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COMPOSITIONS OF CONFLUENT MAPPINGS AND SOME OTHER CLASSES OF FUNCTIONS

BY

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In this paper * we assume all spaces to be compact metric and all mappings to be surjective.

1. Preliminaries. Given a space X and a sequence $A_1, A_2, \ldots$ of subsets of X, we define $\operatorname*{Ls}_{n\to\infty} A_n$ to be the set of all points $x \in X$ for which there exist points $x_k \in X$ such that

$$\lim_{k\to\infty} x_k = x \quad \text{and} \quad x_k \in A_{n_k} \quad (k = 1, 2, \ldots),$$

where $n_1 < n_2 < \ldots$ is a sequence of positive integers. The following proposition can be obtained as an easy consequence of some well-known theorems (see [4], p. 57-61):

1.1. *A mapping f: $X \to Y$ is continuous if and only if*

$$\lim_{n\to\infty} y_n = y \textit{ implies } \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) \subset f^{-1}(y).$$

As usual, we say that a continuous mapping is *open* provided it transforms open sets into open sets.

1.2. *A mapping f: $X \to Y$ is open if and only if*

$$\lim_{n\to\infty} y_n = y \textit{ implies } \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) = f^{-1}(y).$$

Proof. If f is open, the implication holds (see [4], p. 67-68). Now, assume it holds and let U be an open subset of X. Suppose $x \in U$ and $f(x) = y \notin \operatorname{Int} f(U)$. Then there exist points $y_n \in Y \setminus f(U)$ such that $\lim_{n\to\infty} y_n = y$. Hence

$$f^{-1}(y_n) \subset f^{-1}[Y \setminus f(U)] = X \setminus f^{-1}f(U) \subset X \setminus U$$

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and $X \setminus U$ being closed, we conclude that

$$\operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) \subset X \setminus U \subset X \setminus \{x\},$$

whence

$$x \in f^{-1}(y) \setminus \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n),$$

which is impossible. Thus $y \in \operatorname{Int} f(U)$ and $f(U) \subset \operatorname{Int} f(U)$, i.e. the set $f(U)$ is open in Y. Moreover, by 1.1, f is continuous, and, therefore, it is open.

The next proposition states a well-known fact (see [9], p. 10):

1.3. *If $f\colon X \to Y$ is continuous, $y \in Y$ and $U \subset X$ is an open set such that $f^{-1}(y) \subset U$, then $y \in \operatorname{Int} f(U)$.*

Finally, let us yet recall that a continuous mapping is said to be *monotone* (*0-dimensional*) provided the inverses of points under it are connected (0-dimensional, respectively).

2. Quasi-interior mappings. We say that a continuous mapping $f\colon X \to Y$ is *quasi-interior at a point* $y \in Y$ provided, for each open set $U \subset X$ such that a component of $f^{-1}(y)$ is contained in U, we have $y \in \operatorname{Int} f(U)$. A continuous mapping $f\colon X \to Y$ is called *quasi-interior* provided f is quasi-interior at each point of Y (see [9], p. 9). One can use 1.3 to prove the following statement (ibidem):

2.1. *All open mappings and all monotone mappings are quasi-interior.*

An analogue of 1.2 is also possible (1):

2.2. *A continuous mapping $f\colon X \to Y$ is quasi-interior at $y \in Y$ if and only if $\lim_{n\to\infty} y_n = y$ implies that $\operatorname*{Ls}_{n\to\infty} f^{-1}(y_n)$ meets each component of $f^{-1}(y)$.*

Proof. If f is quasi-interior at y and C is a component of $f^{-1}(y)$, then, for each open neighborhood U of C in X, the set $f(U)$ is a neighborhood of y in Y, so that $f(U)$ contains some points y_n. Hence U intersects some sets $f^{-1}(y_n)$ and, taking points from these sets close enough to C, we get

$$C \cap \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) \neq \emptyset.$$

Assume now the condition is satisfied and U is an open subset of X containing a component C of $f^{-1}(y)$. If $y \notin \operatorname{Int} f(U)$, then there would exist points $y_n \in Y \setminus f(U)$ such that $\lim_{n\to\infty} y_n = y$. Hence

$$f^{-1}(y_n) \subset X \setminus U \quad \text{and} \quad \operatorname*{Ls}_{n\to\infty} f^{-1}(y_n) \subset X \setminus U \subset X \setminus C,$$

which is impossible. Thus $y \in \operatorname{Int} f(U)$.

