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ON SCALAR CONCOMITANTS OF GEOMETRIC OBJECTS AND THEIR TRANSITIVE DOMAINS

BY

M. KUCHARZEWSKI (KATOWICE) AND S. WĘGRZYNOWSKI (SZCZECIN)

Determination of concomitants is one of those foremost problems in the theory of geometric objects which are of great importance for differential geometry. In this paper we will present some considerations concerning scalar concomitants and consisting of a definition of a complete and independent system of scalar concomitants for a given abstract geometric object, and of certain relations between such systems and decompositions of the fibre of the object into transitive fibres. These considerations form a basis for the determination of transitive fibres of the tensor $t^{\nu}_{\lambda\mu}$ in a two-dimensional space.

1. Let ω be an abstract special geometric object with the transformation formula

$$\omega' = F(\omega, L), \qquad \omega \in \mathfrak{M}, L \in \mathscr{L}^n_s, \tag{1.1}$$

and with the fibre $\mathfrak{M}$ (cf. [3]).

Let us denote a scalar concomitant of object ω by σ,

$$\sigma = H(\omega), \qquad \omega \in \mathfrak{M}, \qquad \sigma \in \mathfrak{N} \subset R. \tag{1.2}$$

This concomitant is determined by a function H which maps fibre $\mathfrak{M}$ of ω onto fibre $\mathfrak{N}$ of the abstract scalar σ. It is clear that function H is constant on transitive fibres of object ω (see [1]). Since the last property is characteristic for scalar concomitants, we have the following

THEOREM 1.1. *Function $H(\omega)$: $\mathfrak{M} \to \mathfrak{N} \subset R$ determines scalar concomitants of an object ω if and only if it is constant on transitive fibres of this object.*

Let

$$\sigma_i = H_i(\omega), \qquad i = 1, 2, \ldots, p, \tag{1.3}$$

denote system of p-scalar concomitants of object (1.1) with the fibres $\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_p$, respectively, where each $\mathfrak{N}_i \subset R$ $(i = 1, 2, \ldots, p)$ is determined by a function H_i.

As follows from theorem 1.1, each function $\varphi(\sigma_1, \sigma_2, \ldots, \sigma_p)$ transforming cartesian product $\mathfrak{N}_1 \times \mathfrak{N}_2 \times \ldots \times \mathfrak{N}_p$ into a subset $\mathfrak{N}$ of real numbers,

$$(1.4) \qquad \varphi: \mathfrak{N}_1 \times \mathfrak{N}_2 \times \ldots \times \mathfrak{N}_p \to \mathfrak{N} \subset R,$$

determines a scalar concomitant of object ω:

$$(1.5) \qquad \sigma = H(\omega) = \varphi\big(H_1(\omega), H_2(\omega), \ldots, H_p(\omega)\big).$$

Definition 1.1. A scalar concomitant (1.2) is called *dependent on the system of scalar concomitants* (1.3) if there exists a function φ of form (1.4) such that (1.5) holds.

This definition, as well as definitions of a complete and of an independent system of scalar concomitants (see definitions 1.2 and 3.1), are not new, but we shall recall them for the convenience of the reader.

Definition 1.2. A system of scalar concomitants (1.3) of object (1.1) will be called *complete* if every scalar concomitant of (1.1) is dependent on scalar concomitants of (1.3).

2. If a system of scalar concomitants is known, then its transitive fibres can be determined with the aid of theorem 2.1.

Let be given a system (1.3) of scalar concomitants of an object (1.1). For every system of real numbers $C_1, C_2, \ldots, C_p$ let us denote by $\mathfrak{M}_{C_1,C_2,\ldots,C_p}$ the following subset of the fibre $\mathfrak{M}$ of object ω:

$$(2.1) \qquad \mathfrak{M}_{C_1,C_2,\ldots,C_p} = \{\omega: \omega \in \mathfrak{M},\ H_i(\omega) = C_i, \quad i = 1, 2, \ldots, p\}.$$

THEOREM 2.1. *If system* (1.3) *is complete, then each set* $\mathfrak{M}_{C_1,C_2,\ldots,C_p}$ *is either transitive fibre of* ω *or is empty.*

Proof. If the set (2.1) is not empty, there exists an element $\omega_0 \in \mathfrak{M}$ such that $\omega_0 \in \mathfrak{M}_{C_1,C_2,\ldots,C_p}$, i.e. $H_i(\omega_0) = C_i$, $i = 1, 2, \ldots, p$.

