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PERTURBATION OF STRONGLY CONTINUOUS COSINE FAMILY GENERATORS

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1. Introduction. A *strongly continuous cosine family* in the Banach space X is a one-parameter family of bounded linear operators $C(t)$, $t \in \boldsymbol{R}$, in X satisfying

(1.1) $C(t+s)+C(t-s) = 2C(t)C(s)$ for all s, $t \in \boldsymbol{R}$,

(1.2) $C(0) = I$,

(1.3) $C(t)x$ is continuous in t from $\boldsymbol{R}$ to X for each fixed $x \in X$.

The *associated sine family* $S(t)$, $t \in \boldsymbol{R}$, is the one-parameter family of bounded linear operators in X defined by

$$S(t)x = \int_0^t C(s)x\,ds, \qquad x \in X,\ t \in \boldsymbol{R}.$$

The *infinitesimal generator* A of $C(t)$, $t \in \boldsymbol{R}$, is the linear operator in X defined by

$$Ax = \lim_{t\to 0}(2/t^2)\big(C(t)x-x\big)$$

for all x for which this limit exists in X. In analogy with strongly continuous semigroups and first order abstract differential equations, strongly continuous cosine families correspond to abstract linear second order differential equations. Strongly continuous cosine families were studied by many researchers in recent years and we list some of their works in our references.

The purpose of this paper* is to prove some perturbation results for infinitesimal generators of strongly continuous cosine families. That is, we establish sufficient conditions such that if A is the infinitesimal generator of a strongly continuous cosine family in X and P is a closed linear

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operator in X, then $A+P$ is also the infinitesimal generator of a strongly continuous cosine family in X. Our principle result is analogous to a similar result due to Dunford and Schwartz ([2], Theorem 19, p. 631) for the perturbation of strongly continuous semigroup generators.

2. The main result. Our main result is

PROPOSITION. *Let A be the infinitesimal generator of the strongly continuous cosine family $C(t), t \in \boldsymbol{R}$, in X and let P be a closed linear operator in X such that*

(2.1) $\quad S(t)(X) \subset D(P)$ *for all* $t \in \boldsymbol{R}$,

(2.2) $\quad PS(t)x$ *is continuous for* $t \in \boldsymbol{R}$ *for each fixed* $x \in X$.

Then $A+P$ is the infinitesimal generator of a strongly continuous cosine family $\hat{C}(t), t \in \boldsymbol{R}$, in X.

Before giving the proof we first prove some lemmas under the hypothesis of the proposition. It is known that there exist constants $M \geqslant 1$ and $\omega \geqslant 0$ such that

$$|C(t)| \leqslant M\exp(\omega|t|) \quad \text{for all } t \in \boldsymbol{R} \tag{2.3}$$

(see Lemma 5.5 of [3]). We use this fact in the sequel.

LEMMA 1. *For each $t \in \boldsymbol{R}$, $PS(t)$ is bounded and everywhere defined in X, and there exist constants $M_1 \geqslant 1$ and $\omega_1 \geqslant \omega$ such that $|PS(t)| \leqslant M_1\exp(\omega_1|t|)$ for all $t \in \boldsymbol{R}$.*

Proof. Since P is closed and $S(t)$ is bounded, $PS(t)$ is also closed. By (2.1), $PS(t)$ is everywhere defined, and thus also bounded by the Closed Graph Theorem. Let L be a positive constant such that $|PS(t)| \leqslant L$ for $0 \leqslant t \leqslant 1$ (such an L exists by virtue of (2.2) and the Principle of Uniform Boundedness). Let $M_1 = 2ML$ and $\omega_1 = \omega + \log 2M$. One uses induction and the formula

$$S(s+t) = S(s)C(t) + S(t)C(s) \quad \text{for all } s,\ t \in \boldsymbol{R} \tag{2.4}$$

(see (2.9) of [15]) to prove that, for each positive integer n,

$$\begin{aligned} S(t) &= 2S(t/2)C(t/2) = 2^2S(t/4)C(t/4)C(t/2) \\ &= \ldots = 2^nS(t/2^n)C(t/2^n)C(t/2^{n-1})\ldots C(t/2). \end{aligned}$$