(1) The condition which appears in 2.2 has been suggested to us by D. Zaremba who in [10] investigates some metric properties of quasi-interior mappings.

2.3. Theorem. *If $f: X \to Y$ and $g: Y \to Z$ are continuous mappings such that g is 0-dimensional and gf is quasi-interior, then g is open.*

Proof. Let $U \subset Y$ be open and let $y \in U$. Put $z = g(y)$ and take a component C of $f^{-1}(y)$. Then $C \subset f^{-1}g^{-1}(z)$ and let C' be the component of $f^{-1}g^{-1}(z)$ which contains C. We have $gf(C') = \{z\}$, whence $f(C') \subset g^{-1}(z)$. But since g is 0-dimensional, $f(C')$ is degenerate, and so $f(C') = \{y\}$. Thus $C' \subset f^{-1}(y) \subset f^{-1}g^{-1}(z)$ and C' must be a component of $f^{-1}(y)$. In other words, we have $C' = C$. Since

$$C' = C \subset f^{-1}(y) \subset f^{-1}(U),$$

where $f^{-1}(U)$ is open and gf is quasi-interior, we conclude that

$$z \in \operatorname{Int} gff^{-1}(U) = \operatorname{Int} g(U),$$

which implies that $g(U) \subset \operatorname{Int} g(U)$, i.e. the set $g(U)$ is open.

Combining a well-known factorization theorem (see [8], p. 141) with 2.3, we obtain the following corollary (cf. [9], p. 14):

2.4. Corollary. *If h is a quasi-interior mapping, then $h = gf$, where f is monotone and g is 0-dimensional and open.*

A continuous mapping $f: X \to Y$ is called *confluent* provided, for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$, we have $f(C) = K$ (see [1], p. 213). One knows that the following two propositions are true (ibidem):

2.5. *All open mappings and all monotone mappings are confluent.*

2.6. *If $f: X \to Y$ and $g: Y \to Z$ are confluent mappings, then gf is confluent.*

By virtue of 2.4, 2.5 and 2.6, we get our next corollary:

2.7. Corollary. *All quasi-interior mappings are confluent.*

The theorem which follows is suggested by 2.6. In certain particular cases, it can be derived from some results of Whyburn (see [9], p. 12).

2.8. Theorem. *If $f: X \to Y$ and $g: Y \to Z$ are quasi-interior mappings, then gf is quasi-interior.*

Proof. Let $z \in Z$ and let $U \subset X$ be an open set such that there exists a component C of $(gf)^{-1}(z)$ satisfying $C \subset U$. Then $f(C)$ is a continuum and $gf(C) = \{z\}$, whence $f(C) \subset g^{-1}(z)$. Let C' be the component of $g^{-1}(z)$ which contains $f(C)$. We have

$$C \subset f^{-1}f(C) \subset f^{-1}(C') \subset f^{-1}g^{-1}(z) = (gf)^{-1}(z)$$

and, consequently, C is a component of $f^{-1}(C')$. Since f is confluent according to 2.7, it follows that $f(C) = C'$. Given any point $y \in C'$, let $x \in C$ be a point such that $f(x) = y$ and let C_y be the component of $f^{-1}(y)$ which contains x. Thus

$$C \cap C_y \neq \emptyset \quad \text{and} \quad C_y \subset f^{-1}(y) \subset f^{-1}(C'),$$

whence $C_y \subset C \subset U$. Since f is quasi-interior, we obtain $y \in \operatorname{Int} f(U)$ for each $y \in C'$. This means that $C' \subset \operatorname{Int} f(U)$. Now, g being quasi-interior, we get

$$z \in \operatorname{Int} g[\operatorname{Int} f(U)] \subset \operatorname{Int} gf(U).$$

