COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

LITTLEWOOD FUNCTIONS, HANKEL MULTIPLIERS AND POWER BOUNDED OPERATORS ON A HILBERT SPACE

BY

MAREK BOŻEJKO (WROCŁAW)

DEDICATED TO MY TEACHER
PROFESSOR STANISŁAW HARTMAN
ON THE OCCASION OF HIS 70TH BIRTHDAY

1. Introduction. If T is a contraction on a complex Hilbert space (i.e. $\|T\| \leqslant 1$), then by the inequality of J. von Neumann we have

$$\|\varphi(T)\| \leqslant \sup\{|\varphi(z)|: |z| \leqslant 1\} = \|\varphi\|_{H^\infty}$$

for any complex polynomial φ.

B. Sz.-Nagy observed [11] that if T is an invertible bounded operator on a Hilbert space such that $\|T^n\| \leqslant C$ for any integer $n \in Z$, then T is similar to a unitary operator and then for every complex polynomial φ

$$(*) \qquad \|\varphi(T)\| \leqslant C\|\varphi\|_{H^\infty}.$$

A. Lebow [7] showed that there exists a power bounded operator on a Hilbert space (i.e. $\|T^n\| \leqslant C$, $n \geqslant 0$) which is not polynomially bounded (i.e. the inequality $(*)$ does not hold).

Later A. M. Davie and V. V. Peller ([3], [13]) produced other examples of the operators of that type.

In this paper we introduce and describe a subspace of multipliers of the Hardy space H^1 – the Littlewood–Hankel multipliers T_p^H $(p = 1, 2)$.

Among other facts we show that T_2^H is the space of the multipliers from H^1 to H^2.

The main result of this note is the following: *for every sequence $(c_n) \in T_1^H$ there exists a power bounded operator T on a Hilbert space H such that*

$$c_n = \langle T^n \xi, \eta\rangle \quad \textit{for some } \xi, \eta \in H.$$

Using that fact we prove that for any $c > 1$

$$\|\varphi\|_c \geqslant K(c) \sum_{n \geqslant 0} |\hat{\varphi}(2^n)|,$$

where $\|\varphi\|_c = \sup_{n \geq 0}\{\|\varphi(T)\|: \|T^n\| \leqslant c\}$. A similar inequality was obtained by Davie [3] and Peller [13] by different methods.

As a consequence of our Theorem 1 we construct a large family of power bounded operators which are not polynomially bounded.

The author wishes to thank R. Szwarc for valuable discussions and remarks.

2. Preliminaries. Let X be an arbitrary set and let $L(l^2(X))$ denote the Banach space of all bounded operators on $l^2(X)$. Every element $k \in L(l^2(X))$ can be considered as a mapping $k: X \times X \to C$ such that $\|k\| < \infty$, where

$$\|k\|^2 = \sup\{\sum_y |\sum_x k(x, y) u(x)|^2: |u|_{l^2} = 1\}.$$

The algebra of *Schur multipliers* $V_2(X)$ is defined as follows: $a \in V_2(X)$ if $a: X \times X \to C$ and for every $k \in L(l^2(X))$ $\|a \cdot k\| \leqslant |||a||| \, \|k\|$ (the multiplication here is the pointwise multiplication of the matrices). Schur multipliers were extensively studied by G. Bennett [1] who proved the following useful theorem:

$a \in V_2(X)$ if and only if the operator a: $l^1(X) \to l^\infty(X)$ is an absolutely summing operator.

The Grothendieck inequality and the above theorem imply that:

$a \in V_2(X)$ if and only if there exists a Hilbert space H such that the operator a can be factorized through H, i.e.

$$\begin{tikzcd} a\colon l^1(X) \arrow[rr] \arrow[dr, "\alpha"'] & & l^\infty(X) \\ & H \arrow[ur, "\beta^*"'] & \end{tikzcd}$$

or equivalently, a is of the form $a(x, y) = \langle \alpha(x), \beta(y) \rangle$, where α, β: $l^1(X) \to H$ and $\|\alpha(x)\| \leqslant C$, $\|\beta(y)\| \leqslant C$.