Since functions H_i are constant on transitive fibres of ω, we have the relation $H_i(\omega') = C_i$, $i = 1, 2, \ldots, p$, where ω' denotes a component of ω in a new coordinate system $\omega' = F(\omega, L)$.

It is clear that $\mathfrak{M}_{C_1,C_2,\ldots,C_p}$ is an allowable set (cf. [2]). Let us denote by $\mathfrak{M}\omega_0$ the transitive fibre to which ω_0 belongs. Since $\mathfrak{M}_{C_1,C_2,\ldots,C_p}$ is an allowable set and contains ω_0, it also contains $\mathfrak{M}\omega_0$,

$$(2.2) \qquad \mathfrak{M}\omega_0 \subset \mathfrak{M}_{C_1,C_2,\ldots,C_p}.$$

To prove the theorem it suffices to show that

$$(2.3) \qquad \mathfrak{M}\omega_0 \supset \mathfrak{M}_{C_1,C_2,\ldots,C_p}.$$

Assume to the contrary that (2.3) does not hold, i.e. that there exists ω_1 which belongs to $\mathfrak{M}_{C_1,C_2,\ldots,C_p}$ but not to $\mathfrak{M}\omega_0$:

$$(2.4) \qquad \omega_1 \in \mathfrak{M}_{C_1,C_2,\ldots,C_p}, \qquad \omega_1 \notin \mathfrak{M}\omega_0.$$

Consider a function $\tilde{H}$ on $\mathfrak{M}$,

$$\tilde{H}(\omega) = \begin{cases} k_1, & \omega \in \mathfrak{M}\omega_0, \\ k_2, & \omega \in \mathfrak{M} - \mathfrak{M}\omega_0, \end{cases}$$

where k_1 and k_2 are two different real numbers,

$$k_1 \neq k_2. \tag{2.5}$$

Function $\tilde{H}$ determines scalar concomitant $\tilde{\sigma}$, $\tilde{\sigma} = \tilde{H}(\omega)$, because it is constant on transitive fibres of ω. The scalar concomitant $\tilde{\sigma}$ does not depend on system (1.3), because each scalar concomitant dependent on (1.3) is constant on $\mathfrak{M}_{C_1, C_2, \ldots, C_p}$, while, according to (2.4), $\tilde{H}(\omega)$ takes two different values on $\mathfrak{M}_{C_1, C_2, \ldots, C_p}$: $\tilde{H}(\omega_0) = k_1$ and $\tilde{H}(\omega_1) = k_2$.

This is a contradiction to assumption that system (1.3) is complete. Thus (2.3) holds and theorem 2.1 is proved.

Theorem 2.1 leads to a result on the form of transitive fibres of ω.

THEOREM 2.2. *If system* (1.3) *is complete, then every transitive fibre of object* (1.1) *is equal to the set* $\mathfrak{M}_{C_1, C_2, \ldots, C_p}$, *where* $C_1, C_2, \ldots, C_p$ *denote some constants.*

Proof. Let $\mathfrak{M}\omega_0$ be the transitive fibre to which ω_0 belongs. Put

$$C_{0i} = H_i(\omega_0), \quad i = 1, 2, \ldots, p. \tag{2.6}$$

Since the set $\mathfrak{M}_{C_{01}, C_{02}, \ldots, C_{0p}}$ is non-empty, it must be, by theorem 2.1, the transitive fibre of ω. Because to this set belongs ω_0, it must be equal to the transitive fibre $\mathfrak{M}\omega_0$. Thus theorem 2.2 is proved.

Formulas (2.4) permit to find constants $C_1, C_2, \ldots, C_p$ which determine the transitive fibre to which the point ω_0 belongs.

3. There exist many complete systems of scalar concomitants for a given abstract object. Some concomitants of such a system can be dependent on the others. A problem arises to choose a complete system which would contain a minimum number of scalar concomitants. In particular, it is important to find a system which does not contain dependent concomitants. To that end we first define a concept of a dependent concomitant system.