Let $t \geqslant 0$ and let n be the least integer greater than or equal to t. Then $t \leqslant n < t+1 \leqslant 2^n$ and

$$\begin{aligned} |PS(t)| &\leqslant 2^n|PS(t/2^n)|\,|C(t/2^n)|\,|C(t/2^{n-1})| \ldots |C(t/2)| \\ &\leqslant 2^nLM\exp(\omega t/2^n)M\exp(\omega t/2^{n-1}) \ldots M\exp(\omega t/2) \\ &\leqslant (2M)^nL\exp(\omega t) \leqslant 2ML\exp\big((\omega+\log 2M)t\big). \end{aligned}$$

Since $S(t) = -S(-t)$, the conclusion holds for all $t \in \boldsymbol{R}$.

We will use $B(X)$ to denote the Banach algebra of bounded linear everywhere defined operators in X. For a linear operator T in X and $\lambda \in \boldsymbol{R}$ such that $(\lambda I - T)^{-1} \in B(X)$, let $R(\lambda; T)$ denote $(\lambda I - T)^{-1}$.

LEMMA 2. *The following relations hold*:

$$PR(\lambda^2; A) \in B(X) \quad \text{for } \lambda > \omega_1, \tag{2.5}$$

$$R(\lambda^2; A+P) = R(\lambda^2; A) \sum_{n=0}^{\infty} \big(PR(\lambda^2; A)\big)^n \quad \text{for } \lambda > \omega_1 + M_1. \tag{2.6}$$

Proof. It is known that, for $x \in X$ and $\lambda > \omega$,

$$R(\lambda^2; A)x = \int_0^{\infty} e^{-\lambda s} S(s)x\,ds$$

(see (5.12) of [3]). For $x \in X$ and $\lambda > \omega_1$, $\int_0^{\infty} e^{-\lambda s} PS(s)x\,ds$ exists by (2.2). Since $\omega_1 \geqslant \omega$, (2.5) follows by virtue of the closedness of P. For $x \in X$ and $\lambda > \omega_1 + M_1$,

$$\begin{aligned}\|PR(\lambda^2; A)x\| &= \Big\|\int_0^{\infty} e^{-\lambda s} PS(s)x\,ds\Big\| \leqslant \int_0^{\infty} e^{-\lambda s} M_1 \exp(\omega_1 s)\|x\|\,ds \\ &\leqslant M_1 \|x\| (\lambda - \omega_1)^{-1}.\end{aligned}$$

Then

$$\big(I - PR(\lambda^2; A)\big)^{-1} = \sum_{n=0}^{\infty} \big(PR(\lambda^2; A)\big)^n \in B(X)$$

and (2.6) follows.

Proof of the Proposition. Let $t \geqslant 0$ and $x \in X$. Then for $n = 0, 1, 2, \ldots$ we define by induction

$$\hat{S}_0(t)x = S(t)x, \quad \hat{S}_n(t)x = \int_0^t S(t-s)P\hat{S}_{n-1}(s)x\,ds. \tag{2.7}$$

Observe that $\hat{S}_n(t)x$ is well defined and $P\hat{S}_n(t)x$ is continuous for $t \geqslant 0$ and x fixed by virtue of (2.1), (2.2), and the closedness of P. For $t \geqslant 0$ and $x \in X$ we also define

$$\hat{C}_0(t)x = C(t)x, \quad \hat{C}_n(t)x = \frac{d}{dt}\hat{S}_n(t)x = \int_0^t C(t-s)P\hat{S}_{n-1}(s)x\,ds. \tag{2.8}$$

