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ON l_1-SINGULAR OPERATORS

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Using a result of Rosenthal's [9], Lohman [4] has shown that, for any Banach spaces X and Y, a continuous linear operator $T: X \to Y$ maps bounded sequences into sequences having a weak Cauchy subsequence (called a *weak Cauchy operator*) or there exists a subspace Z of X, isomorphic to l_1, for which $T|_Z$ is an isomorphism.

To show this, assume that $\{x_n\}$ is a bounded sequence in X such that $\{Tx_n\}$ has no weak Cauchy subsequence. Then there exists a basic subsequence $\{Tx_{n_k}\}$ equivalent to the unit vector basis of l_1. So there is a constant $M > 0$ such that for all m and for scalars $a_1, \ldots, a_m$ we have

$$M \sum_{k=1}^{m} |a_k| \leqslant \Big\| \sum_{k=1}^{m} a_k T x_{n_k} \Big\| \leqslant \|T\| \sup_n \|x_n\| \sum_{k=1}^{m} |a_k|.$$

If $Z = [x_{n_k}]$, then Z is isomorphic to l_1, and $T|_Z$ is an isomorphism.

The purpose of this note is to show how this result can be applied in both operator theory and dual space theory. Note that properties of the weak Cauchy operator can be found in [2].

All operators are to be continuous linear operators and X, Y are to denote Banach spaces. An operator $T: X \to Y$ is *l_1-singular* if it has no bounded inverse on a subspace isomorphic to l_1. Let $K(X)$ denote the weak* sequential closure of JX in X'' (J is the natural embedding map). Define $\hat{K}(X)$ by (the bar denotes weak* closure)

$$\hat{K}(X) = \bigcup_{F \subseteq X} \overline{JF},$$

where JF is weak* sequentially compact in X''. Note that $K(X) \subseteq \hat{K}(X)$.

THEOREM. *Let $T: X \to Y$ be a continuous linear operator. The following are equivalent*:

(a) *T is l_1-singular.*

(b) *T is weak Cauchy.*

(c) *$T''JS$ is weak* sequentially compact in Y'', where S is the unit disk in X.*

(d) $T''(X'') \subseteq \hat{K}(Y)$.

Proof. The implication (a) $\Rightarrow$ (b) follows from Lohman's result given at the beginning of the paper.

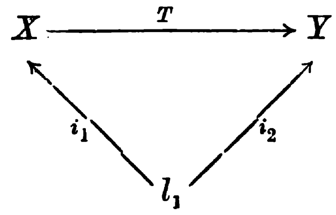
(b) $\Rightarrow$ (c). TS is such that every sequence has a weak Cauchy subsequence, hence $J(TS) = T''(JS)$ is weak* sequentially compact.

(c) $\Rightarrow$ (d). Let S'' be the unit disk of X''. Then

$$T''S'' = T''(\overline{JS}) \subseteq \overline{T''(JS)} = \overline{J(TS)}.$$

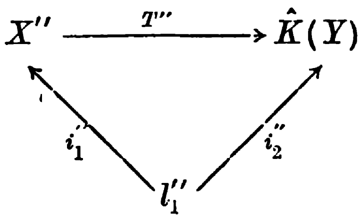
Since $J(TS)$ is weak* sequentially compact, $T''(X'') \subseteq \hat{K}(Y)$.

(d) $\Rightarrow$ (a). Assume that T is not l_1-singular. Then there exist isomorphic embeddings $i_1 : l_1 \to X$ and $i_2 : l_1 \to Y$ such that the diagram



is commutative.

Then the diagram



is commutative, where i_1'' and i_2'' are isomorphic embeddings. Now $\hat{K}(l_1) = l_1$ since l_1 is weakly complete. By the diagram above, we have

$$i_2''(l_1'') \subseteq \hat{K}(i_2(l_1)) = i_2(l_1).$$

This implies that l_1 has a non-separable subspace isomorphic to l_1''. This contradiction gives the desired result.

COROLLARY. *Y has no subspace isomorphic to l_1 if and only if $Y'' = \hat{K}(Y)$.*

Remark. If Y is separable, $\hat{K}(Y)$ can be replaced by $K(Y)$ (see [7]). An example for which $K(Y)$ is a proper subspace of $\hat{K}(Y)$ is $c_0(\Gamma)$ for Γ uncountable [6].

Let $L(X, Y)$ denote the set of all bounded operators from X to Y.

PROPOSITION. *The l_1-singular operators are closed in the uniform operator topology of $L(X, Y)$*

We prove first

LEMMA. *$\hat{K}(Y)$ is norm closed in Y''.*

Proof. Let A be any set in Y such that JA is weak* sequentially compact. For every $y'' \in \overline{JA}$, we can construct a sequence $\{G_n\}$ such that each $G_n \in K(Y)$, $\|G_n\| = \|y''\|$, and G_n converges weak* to y''. The method of construction (an inductive procedure) is the same as that given in [10].

Let F belong to the norm closure of $\hat{K}(Y)$ (see [5]). Then there is a sequence $\{F_n\}$ in $\hat{K}(Y)$ such that $F_n \to F$ in norm, and $\|F_n - F_{n-1}\| < 2^{-n}$ for each $n > 1$. If we let $F_0 = 0$, then by our construction above there exists, for each $n \geqslant 1$, a sequence G_{n_k} in $K(Y)$ such that $\|G_{n_k}\| = \|F_n - F_{n-1}\|$ for all k and such that $F_n - F_{n-1}$ is the weak* limit of $\{G_{n_k}\}$.

For each k the series $\sum_{n=1}^{\infty} G_{n_k}$ converges to an element $G_k \in K(Y)$ such that

$$\Big\| G_k - \sum_{n=1}^{j} G_{n_k} \Big\| < 2^{-j} \quad \text{for each } j.$$

Given $0 \neq f \in X^*$ and $\varepsilon > 0$, there exist positive integers J and K such that

$$2^{-j} < \frac{\varepsilon}{3\|f\|} \quad \text{and} \quad \Big| F_J(f) - \sum_{n=1}^{J} G_{n_k}(f) \Big| < \frac{\varepsilon}{3} \text{ for all } k \geqslant K.$$

Hence, for $k \geqslant K$,

$$|F(f) - G_k(f)| \leqslant |(F-F_J)(f)| + \Big| F_J(f) - \sum_{n=1}^{J} G_{n_k}(f) \Big| + \\ + \Big| \Big(\sum_{n=1}^{J} G_{n_k} - G_k \Big)(f) \Big| < \varepsilon,$$

so that F is the weak* limit of $\{G_k\}$. By the definition of $\hat{K}(Y)$, $F \in \hat{K}(Y)$.

The proof of the Proposition is the same as that for weakly compact operators (see [1], p. 483), except to use the Lemma.

Remark. An alternative proof of the Proposition and additional properties for l_1-singular operators can be found by using results of [3] From Proposition 3.3 of [8] we know that if T' is l_1-singular, then T is also l_1-singular. An example in [2] shows that the converse is not true.

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