If Y and Z are two spaces of functions on some set X, let $M(Y, Z)$ denote the space of the *multipliers* from Y into Z, i.e. the space of the functions ψ on X such that $\psi \cdot f \in Z$ for every $f \in Y$.

If A, B are two Banach spaces let $L(A, B)$ denote the space of the linear bounded maps from A into B.

In our notation $V_2(X) = M(L(l^2(X)))$.

N. Varopoulos [15] introduced the space of Littlewood functions – $T_2(X)$. A function $c \in l^\infty(X \times X)$ is called a *Littlewood function* if c can be decomposed as

$$c = a + b,$$

where

$$a: l^1(X) \to l^2(X) \quad \text{and} \quad b: l^2(X) \to l^\infty(X)$$

or, in other words,

$$T_2(X) = L(l^1(X), l^2(X)) + L(l^2(X), l^\infty(X)).$$

By the Grothendieck inequality we see that

$$T_2(X) \subset V_2(X).$$

Now we recall the theorem of Varopoulos on the characterization of Littlewood functions (see [15]):

$a \in T_2(X)$ *if and only if for any choice of finite subsets* F_1, F_2 *of the set* X

$$\{\sum |a(x, y)|^2 \colon x \in F_1, y \in F_2\} \leqslant C^2 \max_{j=1,2}(|F_j|).$$

Now we introduce a subclass $T_1(X)$ of the Littlewood functions (which is more important for this note), defined as

$$T_1(X) = L(l^1(X), l^1(X)) + L(l^\infty(X), l^\infty(X)).$$

It is not difficult to observe (see [2]) that a function a belongs to $T_1(X)$ if and only if for any choice of finite subsets F_1, F_2 of X we have

$$\{\sum |a(x, y)| \colon x \in F_1, y \in F_2\} \leqslant C \max_{j=1,2}(|F_j|).$$

3. **Littlewood–Hankel multipliers.** From now we put $X = N$, the set of all nonnegative integers. We shall say that a sequence (c_n) is the *Hankel multiplier* ($(c_n) \in M^H$) if the matrix (c_{i+j}) is a Schur multiplier.

If $H^1(T)$ denotes the *Hardy class*, i.e. $H^1(T) = \{f \in L^1(T) \colon \hat{f}(n) = 0$ for $n < 0\}$, where $\hat{f}(n)$ is the n-th Fourier coefficient of f, then by the Z. Nehari theorem [9] we have

$$M^H \subset M(H^1).$$

Definition. A sequence $c = (c_n)$ is called a *p-Littlewood–Hankel multiplier* ($c \in T_p^H$) if $(c_{i+j}) \in T_p(N)$ ($p = 1$ or 2).

Proposition 1. *Let $p = 1$ or 2. The following conditions are equivalent*:

(i) $$(c_n) \in T_p^H;$$

(ii) $$\sup_{m \geqslant 1} (1/m) \sum_{k=1}^{m} k |c_k|^p < \infty;$$

(iii) $$\sup_{m \geqslant 1} (1/m^2) \sum_{k=1}^{m} k^2 |c_k|^p < \infty;$$

(iv) $$\sup_{m \geqslant 1} \sum_{k=m}^{2m} |c_k|^p < \infty;$$

(v) *If* $c_{n+m} = a(n, m)+b(n, m)$, *where*

$$a(n, m) = \begin{cases} c_{n+m} & \text{for } n \geq m, \\ 0 & \text{for } n < m, \end{cases}$$

then $a \in L(l^1, l^p)$ *and* $b \in L(l^{p'}, l^\infty)$.