We claim that

$$|P\hat{S}_n(t)| \leqslant M_1^{n+1}(t^n/n!)\exp(\omega_1 t) \quad \text{for } t \geqslant 0,\ n = 0, 1, 2, \ldots, \tag{2.9}$$

$$|\hat{S}_n(t)| \leqslant MM_1^n(t^n/n!)t\exp(\omega t)\exp(\omega_1 t) \quad \text{for } t \geqslant 0, n = 0, 1, 2, \ldots, \tag{2.10}$$

$$|\hat{C}_n(t)| \leqslant MM_1^n(t^n/n!)\exp(\omega t)\exp(\omega_1 t) \quad \text{for } t \geqslant 0,\ n = 0, 1, 2, \ldots \tag{2.11}$$

To prove (2.9) we use induction as follows: $|P\hat{S}_0(t)| \leqslant M_1\exp(\omega_1 t)$ by Lemma 1. Assume (2.9) holds, and then for $x \in X$ we have

$$\begin{aligned} \|P\hat{S}_{n+1}(t)x\| &\leqslant \int_0^t |PS(t-s)|\,|P\hat{S}_n(s)|\,\|x\|ds \\ &\leqslant \int_0^t M_1\exp\big(\omega_1(t-s)\big)M_1^{n+1}\exp(\omega_1 s)(s^n/n!)\,\|x\|ds \\ &= M_1^{n+2}\big(t^{n+1}/(n+1)!\big)\exp(\omega_1 t)\,\|x\|. \end{aligned}$$

To prove (2.10) use (2.9) and the fact that $|S(t)| \leqslant Mt\exp(\omega t)$ for $t \geqslant 0$ as follows:

$$\begin{aligned} \|\hat{S}_{n+1}(t)x\| &\leqslant \int_0^t |S(t-s)|\,|P\hat{S}_n(s)|\,\|x\|ds \\ &\leqslant Mt\exp(\omega t)\int_0^t M_1^{n+1}(s^n/n!)\exp(\omega_1 s)\,\|x\|ds \\ &\leqslant Mt\exp(\omega t)\,M_1^{n+1}\exp(\omega_1 t)\big(t^{n+1}/(n+1)!\big)\,\|x\|. \end{aligned}$$

One proves (2.11) in a similar fashion using (2.3). Next, we write

$$\hat{S}(t)x = \sum_{n=0}^{\infty}\hat{S}_n(t)x \quad \text{for } t \geqslant 0, x \in X, \tag{2.12}$$

$$\hat{C}(t)x = \sum_{n=0}^{\infty}\hat{C}_n(t)x \quad \text{for } t \geqslant 0, x \in X, \tag{2.13}$$

where we observe that both series converge absolutely in $B(X)$ by (2.10) and (2.11). Extend $\hat{S}(t)$ and $\hat{C}(t)$ to $\boldsymbol{R}$ by defining $\hat{S}(t) = -\hat{S}(-t)$ and

$\hat{C}(t) = \hat{C}(-t)$ for $t \leqslant 0$. Observe that (2.10) and (2.11) imply

$$|\hat{S}(t)| \leqslant Mt\exp\big((\omega+\omega_1+M_1)|t|\big), \quad t \in \boldsymbol{R}, \tag{2.14}$$

$$|\hat{C}(t)| \leqslant M\exp\big((\omega+\omega_1+M_1)|t|\big), \quad t \in \boldsymbol{R}. \tag{2.15}$$

Also, by absolute convergence of the series, we have

$$\frac{d}{dt}\hat{S}(t)x = \hat{C}(t)x \quad \text{for } t \in \boldsymbol{R},\ x \in X. \tag{2.16}$$

We next claim that, for $\lambda > \omega+\omega_1$, $x \in X$, and $n = 1, 2, \ldots$,

$$\int_0^\infty e^{-\lambda s}\hat{S}_n(s)x\,ds = R(\lambda^2; A)\int_0^\infty e^{-\lambda s}P\hat{S}_{n-1}(s)x\,ds. \tag{2.17}$$