3. OM-mappings and MO-mappings. A continuous mapping h is said to be an *OM-mapping* (or an *MO-mapping*) provided there exist mappings f and g such that $h = gf$, where f is monotone and g is open (or f is open and g is monotone, respectively). Using 2.1, 2.4 and 2.8, we obtain the following three corollaries, the first of them being essentially due to Whyburn [9]:

3.1. COROLLARY. *If h is a continuous mapping, then the following three conditions are equivalent to each other:*

(i) *h is quasi-interior,*

(ii) *h is an OM-mapping,*

(iii) *h is representable as the composition $h = gf$ of two mappings such that f is monotone and g is 0-dimensional and open.*

3.2. COROLLARY. *All MO-mappings are OM-mappings.*

3.3. COROLLARY. *If $n_1, \ldots, n_k$ is a sequence of 0's and 1's, let us distinguish a class of continuous mappings f which are representable as compositions $f = f_1 \ldots f_k$ such that f_i is open or monotone depending on whether $n_i = 0$ or $n_i = 1$, respectively ($i = 1, \ldots, k$). Then each such a class is equal to one of the following four classes of mappings: monotone mappings, open mappings, MO-mappings, and OM-mappings.*

3.4. Example. *There exists a mapping h: $[0,1] \to [0,1]$ such that h is an OM-mapping and h is not an MO-mapping.*

Proof. We define h by the following formula:

$$h(x) = \begin{cases} 3x & \text{for } 0 \leqslant x \leqslant \frac{1}{3}, \\ 2-3x & \text{for } \frac{1}{3} \leqslant x \leqslant \frac{2}{3}, \\ 0 & \text{for } \frac{2}{3} \leqslant x \leqslant 1. \end{cases}$$

Then we can write $h = gf$, where f and g are defined by the formulae:

$$f(x) = \begin{cases} \frac{3}{2}x & \text{for } 0 \leqslant x \leqslant \frac{2}{3}, \\ 1 & \text{for } \frac{2}{3} \leqslant x \leqslant 1; \end{cases}$$

$$g(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2-2x & \text{for } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Clearly, f is monotone and g is open. Thus h is an OM-mapping; to prove that h is not an MO-mapping, let us suppose, on the contrary, that $h = g_0 f_0$, where f_0 is open and g_0 is monotone. As is well-known (see [4], p. 293, and [8], p. 147), open mappings transform arcs into arcs and end-points into end-points. Then $A = f_0([0,1])$ is an arc and $b = f_0(1)$ is an end-point of A. Let $U = \{x\colon 2/3 < x \leqslant 1\}$. Since f_0 is open, the set $f_0(U)$ is open in A, and the set $V = f_0^{-1}f_0(U)$ is open in $[0,1]$. The point $a = f_0(0)$ is also an end-point of A with $g_0(a) = h(0) = h(1) = g_0(b)$. Since g_0 is monotone, if a and b were different end-points of A, we would have $A \subset g_0^{-1}g_0(a)$, whence

$$h([0,1]) = g_0(A) = \{g_0(a)\} = \{h(0)\} = \{0\},$$

which is impossible. Consequently, we have $a = b$ and thus

$$0 \in f_0^{-1}(b) = f_0^{-1}f_0(1) \subset f_0^{-1}f_0(U) = V,$$

which means that V is a neighborhood of 0 in $[0,1]$ and

$$h(V) = g_0 f_0 f_0^{-1} f_0(U) = g_0 f_0(U) = h(U) = \{0\},$$

which is not the case, h being constant on no neighborhood of 0.

3.5. Example. *There exist MO-mappings $f, g\colon [0,1]\to[0,1]$ such that the composition gf is not an MO-mapping.*

Proof. Take f and g as in 3.4. Since f is monotone and g is open, both f and g are MO-mappings. But their composition h is not an MO-mapping.