Proof. Since

$$\sum_{k=m}^{2m} |c_k| \leq (1/m^2) \sum_{k=m}^{2m} k^2 |c_k| \leq 2/m \sum_{k=m}^{2m} k|c_k|,$$

we infer that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Let $c = (c_n) \in T_p^H$; the Varopoulos theorem implies that

$$\sum_{i,j=1}^{m} |c_{i+j}|^p \leq Cm \quad \text{for any } m = 1, 2, \ldots,$$

and by an elementary calculation we get

$$\sum_{k=1}^{m} k|c_k|^p \leq \sum_{i,j=0}^{m} |c_{i+j}|^p,$$

hence (i) $\Rightarrow$(ii). To obtain the implication (iv) $\Rightarrow$ (v) let us observe that the decomposition $c_{i+j} = a_{ij}+b_{ij}$, where $a_{ij} = c_{i+j}$ for $i \geq j$ and 0 for $i < j$, yields

$$\sum_j |a_{ij}|^p = \sum_{k=i}^{2i} |c_k|^p \quad \text{and} \quad \sum_i |b_{ij}|^p \leq \sum_{k=j}^{2j} |c_k|^p.$$

The implication (v) $\Rightarrow$ (i) now follows from the definition of T_p^H.

Corollary 1. *Let* $E = (n_k)$ *be any sequence of positive integers with* $n_{k+1}/n_k \geq \lambda > 1$. *Then* $l^\infty(E) \subset T_1^H$.

For the proof let us note that for any natural m we have

$$|\{n \in E\colon m \leq n \leq 2m\}| \leq \text{Const}.$$

Proposition 2. $T_2^H = M(H^1, H^2)$.

The proof of that fact follows from Proposition 1 and the well-known characterization of $M(H^1, H^2)$ (see Duren, *Theory of H^p spaces*). We shall present the proof which does not use the theory of analytic functions.

Proof. Let $(c_n) \in M(H^1, H^2) = M(H^1, l^2)$ and let F_m denote the Fejer kernel

$$F_m(t) = \sum_{|k| \leq m} (1-|k|/m) e^{ikt}.$$

Let us define $G_m(t) = F_m(t) e^{imt}$. It is clear that

(i) $G_m \in H^1$, $\|G_m\|_{L^1} = 1$;

(ii) $\hat{G}_m(k) = k/m$ for $1 \leq k \leq m$.

Since $(c_n) \in M(H^1, l^2)$, by the Closed Graph Theorem we have

$$\sum_{k=0}^{\infty} |c_k \hat{G}_m(k)|^2 \leqslant C^2 \|G_m\|_{L^1}^2 = C^2.$$

Hence

$$(1/m^2) \sum_{k=1}^{m} k^2 |c_k|^2 \leqslant C^2$$

and, by Proposition 1,

$$M(H^1, l^2) \subset T_2^H.$$

On the other hand, if $(c_n) \in T_2^H$, then for any sequence (α_n), $|\alpha_n| = 1$,

(**) $$(\alpha_n c_n) \in T_2^H \subset M(H^1).$$

Let us take $\alpha_n = r_n(\omega)$, where $\omega \in [0, 1]$ and r_n is the sequence of the Rademacher functions. By (**), for any $\omega \in [0,1]$ the following inequality holds:

$$\int_T \Big| \sum_{n \geqslant 0} c_n r_n(\omega) \hat{f}(n) e^{int} \Big| dt \leqslant C \|f\|_{H^1}.$$

Using the Khinchine inequality we get

$$\sum_{n \geqslant 0} |c_n \hat{f}(n)|^2 \leqslant C_1^2 \|f\|_{H^1}^2.$$

This completes the proof.

4. The construction of the power bounded operators. In this section we construct a large family of power bounded operators on a Hilbert space. The main result is the following:

THEOREM 1. *Let $c \in T_1^H$. Then there exists a linear operator T on a Hilbert space H such that*

$$\|T^n\| \leqslant K,$$

$$c_n = \langle T^n \xi, \eta \rangle \quad \textit{for some } \xi, \eta \in H,$$

$$K = K(\|c\|_{T_1^H}) > 1 \quad \textit{and} \quad K(\alpha) \to 1 \quad \textit{if } \alpha \to 0.$$

Proof. Let $c_{n+m} = a(n, m) + b(n, m)$, where $a\colon l^1 \to l^1$, $b\colon l^\infty \to l^\infty$ and $a(n, 0) = c_n$,

$$\sum_m |a(n, m)| \leqslant C, \quad \sum_n |b(n, m)| \leqslant C, \quad C = \|c\|_{T_1}$$

(see Proposition 1 (v)).