The integrals in (2.17) exist by (2.9) and (2.10). Equality (2.17) holds because

$$\begin{aligned}
\int_0^\infty e^{-\lambda s}\hat{S}_n(s)x\,ds &= \int_0^\infty e^{-\lambda s}\int_0^s S(s-\sigma)P\hat{S}_{n-1}(\sigma)x\,d\sigma\,ds \\
&= \int_0^\infty\int_\sigma^\infty e^{-\lambda s}S(s-\sigma)P\hat{S}_{n-1}(\sigma)x\,ds\,d\sigma \\
&= \int_0^\infty\int_0^\infty e^{-\lambda(s+\sigma)}S(s)P\hat{S}_{n-1}(\sigma)x\,ds\,d\sigma \\
&= \int_0^\infty e^{-\lambda\sigma}\int_0^\infty e^{-\lambda s}S(s)P\hat{S}_{n-1}(\sigma)x\,ds\,d\sigma \\
&= \int_0^\infty e^{-\lambda\sigma}R(\lambda^2; A)P\hat{S}_{n-1}(\sigma)x\,d\sigma \\
&= R(\lambda^2; A)\int_0^\infty e^{-\lambda\sigma}P\hat{S}_{n-1}(\sigma)x\,d\sigma.
\end{aligned}$$

Next, we claim that, for $\lambda > \omega+\omega_1+M_1$ and $x \in X$,

$$\int_0^\infty e^{-\lambda s}\hat{S}(s)x\,ds = R(\lambda^2; A+P)x. \tag{2.18}$$

From (2.17) we infer that for $n = 1, 2, \ldots$, $x \in X$, and $\lambda > \omega + \omega_1$,

$$\begin{aligned}\int_0^\infty e^{-\lambda s}\hat{S}_n(s)x\,ds &= R(\lambda^2; A)P\int_0^\infty e^{-\lambda s}\hat{S}_{n-1}(s)x\,ds \\ &= R(\lambda^2; A)P\Big[R(\lambda^2; A)\int_0^\infty e^{-\lambda s}P\hat{S}_{n-2}(s)x\,ds\Big] \\ &= R(\lambda^2; A)[PR(\lambda^2; A)]P\int_0^\infty e^{-\lambda s}\hat{S}_{n-2}(s)x\,ds \\ &= \ldots = R(\lambda^2; A)[PR(\lambda^2; A)]^n x.\end{aligned}$$

Then (2.18) follows from (2.12), (2.14), and Lemma 2.

Lastly, we claim that, for $\lambda > \omega + \omega_1 + M_1$ and $n = 1, 2, \ldots$,

$$\left|\frac{d^n}{d\lambda^n}\big(\lambda R(\lambda^2; A+P)\big)\right| \leqslant M_1 n!/(\lambda - \omega - \omega_1 - M_1)^{n+1}. \tag{2.19}$$

Using (2.18), (2.16), (2.15), and integration by parts, for $x \in X$ and $\lambda > \omega + \omega_1 + M_1$ we obtain

$$\lambda R(\lambda^2; A+P)x = \int_0^\infty e^{-\lambda s}C(s)x\,ds. \tag{2.20}$$

Thus, for $n = 1, 2, \ldots$, $x \in X$, and $\lambda > \omega + \omega_1 + M_1$ we get

$$\frac{d^n}{d\lambda^n}\big(\lambda R(\lambda^2; A+P)\big)x = \int_0^\infty (-s)^n e^{-\lambda s}\hat{C}(s)x\,ds.$$

Then (2.19) follows by (2.15) and the estimate

$$\begin{aligned}\left\|\frac{d^n}{d\lambda^n}\big(\lambda R(\lambda^2; A+P)\big)x\right\| &\leqslant \int_0^\infty s^n e^{-\lambda s}M\exp\big((\omega+\omega_1+M_1)s\big)ds\,\|x\| \\ &= Mn!/(\lambda-\omega-\omega_1-M_1)^{n+1}\,\|x\|.\end{aligned}$$

By the generation theorem of Sova-DaPrato, Giusti-Fattorini (see [4], Theorem 3.1), (2.19) is a necessary and sufficient condition that $A+P$ be the infinitesimal generator of a strongly continuous cosine family in X. In fact, by (2.20) and Lemma 5.8 of [3], the cosine family generated by $A+P$ must be $\hat{C}(t)$, $t \in \boldsymbol{R}$, as in (2.13). The proof of the Proposition is thus complete.