Definition 3.1. System (1.3) of scalar concomitants of object (1.1) will be called *dependent* if one (at least) of concomitants of this system is dependent on the others. In the opposite case the system will be called *independent.*

Definition 3.2. If function H is constant, the scalar concomitant σ will be called *trivial.*

A necessary condition for a system to be dependent is given by the following

THEOREM 3.1. *If system* (1.3) *of non-trivial scalar concomitants of object* (1.1) *is dependent, then there exists a system of constants* $C_1, C_2, \ldots, C_p$ *satisfying conditions*

$$C_i \in \mathfrak{N}_i, \quad i = 1, 2, \ldots, p, \tag{3.1}$$

for which set $\mathfrak{M}_{C_1, C_2, \ldots, C_p}$ *is empty.*

Proof. Let system (1.3) be dependent. Assume that σ_p is dependent on the others. Then there exists a function φ such that

$$\sigma_p = \varphi(\sigma_1, \sigma_2, \ldots, \sigma_{p-1}).$$

By the assumption, σ_p is a non-trivial concomitant, i.e., it takes at least two different values k_1 and k_2. Let $C_{11}, C_{12}, \ldots, C_{1,p-1}$ denote the system of real numbers for which function φ takes value k_1. Then the set $\mathfrak{M}_{C_{11}, C_{12}, \ldots, C_{1,p-1}, k_2}$ is empty. Constants $C_{11}, C_{12}, \ldots, C_{1p-1}, k_2$ satisfy condition (3.1). The theorem is proved.

4. Examples. Let be given the Weyl density $\omega' = |J|\omega$ with the fibre $\mathfrak{M} = R$. One of scalar concomitants of this density is the object σ determined by the function

$$H(\omega) = \begin{cases} \operatorname{sgn} \omega, & \omega \in R - \{0\}, \\ 0, & \omega = 0. \end{cases} \tag{4.1}$$

The set $\mathfrak{N} = \{-1, 0, 1\}$ is the fibre of σ. The most general scalar concomitant of ω is determined by the function defined as follows:

$$\tilde{H}(\omega) = \begin{cases} C_1, & \omega > 0, \\ C_2, & \omega = 0, \\ C_3, & \omega < 0, \end{cases} \qquad \tilde{\sigma} = \tilde{H}(\omega).$$

The scalar σ forms a complete system of scalar concomitants of the Weyl density, because each scalar concomitant $\tilde{\sigma}$ is dependent on σ. In fact, this dependence is determined by the function φ defined as follows: $\varphi(-1) = C_3$, $\varphi(0) = C_2$, $\varphi(1) = C_1$. Then we have $\tilde{\sigma} = \varphi(\sigma)$ and $\tilde{H}(\omega) = \varphi(H(\omega))$.

The trivial concomitant is determined by the function constant on the fibre $\mathfrak{M}$, i.e., such that for all real numbers we have $H_0(\omega) = C$, $\omega \in R$, and $\sigma_0 = H_0(\omega)$.

The trivial concomitant does not form a complete system, σ being independent of it because there is no function φ which satisfies $H(\omega) = \varphi(H_0(\omega))$. It follows that every scalar concomitant of the Weyl density is dependent on (4.1).

5. Transitive domains of tensor $t^{\nu}_{\lambda\mu}$ in the two-dimensional space. Let be given a tensor $t^{\nu}_{\lambda\mu}$ in some point of a two-dimensional manifold of class C_1,

$$t^{\nu'}_{\lambda'\mu'} = A^{\lambda}_{\lambda'} A^{\mu}_{\mu'} A^{\nu'}_{\nu} t^{\nu}_{\lambda\mu}.$$

The fibre of tensor $t^{\nu}_{\lambda\mu}$ is $\mathfrak{M} = R^8$.