For $f \in l^\infty(N)$ define

$$(T_k f)(n) = f(k+n), \quad k, n \in N;$$

$$(T_k^* f)(n) = \begin{cases} f(n-k) & \text{for } n \geqslant k, \\ 0 & \text{for } n < k. \end{cases}$$

It follows from the assumption that for all natural k

$$a(n+k, m) - a(n, k+m) = b(n, k+m) - b(n+k, m);$$

the last equality means that for any $f \in l^1(N)$ we have

$$A_k f = a(T_k^* f) - T_k(a(f)) = T_k(b(f)) - b(T_k^* f).$$

Since T_k is uniformly bounded (with respect to k) on $l^1(N)$ and $l^\infty(N)$, we obtain that for any $k \in N$

$$A_k \in L(l^1, l^1) \cap L(l^\infty, l^\infty(N)).$$

Hence by the interpolation (in fact, by the Schwarz inequality) we get

$$A_k\colon l^2 \to l^2(N) \quad \text{and} \quad \|A_k\| \leqslant 2C \quad \text{for any } k \in N.$$

Now consider $H_0 = l^1(N) \times l^1(N)$ as a prehilbert space with the inner product given by the following formula:

$$\langle (f_1, g_1), (f_2, g_2) \rangle_0 = \langle f_1, f_2 \rangle + \langle a(f_1) + g_1, a(f_2) + g_2 \rangle,$$
$$(f_i, g_i) \in H_0, \ i = 1, 2,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $l^2(N)$.

Let us define the operator $S_k\colon H_0 \to H_0$ by the formula

$$S_k(f, g) = (T_k^* f, T_k g).$$

We shall show now that $\|S_k\| \leqslant K$, where

$$K^2 = \max(1 + 1/\beta^2, 1 + 4C^2 + 4C^2\beta^2)$$

for any $\beta > 0$.

This follows immediately from the inequalities

$$\begin{aligned} \|S_k(f, g)\|_0^2 &= \|T_k^* f\|^2 + \|a(T_k^* f) + T_k g\|^2 \\ &\leqslant \|f\|^2 + \big(\|A_k f\|^2 + \|T_k(a(f)) + T_k g\|\big)^2 \\ &\leqslant (1 + 4C^2)\|f\|^2 + \|a(f) + g\|^2 + 2\beta \|A_k f\| \, \|a(f) + g\| \, \beta^{-1}. \end{aligned}$$

Put $T = S_1$ and $\xi = (\delta_0, 0)$, $\eta = (0, \delta_0)$; we then get $\|T^n\| \leqslant K$ and $c_n = \langle T^n \xi, \eta \rangle = a(n, 0)$.

Remark. The idea of that construction was first given by G. Fendler [4], who produces, in a special case, the uniformly bounded representation of the free noncommutative group.

5. The application to polynomially bounded operators on Hilbert space. A direct consequence of Theorem 1 is the following

COROLLARY 2. *For any bounded sequence* (c_n), $\sup|c_n| = 1$, *there exist an operator* T *on some Hilbert space* H *and a constant* $C > 0$ *such that* $\sup_n \|T^n\| < \infty$ *and that for any complex polynomial* φ *we have*

$$\|\varphi(T)\| \geqslant C\Big|\sum_{n\geqslant 0} \hat{\varphi}(2^n)c_n\Big|.$$

Proof. By Corollary 1, if $E = \{2^n\colon n = 1, 2, \ldots\}$, then $l^\infty(E) \subset T_1^H$. Hence, by Theorem 1, there exists an operator T such that

$$\langle T^n\xi, \eta\rangle = \begin{cases} c_k & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k, \end{cases}$$

and $\|T^n\| \leqslant C_1$. Therefore if $\varphi = \sum_{n\geqslant 0} \hat{\varphi}(n)z^n$, then

$$\Big|\sum_{k\geqslant 0} \hat{\varphi}(2^k)c_k\Big| = |\langle\varphi(T)\xi, \eta\rangle| \leqslant C_2\|\varphi(T)\|.$$

Corollary 2 implies the following inequality which was obtained by a different method by V. V. Peller [13].