3. Remarks and open questions. As consequences of the Proposition we have the following

Remark 1. Suppose the hypothesis of the Proposition is satisfied. By Proposition 2.3 of [16], $S(t)$ is compact for every $t \in \boldsymbol{R}$ if and only if $R(\lambda; A)$ is compact for some $\lambda \in \boldsymbol{R}$. By Lemma 2, $R(\lambda^2; A+P)$ is compact if $R(\lambda^2; A)$ is compact. Thus we infer that if $S(t)$ is compact for every $t \in \boldsymbol{R}$, then $\hat{S}(t)$ is also compact for every $t \in \boldsymbol{R}$.

Remark 2. Conditions (2.1) and (2.2) are obviously satisfied provided $P \in B(X)$. Thus, the Proposition generalizes Theorem 1 in [11].

As corollaries to the Proposition we have the following

COROLLARY 1. *Let $C(t)$, $t \in \boldsymbol{R}$, be a strongly continuous cosine family in X, let P be a closed linear operator in X, and let*

(3.1) $\quad D(A) \subset D(P),$

(3.2) $\quad \|PS(t)x\| \leqslant k_t \|x\|$ *for all* $x \in D(A)$, $t \in \boldsymbol{R}$, *where k_t is a constant.*

Then $A+P$ is the infinitesimal generator of a strongly continuous cosine family in X.

Proof. We use the fact that $D(A)$ is dense in X (see (2.14) in [15]). First, let $t \in \boldsymbol{R}$, $x \in X$, and let $\{x_n\} \subset D(A)$, where $x_n \to x$. Then

$$\{S(t)x_n\} \to S(t)x \quad \text{and} \quad \|PS(t)x_n - PS(t)x_m\| \leqslant k_t \|x_n - x_m\|.$$

Consequently, (2.1) holds since $\{PS(t)x_n\}$ is Cauchy, and by the closedness of P we must have $S(t)x \in D(P)$. By the Closed Graph Theorem, $PS(t) \in B(X)$. Also, A is closed in X (see (2.14) in [15]) and P is closed from $[D(A)]$ to X, where $[D(A)]$ denotes $D(A)$ with the graph norm $\|x\|+\|Ax\|$. By the Closed Graph Theorem, P must be bounded from $[D(A)]$ to X, so there exists a constant a such that $\|Px\| \leqslant a(\|x\|+\|Ax\|)$ for $x \in D(A)$. Then (2.2) holds by virtue of (1.2), (1.3), (2.4), and the estimate

$$\|PS(t+h)x - PS(t)x\| \leqslant |PS(t)|\,\|C(h)x - x\| + a\big(\|S(h)C(t)x\| + \|S(h)AC(t)x\|\big).$$

The next corollary is similar to a result of Goldstein ([5], Theorem 8.9, p. 91).

COROLLARY 2. *Let $C(t)$, $t \in \boldsymbol{R}$, be a strongly continuous cosine family in X and let A be the infinitesimal generator $C(t)$, $t \in \boldsymbol{R}$. Further, suppose P is a closed linear operator in X and*

(3.3) $\quad$ *$C(t)$, $t \in \boldsymbol{R}$, satisfies condition* (F) *as in* [15] *(that is, there exists a closed linear operator B in X such that $B^2 = A$, $S(t)(X) \subset D(B)$ for all $t \in \boldsymbol{R}$, and $BS(t)x$ is continuous for $t \in \boldsymbol{R}$ for fixed $x \in X$);*

(3.4) $D(B) \subset D(P)$.

Then $A+P$ is the infinitesimal generator of a strongly continuous cosine family in X.