3.6. Example. *There exists a mapping $f\colon X\to Y$ such that X and Y are arc-like continua, f is confluent and f is not an OM-mapping.*

Proof. Let X be the subset of the Euclidean plane defined by the formula

$$X = \{(x,1)\colon |x| \leqslant 1\} \cup \{(1,y)\colon |y| \leqslant 1\} \cup \left\{\left(x, \sin\frac{1}{x-1}\right)\colon 1 < x \leqslant 2\right\},$$

and let us consider an equivalence relation R in X which we define by the following formula:

$$R = \big\{\big((t,1),(1,t)\big)\colon |t| \leqslant 1\big\} \cup \big\{\big((1,t),(t,1)\big)\colon |t| \leqslant 1\big\} \cup \{(p,p)\colon p \in X\}.$$

It is not difficult to see that both the space X and the quotient space $Y = X/R$ are arc-like continua. We define f to be the natural projection of X onto Y. Because there are only two types of continua contained in Y, namely, arcs and copies of Y itself, a rather apparent argument can be used to show that f is confluent. On the other hand, the set $U = \{(x,1)\colon |x| < 1\}$ is open in X and the point $p = (0,1)$ belongs to U. The set $\{p\}$ being a component of $f^{-1}f(p)$ and the interior of $f(U)$ in Y being empty, we conclude that f is not quasi-interior at $f(p)$. Thus, by 3.1, f is not an OM-mapping.

Remark. It follows from 5.2 of this paper that the continuum Y, hence also the continuum X, as they appear in 3.6, cannot be made locally connected.

4. Weakly and locally confluent mappings. We say that a continuous mapping $f\colon X \to Y$ is *weakly confluent* (2) provided, for each continuum $K \subset Y$, there exists a component C of $f^{-1}(K)$ such that $f(C) = K$. We say that a continuous mapping $f\colon X \to Y$ is *locally confluent at a point* $y \in Y$ provided there exists a closed neighborhood V of y in Y such that $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V. A continuous mapping $f\colon X \to Y$ is called *locally confluent* provided f is locally confluent at each point of Y (see [3], p. 239).

4.1. Example. *There exists a mapping* $f\colon [0,1] \to [0,1]$ *such that* f *is weakly confluent and* f *is not locally confluent.*

Proof. The function f defined by the formula

$$f(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{3}{2} - x & \text{for } \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

obviously, is a weakly confluent mapping, but f is not locally confluent at $\frac{1}{2}$.

Remark. The weak confluency of the function from 4.1 is a special case of a more general phenomenon. It is known that all continuous mappings of continua onto arc-like continua are weakly confluent (see [6], Theorem 11).

4.2. Example. *There exists a mapping* $f\colon X \to Y$ *of an arc-like continuum* X *onto a triod-like continuum* Y *such that* f *is locally confluent and* f *is not weakly confluent.*

Proof. An example of this kind can be obtained by a modification of the example described in 3.6 above. Let X be the set

$$X = \{(x, 2)\colon\ -1 \leqslant x \leqslant 2\} \cup \left\{\left(\sin\frac{1}{y-2}, y\right)\colon\ 2 < y \leqslant 3\right\} \cup$$
$$\cup \{(2, y)\colon\ -1 \leqslant y \leqslant 2\} \cup \left\{\left(x, \sin\frac{1}{x-2}\right)\colon\ 2 < x \leqslant 3\right\},$$

and let R be the following equivalence relation:

$$R = \left\{\big((t, 2), (2, t)\big)\colon\ -1 \leqslant t \leqslant 2\right\} \cup$$
$$\cup \left\{\big((2, t), (t, 2)\big)\colon\ -1 \leqslant t \leqslant 2\right\} \cup \{(p, p)\colon\ p \in X\}.$$

Clearly, X is an arc-like continuum and $Y = X/R$ is a triod-like continuum with the projection f of X onto Y being locally confluent. Now, denoting by X_1 and X_2 the intersections of X with the strips of the plane determined by the inequalities $|x| \leqslant 1$ and $|y| \leqslant 1$, respectively,

(2) The class of weakly confluent mappings has been introduced by the first author in [5] where almost all results of this paper are announced. The second author, in his doctoral dissertation [6], establishes, among other things, most results which constitute Sections 4 and 5 here.

we see that the set $K = f(X_1 \cup X_2)$ is a continuum contained in Y. The components of $f^{-1}(K)$ are X_1 and X_2 but $f(X_i) \neq K$ for $i = 1, 2$. Thus f is not weakly confluent.