COROLLARY 3. *For any complex polynomial* φ *and any* $c > 1$

$$\|\varphi\|_c \geqslant D(c)\sum_{n\geqslant 0} |\hat{\varphi}(2^n)|.$$

Now we can state our application theorem.

THEOREM 2. *Let* $c = (c_n) \in l^\infty(E)$, $E = \{2^n\colon n = 1, 2, \ldots\}$ *and* T *be a power bounded operator such that*

$$\langle T^n\xi, \eta\rangle = c_n, \quad n \geqslant 0.$$

If T *is polynomially bounded, then* $c \in l^2(E)$.

Proof. If T is polynomially bounded, then for any polynomial φ by Corollary 2 we get

$$\Big|\sum_{n\in E} \hat{\varphi}(n)c_n\Big| \leqslant K\|\varphi\|_\infty.$$

Since E is the Sidon set, it follows that for any $(\alpha_n) \in l^\infty(E)$, $|\alpha_n| \leqslant 1$, there exists a bounded Borel measure $\mu \in M(T)$ such that

$$\hat{\mu}(n) = \alpha_n \quad \text{for } n \in E \text{ and } \|\mu\| \leqslant 2.$$

If we choose α_n in such a way that

$$\hat{\varphi}(n)c_n\alpha_n = |\hat{\varphi}(n)c_n| \quad \text{for } n \in E,$$

then we get

$$\sum_{n\geq 0} |\hat{\varphi}(n) c_n| \leqslant K \|\mu * \varphi\|_\infty \leqslant 2K \|\varphi\|_\infty .$$

This implies that the sequence (c_n) belongs to the space of multipliers from the disc algebra $A(D)$ to $l^1(N)$. By the theorem of Paley [12], $(c_n) \in l^2$.

COROLLARY 4. *For any sequence* $c = (c_n) \in l^\infty(E) \setminus l^2(E)$, $E = \{2^n: n = 1, 2, \ldots\}$, *the operator* $T = T_c$ (existing by Theorem 1) *is a power bounded operator which is not polynomially bounded.*

REFERENCES

[1] G. Bennett, *Schur multipliers*, Duke Mathematical Journal 44 (1977), p. 603–639.

[2] M. Bożejko and J. Wysoczański, *Littlewood functions and the characterization of discrete groups*, Preprint.

[3] A. M. Davie, *Power-bounded elements in Q-algebra*, Bulletin of the London Mathematical Society 6 (1974), p. 61–65.

[4] G. Fendler, *A letter to the author* (1981).

[5] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Boletim da Sociedade de Matématica de São Paulo 8 (1955), p. 1–79.

[6] G. H. Hardy and J. E. Littlewood, *Theorem concerning mean values of analytic and harmonic functions*, The Quarterly Journal of Mathematics, Oxford, 12 (1841), p. 221–256.

[7] A. Lebow, *A power-bounded operator that is not polynomially bounded*, Michigan Mathematical Journal 15 (1968), p. 397–399.

[8] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Mathematica 29 (1968), p. 275–326.

[9] Z. Nehari, *On bounded bilinear forms*, Annals of Mathematics 65 (1957), p. 153–162.

[10] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Mathematische Nachrichten 4 (1951), p. 258–281.

[11] B. Sz.-Nagy, *On uniformly bounded linear transformations in Hilbert space*, Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum 11 (1947), p. 152–157.

[12] R. E. A. C. Paley, *A note on power series*, The Journal of the London Mathematical Society 7 (1932), p. 122–130.

[13] V. V. Peller, *Estimates of functions of power bounded operators on Hilbert spaces*, Journal of Operator Theory 7 (1982), p. 341–372.

[14] N. Th. Varopoulos, *Tensor algebras over discrete spaces*, Journal of Functional Analysis 3 (1969), p. 321–335.

[15] –, *On an inequality of von Neumann and an application of the metric theory of tensor products to operator theory*, ibidem 16 (1974), p. 83–100.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
WROCŁAW, POLAND

Reçu par la Rédaction le 25. 04. 1984