Proof. We show that (2.1) and (2.2) are satisfied. Obviously, (2.1) follows from (3.3) and (3.4). To see that (2.2) holds observe that P is closed from $[D(B)]$ to X, where $[D(B)]$ denotes $D(B)$ with the graph norm $\|x\| + \|Bx\|$. By the Closed Graph Theorem, P must be bounded from $[D(B)]$ to X, so there exists a constant b such that $\|Px\| \leqslant b(\|x\| + \|Bx\|)$ for all $x \in D(B)$. Since $S(t)x$ and $BS(t)x$ are continuous in t for each fixed $x \in X$, (2.2) follows immediately.

COROLLARY 3. *Let $C(t)$, $t \in \boldsymbol{R}$, be a strongly continuous cosine family in X and let A be the infinitesimal generator of $C(t)$, $t \in \boldsymbol{R}$. Further, suppose* (3.3) *holds, P is a closed linear operator in X, and*

(3.5) $D(A) \subset D(P)$;

(3.6) *there exist constants a and b such that $\|Px\| \leqslant a\|x\| + b\|Bx\|$ for all $x \in D(A)$.*

Then $A+P$ is the infinitesimal generator of a strongly continuous cosine family in X.

Proof. The Closed Graph Theorem and (3.3) imply that $BS(t) \in B(X)$ for all $t \in \boldsymbol{R}$. Then (3.5) and (3.6) imply (3.1) and (3.2) immediately, with $k_t = a|S(t)| + b|BS(t)|$.

We conclude with some open questions concerning the perturbation of cosine family generators.

If A_1 and A_2 are both infinitesimal generators of strongly continuous cosine families and A_1 and A_2 commute, is $A_1 + A_2$ the infinitesimal generator of a strongly continuous cosine family? (**P 1238**)

If A is the infinitesimal generator of a strongly continuous cosine family, does there exist a positive number b such that the cosine family $C_b(t)$, $t \in \boldsymbol{R}$, generated by $A - bI$ is uniformly bounded in the sense that there exists an $M \geqslant 1$ and $|C_b(t)| \leqslant M$ for all $t \in \boldsymbol{R}$? (**P 1239**)

REFERENCES

[1] G. DaPrato and E. Giusti, *A characterization of generators of abstract cosine functions* (in Italian), Bolletino della Unione Matematica Italiana 22 (1967), p. 357-362.

[2] N. Dunford and J. T. Schwartz, *Linear operators*, Part I: *General theory*, New York 1958.

[3] H. O. Fattorini, *Ordinary differential equations in linear topological spaces, I*, Journal of Differential Equations 5 (1968), p. 72-105.
[4] — *Ordinary differential equations in linear topological spaces, II*, ibidem 6 (1969), p. 50-70.
[5] J. Goldstein, *Semigroups of operators and abstract Cauchy problems*, Monograph, Tulane University, New Orleans, 1970.
[6] J. Kisyński, *On operator-valued solutions of d'Alembert's functional equation, I*, Colloquium Mathematicum 23 (1971), p. 107-114.
[7] — *On cosine operator functions and one-parameter groups of operators*, Studia Mathematica 44 (1972), p. 93-105.
[8] S. Kurepa, *On some functional equations in Banach spaces*, ibidem 19 (1960), p. 149-158.
[9] — *A cosine functional equation in Banach algebras*, Acta Scientiarum Mathematicarum (Szeged) 23 (1962), p. 255-267.
[10] B. Nagy, *On the generators of cosine operator functions*, Publicationes Mathematicae 21 (1974), p. 151-154.
[11] — *On cosine operator functions in Banach spaces*, Acta Scientiarum Mathematicarum (Szeged) 36 (1974), p. 281-289.
[12] — *Cosine operator functions and the abstract Cauchy problem*, Periodica Mathematica Hungarica 7 (3) (1976), p. 15-18.
[13] M. Sova, *Cosine operator-functions*, Rozprawy Matematyczne 49 (1966), p. 1-47.
[14] — *Semigroups and cosine functions of normal operators in Hilbert spaces*, Časopis pro Pěstování Matematiky 93 (1968), p. 437-458.
[15] C. Travis and G. Webb, *Cosine families and abstract nonlinear second order differential equations* (to appear).
[16] — *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families* (to appear).

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