Consider two covariant vectors formed by the contraction of $t^{\nu}_{\lambda\mu}$:

$$\overset{1}{v}_{\lambda} \overset{\mathrm{df}}{=} t^{\mu}_{\lambda\mu} \quad \text{and} \quad \overset{2}{v}_{\lambda} \overset{\mathrm{df}}{=} t^{\mu}_{\mu\lambda}. \tag{5.2}$$

Put

$$\begin{aligned} \mathfrak{M}_2 &= \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \det\|\overset{a}{v}_{\lambda}\| \neq 0\}, \\ \mathfrak{M}_{\varkappa 1} &= \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} \neq 0,\ \overset{2}{v} = \varkappa \overset{1}{v}\}, \\ \mathfrak{M}_{10} &= \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} = 0,\ \overset{2}{v} \neq 0\}, \\ \mathfrak{M}_{00} &= \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} = \overset{2}{v} = 0\}. \end{aligned} \tag{5.3}$$

Subsets $\mathfrak{M}_2, \mathfrak{M}_{\varkappa 1}, \mathfrak{M}_{10}$ and $\mathfrak{M}_{00}$ are allowable sets of the fibre $\mathfrak{M}$ of tensor $t^{\nu}_{\lambda\mu}$ in the two-dimensional space. In the paper [4] a complete system of scalar concomitants for $t^{\nu}_{\lambda\mu}$ in several allowable sets has been determined. Making use of [4], we can determine transitive fibres for tensor $t^{\nu}_{\lambda\mu}$ in the two-dimensional space.

It follows from theorem 1 in [4] that the most general scalar concomitant of tensor $t^{\nu}_{\lambda\mu}$ in the allowable set $\mathfrak{M}_2$ is a function of the form

$$f(t^{\nu}_{\lambda\mu}) = \varphi(\underset{11}{\overset{1}{\omega}}, \underset{11}{\overset{2}{\omega}}, \underset{22}{\overset{1}{\omega}}, \underset{22}{\overset{2}{\omega}}), \tag{5.4}$$

where

$$\underset{\varrho\sigma}{\overset{a}{\omega}} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu} \underset{\varrho}{\overset{-1}{v}}{}^{\lambda} \underset{\sigma}{\overset{-1}{v}}{}^{\mu} \overset{a}{v}_{\nu} \tag{5.5}$$

and $\overset{-1}{v}{}^{\lambda}$ are contravariant vectors satisfying conditions

$$\overset{\varrho}{v}_{a} \underset{\sigma}{\overset{-1}{v}}{}^{a} = \delta^{\varrho}_{\sigma}.$$

By theorem 2.1 we infer that, in a two-dimensional space, every set $\mathfrak{M}_{C_1,C_2,C_3,C_4}$ of the form

$$\mathfrak{M}_{C_1,C_2,C_3,C_4} = \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M}_2,\ \underset{11}{\overset{1}{\omega}} = C_1,\ \underset{11}{\overset{2}{\omega}} = C_2,\ \underset{22}{\overset{1}{\omega}} = C_3,\ \underset{22}{\overset{2}{\omega}} = C_4\}, \tag{5.6}$$

where C_1, C_2, C_3 and C_4 are arbitrary real numbers, is either the transitive fibre of tensor $t^{\nu}_{\lambda\mu}$ or is empty. It can be proved that this set is never empty. For that purpose it is enough to show that the system of equations

$$\underset{11}{\overset{1}{\omega}} = C_1, \quad \underset{11}{\overset{2}{\omega}} = C_2, \quad \underset{22}{\overset{1}{\omega}} = C_3, \quad \underset{22}{\overset{2}{\omega}} = C_4 \tag{5.7}$$

has solutions for arbitrary values C_1, C_2, C_3 and C_4.

It suffices to take the following values of the components of tensor $t^{\nu}_{\lambda\mu}$:

$$(5.8)\qquad \begin{gathered} t^1_{11} = C_1, \quad t^2_{11} = C_2, \quad t^1_{12} = 1-C_4, \quad t^2_{12} = 1-C_1, \\ t^1_{22} = -C_4, \quad t^2_{21} = -C_1, \quad t^1_{22} = C_3, \quad t^2_{22} = C_4. \end{gathered}$$

