+x) \leqslant 1/m$ for all m and, therefore,

$$\mu(N+x_n+x) = 0.$$

By Lemma 3, given $\varepsilon > 0$ and any positive integer n, there exists an open set A_n which contains $N+x_n$ and such that

$$\mu(A_n+x) < \frac{\varepsilon}{2^n} \quad \text{for all } x \in G.$$

Let

$$A = \bigcup_1^\infty A_n.$$

Then A is open and dense, since

$$A \supset \bigcup_{1}^{\infty} (N+x_n).$$

Finally,

$$\mu(A+x) = \mu\Big(\bigcup_{1}^{\infty} (A_n+x)\Big) \leqslant \sum_{n=1}^{\infty} \mu(A_n+x) \leqslant \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon \quad \text{for all } x \in G.$$

The theorem just proved for any continuous positive measure $\mu \in M_c^+(G)$ yields for such elements of $M(G)$ a result which is analogous to the result proved for the Haar measure by Rudin in Theorem 2.4 of [3].

5. Characterization of the central measures. Let G be a locally compact, compactly generated, not discrete Abelian group. We know that discrete measures are central. In this section we will prove that only discrete measures belong to the center.

The remark at the end of Section 2 and Lemma 2 allow us to reduce this problem to that of proving that if μ is continuous and $\mu(G) \neq 0$, then μ does not belong to the center. In order to prove this fact, we will use Theorem 2.

We recall that an *invariant mean* on $L^\infty(G)$, where G is any group, is a continuous linear functional on $L^\infty(G)$ such that

(i) $M(L_s f) = M(f) = M(R_s f)$ for all $f \in L^\infty(G)$ and $s \in G$, where $L_s(f)(x) = f(s+x)$ and $R_s(f)(x) = f(x+s)$;

(ii) $M(1) = 1$;

(iii) $|M(f)| \leqslant \operatorname*{supess}_{x\in G} |f(x)|$ for all $f \in L^\infty(G)$.

A familiar argument shows that (ii) and (iii) imply

(iv) $M(f) \geqslant 0$ if $f \geqslant 0$ and $f \in L^\infty(G)$.

Let A be a dense open subset of G and let χ_A be its characteristic function. As a consequence of Theorem 3.4 of [3], there exists an invariant mean M on $L^\infty(G)$ such that $M(\chi_A) = 1$. Let $S \in \mathrm{Hom}\big(L^\infty(G), L^\infty(G)\big)$ be the operator on $L^\infty(G)$ defined by

$$S(f)(x) = M(f) \quad \text{for all } f \in L^\infty(G) \text{ and all } x \in G.$$

THEOREM 3. *Let G be a locally compact, compactly generated, not discrete Abelian group. Let $\mu \in M_c(G)$ and $\mu(G) \neq 0$. Then there exists a dense open set A in G such that*

$$S(\check{\mu} * g) \neq \check{\mu} * S(g),$$

where $g = 1-\chi_A$ and S is the operator $S(f)(x) = M(f)$ defined above.

Proof. Since $\mu(G) \neq 0$, we can suppose that $\mu(G) = 1$. If A is any dense open set in G, then, by the definition of S, $S(g) = 0$ and, therefore,

$\check{\mu} * S(g) = 0$. We prove that it is possible to choose A so that

$$S(\check{\mu} * g) \neq 0.$$

Since

$$(\check{\mu} * 1)(x) = \int_G d\mu(y) = \mu(G) = 1,$$

we have

$$S(\check{\mu} * g) = M(\check{\mu} * g) = M(\check{\mu} * 1 - \check{\mu} * \chi_A) = M(1) - M(\check{\mu} * \chi_A) = 1 - M(\check{\mu} * \chi_A),$$

whence

$$|S(\check{\mu} * g)| \geqslant \big|1 - |M(\check{\mu} * \chi_A)|\big|.$$

By the definition of a mean,

$$|M(\check{\mu} * \chi_A)| \leqslant M(|\check{\mu} * \chi_A|).$$

Moreover, since $|\check{\mu} * \chi_A| \leqslant |\check{\mu}| * \chi_A$, we have

$$|M(\check{\mu} * \chi_A)| \leqslant M(|\check{\mu}| * \chi_A).$$

We also note that, for all $x \in G$,

$$|\check{\mu}| * \chi_A(x) = \int_G \chi_A(x+y)\, d|\mu|(y) = \int_G \chi_{A-x}(y)\, d|\mu|(y) = |\mu|(A-x).$$

Since $|\mu| \in M_c^+(G)$, by Theorem 2 it is possible to choose A so that $|\mu|(A-x) < 1/2$ for all $x \in G$. Thus for this choice of A we have $|S(\check{\mu} * g)| \geqslant 1/2$ and, therefore, $S(\check{\mu} * g) \neq 0$. This completes the proof of the theorem.

By Lemma 2 and Theorem 3 we have immediately

Corollary 3. *Let G be a locally compact, compactly generated, not discrete Abelian group. Then*

$$M(G) \cap Z_\infty = M_d(G).$$

6. Characterization of Z_∞ for compact G and $G = \boldsymbol{R}$. Let $C_u(G)$ be the space of all uniformly continuous bounded functions on G and let $\mathrm{Hom}(C_u(G), C_u(G))$ denote the algebra of all continuous linear operators on $C_u(G)$ which commute with translations by elements of G. Furthermore, let R_∞ denote the algebra of all continuous linear operators on $L^\infty(G)$ which commute with convolutions by elements of $L^1(G)$. From Theorem 3.3 of [1] we can easily derive the relations

$$\mathrm{Hom}(L^\infty(G), L^\infty(G)) \supset R_\infty \cong \mathrm{Hom}(C_u(G), C_u(G)) \cong C_u^*(G),$$

where $C_u^*(G)$ is the convolution algebra of continuous linear functionals on $C_u(G)$ (see [2]). The isomorphism is given by the map $T \mapsto \varphi_T$ defined for all $T \in R_\infty$ by

$$\varphi_T(f) = T(f)(0) \quad \text{for all } f \in C_u(G).$$

(Obviously, the inclusion is an identity if G is a discrete group.) If G is a compact Abelian group, then the algebra $C_u^*(G)$ is commutative, since $C_u^*(G) = C_0^*(G) = M(G)$. It has been proved in [4] that if G is locally compact Abelian but not compact, then $C_u^*(G)$ is non-commutative and the center Z_u of $C_u^*(G)$ contains $M(G)$. We have also proved that $Z_u = M(G)$ when G is $\boldsymbol{R}$ or a discrete subgroup of $\boldsymbol{R}$ and when G is the dual of the Cantor group. As a consequence we have

COROLLARY 4. *Let G be a compact Abelian group or $G = \boldsymbol{R}$. Then* $Z_\infty = M_d(G)$.

Proof. It is clear that the restrictions of elements of Z_∞ to the algebra $\operatorname{Hom}(C_u(G), C_u(G))$ are in Z_u; so in case G compact or $G = \boldsymbol{R}$ they are measures. Therefore

$$M_d(G) \subset Z_\infty \subset M(G).$$

Thus, by Theorem 2, $Z_\infty = Z_\infty \cap M(G) = M_d(G)$.

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