By [4] it is clear that the general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in the allowable set $\mathfrak{M}_{\varkappa 1}$ can have one of the three forms

$$(5.9)\qquad f(t^{\nu}_{\lambda\mu}) = \varphi\left(\varkappa, \operatorname{sgn}\mathfrak{g}, \frac{\mathfrak{k}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2}, \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}}\right), \qquad \mathfrak{g} \neq 0,$$

$$(5.10)\qquad f(t^{\nu}_{\lambda\mu}) = \varphi(\varkappa, \tau), \qquad \mathfrak{g} = 0, \ \det\,[a_{\lambda\mu}] \neq 0,$$

$$(5.11)\qquad f(t^{\nu}_{\lambda\mu}) = \varphi(\varkappa, \eta), \qquad \mathfrak{g} = 0, \ \det\,[a_{\lambda\mu}] \neq 0,$$

where

$$(5.12)\qquad \mathfrak{g} \stackrel{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\varepsilon^{\lambda\varrho}\varepsilon^{\mu\sigma}\overset{1}{v}_{\varrho}\overset{1}{v}_{\sigma}\overset{1}{v}_{\nu},$$

$$(5.13)\qquad \mathfrak{k} \stackrel{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}t^{\lambda}_{\alpha\beta}t^{\mu}_{\gamma\delta}\varepsilon^{\alpha\varrho}\varepsilon^{\beta\sigma}\varepsilon^{\gamma\tau}\varepsilon^{\sigma\omega}\overset{1}{v}_{\varrho}\overset{1}{v}_{\sigma}\overset{1}{v}_{\tau}\overset{1}{v}_{\omega}\overset{1}{v}_{\nu},$$

$$(5.14)\qquad \mathfrak{w} \stackrel{\mathrm{df}}{=} -t^{\nu}_{\lambda\mu}t^{\lambda}_{\alpha\beta}t^{\mu}_{\gamma\delta}t^{\pi}_{\varkappa l}\varepsilon_{\nu\pi}\varepsilon^{\alpha\varrho}\varepsilon^{\beta\sigma}\varepsilon^{\gamma\tau}\varepsilon^{\delta\omega}\varepsilon^{\varkappa s}\varepsilon^{lp}\overset{1}{v}_{\varrho}\overset{1}{v}_{\sigma}\overset{1}{v}_{\tau}\overset{1}{v}_{\omega}\overset{1}{v}_{s}\overset{1}{v}_{p},$$

$$(5.15)\qquad a_{\lambda\mu} \stackrel{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\overset{1}{v}_{\nu},$$

$$(5.16)\qquad \tau \stackrel{\mathrm{df}}{=} \det\left[2t_{\lambda}{}^{[1}_{[1}t^{2]}_{2]}\cdot a^{\varrho\mu}\right],$$

$$(5.17)\qquad a^{\varrho\mu}\cdot a_{\mu\sigma} = \delta^{\varrho}_{\sigma},$$

and where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\lambda\mu}$ are Ricci symbols. Scalar η is a proportional coefficient of $a_{\lambda\mu}$ (in the case of $\det[a_{\lambda\mu}] = 0$) and $t^{\alpha}_{\lambda\alpha}t^{\beta}_{\mu\beta}$.

If $t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{\varkappa 1}$ and $\mathfrak{g} \neq 0$, then, according to theorem 2.1, in a two-dimensional space every set

$$(5.18)\qquad \mathfrak{M}_{\varkappa, \pm 1, C_1, C_2} = \left\{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{\varkappa 1}, \ \mathfrak{g} \neq 0, \ \operatorname{sgn}\,\mathfrak{g} = \pm 1, \ \frac{\mathfrak{k}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2} = C_1, \ \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}} = C_2\right\},$$

where $\varkappa$, C_1 and C_2 are arbitrary real numbers, is either the transitive fibre of tensor $t^{\nu}_{\lambda\mu}$ or is empty. Thus we can determine constants (for example $\operatorname{sgn}\mathfrak{g} = 1$, $C_2 < 0$) such that the set $\mathfrak{M}_{\varkappa, 1, C_1, C_2}$ will be empty.

Assuming that $C_2\cdot\operatorname{sgn}\mathfrak{g} > 0$, the system

$$(5.19)\qquad \operatorname{sgn}\mathfrak{g} = \pm 1, \qquad \frac{\mathfrak{k}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2} = C_1, \qquad \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}} = C_2$$

has a solution. One of solutions is

$$(5.20)\qquad \begin{gathered} t^1_{11} = C_1\operatorname{sgn}t^1_{22}, \quad t^2_{11} = C_2\operatorname{sgn}t^1_{22}\cdot|t^1_{22}|^{-1/2}, \quad t^1_{12} = 0, \\ t^2_{12} = 1-C_1\operatorname{sgn}t^1_{22}, \quad t^1_{21} = 0, \quad t^2_{21} = \varkappa - C_1\operatorname{sgn}t^1_{22}, \\ t^1_{22} \neq 0, \quad t^2_{22} = 0. \end{gathered}$$

If $t^{\nu}_{\lambda\mu} \epsilon \mathfrak{M}_{\varkappa 1}$, $\mathfrak{g} = 0$ and $\det[a_{\lambda\mu}] \neq 0$, then every set

$$\mathfrak{M}_{\varkappa\tau} = \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \epsilon \mathfrak{M}_{\varkappa 1},\ \mathfrak{g} = 0,\ \det[a_{\lambda\mu}] \neq 0,\ \det[2t_{\lambda}{}^{[1}_{[1}\, t^{2]}_{2]\mu} \cdot a^{\mu\nu}] = \tau\}, \tag{5.21}$$

where $\varkappa$ and τ are arbitrary real numbers, is a transitive fibre of tensor $t^{\nu}_{\lambda\mu}$ (in a two-dimensional space). It is clear that for arbitrary $\varkappa$ and τ the tensor $t^{\nu}_{\lambda\mu}$, whose components are

$$\begin{aligned} &t^1_{11} = 0, \quad t^2_{11} = -\tau, \quad t^1_{12} = 1, \quad t^2_{12} = 1,\\ &t^1_{21} = 1, \quad t^2_{21} = \varkappa, \quad t^1_{22} = 0, \quad t^2_{22} = -1, \end{aligned} \tag{5.22}$$

is an element of $\mathfrak{M}_{\varkappa\tau}$.

In the case $t^{\nu}_{\lambda\mu} \epsilon \mathfrak{M}_{\varkappa 1}$, $\mathfrak{g} = 0$ and $\det[a_{\lambda\mu}] = 0$, every set

$$\mathfrak{M}_{\varkappa\eta} = \{t^{\nu}_{\lambda\mu} : t^{\nu}_{\lambda\mu} \epsilon \mathfrak{M}_{\varkappa 1},\ \mathfrak{g} = 0,\ \det[a_{\lambda\mu}] = 0,\ a_{\lambda\mu} = \eta \cdot \overset{1}{v}_{\lambda} \cdot \overset{1}{v}_{\mu}\}, \tag{5.23}$$

where $\varkappa$ and η are arbitrary real numbers fulfilling condition $\varkappa \neq 3\eta - 1$, is a transitive fibre of tensor $t^{\nu}_{\lambda\mu}$ (in a two-dimensional space). One of the points of this family of transitive fibres is

$$\begin{aligned} &t^1_{11} = \eta, \quad t^2_{11} = 0, \quad t^1_{12} = 0, \quad t^2_{12} = 1 - \eta,\\ &t^1_{21} = 0, \quad t^2_{21} = \varkappa - \eta, \quad t^1_{22} = 0, \quad t^2_{22} = 0. \end{aligned} \tag{5.24}$$

For $\varkappa = 3\eta - 1$ we obtain two families of transitive fibres for arbitrary η: one for $t^2_{11} \overset{*}{=} 0$, and the other for $t^2_{11} \overset{*}{\neq} 0$ (calculated in coordinate systems in which $\overset{1}{v}_1 \overset{*}{=} 1$ and $\overset{1}{v}_2 \overset{*}{=} 0$) (see [4], p. 21).

Similarly, with the aid of theorem 2.1 and the forms of scalar concomitants of $t^{\nu}_{\lambda\mu} \epsilon \mathfrak{M}_{10}$ (see [4]), the set $\mathfrak{M}_{10}$ can be decomposed into transitive fibres.

Allowable set $\mathfrak{M}_{00}$ can be decomposed into five transitive fibres (see [4]).

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