Remarks. A further modification leads to an example of a mapping f': $X' \to Y'$ such that X' and Y' are arc-like continua, f' is locally confluent and f' is not confluent. Namely, one can take X' to be the intersection of the continuum X from 4.2 with the half-plane $y \leqslant 2$, and put $Y' = f(X')$ and $f' = f|X'$. The mapping f', however, is weakly confluent. Another easy modification of 4.2 yields a locally confluent mapping which is not weakly confluent and transforms an arcwise connected continuum onto an arcwise connected continuum. We observe that the latter continuum cannot be made locally connected, by 4.3 and 5.2, or hereditarily arcwise connected, by 5.3.

Using 2.7, 3.1, 3.2, 3.4, 3.6, 4.1 and 4.2, we get this corollary:

4.3. Corollary. *The following implications hold for mappings and none of them can be reversed*:

$$\begin{tikzcd}
(\text{monotone}) \arrow[dr, Rightarrow] & & & & (\text{weakly confluent}) \\
& (\text{MO}) \arrow[r, Rightarrow] & (\text{OM}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\text{confluent}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
(\text{open}) \arrow[ur, Rightarrow] & & (\text{quasi-interior}) & & (\text{locally confluent})
\end{tikzcd}$$

Remark. Most of the above-mentioned seven classes of mappings are multiplicative, i.e. preserved by taking the compositions of their elements. More precisely, it follows from 2.6, 2.8 and 4.4 that all these classes are multiplicative except two – the class of MO-mappings and the class of locally confluent mappings – which are not multiplicative, by 3.5 and 4.5, respectively.

4.4. *If f: $X \to Y$ and g: $Y \to Z$ are weakly confluent mappings, then gf is weakly confluent.*

Proof. Let $K \subset Z$ be a continuum. There exists a component C' of $g^{-1}(K)$ such that $g(C') = K$. Again, there exists a component C of $f^{-1}(C')$ such that $f(C) = C'$. Thus $gf(C) = K$, and, obviously, C is a component of $(gf)^{-1}(K)$.

4.5. Example. *There exists a locally confluent mapping f: $X \to Y$, where X is an arc-like continuum, and a monotone mapping g: $Y \to Z$ such that the composition gf is neither weakly confluent nor locally confluent.*

Proof. Take X and f as defined in 4.2. Let $A = \{(x, 2)\colon |x| \leqslant 1\}$ and let us consider an equivalence relation R' in $Y = f(X)$ whose only non-degenerate equivalence class is the set $f(A)$. It is rather easy to see that the quotient space $Z = Y/R'$ and the natural projection g: $Y \to Z$

satisfy the requirements of 4.5. Specifically, the mapping gf is not weakly confluent on the inverse image of any neighborhood of the point $gf(A)$ in Z.

4.6. *If f: $X \to Y$ is a confluent mapping and g: $Y \to Z$ is a locally confluent mapping, then gf is locally confluent.*

Proof. Let $z \in Z$ be a point and let V be a closed neighborhood of z in Z such that $g_0 = g|g^{-1}(V)$ is confluent. Let $K \subset V$ be a continuum and let C be a component of $h^{-1}(K)$, where $h = (gf)|(gf)^{-1}(V)$. Then $f(C) \subset g_0^{-1}(K)$ and, denoting by C' the component of $g_0^{-1}(K)$ which contains $f(C)$, we get

$$C \subset f^{-1}f(C) \subset f^{-1}(C') \subset f^{-1}g_0^{-1}(K) = h^{-1}(K),$$

whence C is a component of $f^{-1}(C')$. Since f and g_0 are confluent, we conclude that

$$h(C) = gf(C) = g(C') = g_0(C') = K,$$

which means that h is confluent. Thus gf is locally confluent at z, and z being arbitrary, the proof of 4.6 is complete.

4.7. *If f: $X \to Y$ is a weakly confluent (locally confluent) mapping and $B \subset Y$ is a closed subset, then the mapping*

$$f|f^{-1}(B)\colon f^{-1}(B) \to B$$

is weakly confluent (locally confluent, respectively).

Proof. Write $f_0 = f|f^{-1}(B)$ and assume f is weakly confluent. Then, given a continuum $K \subset B$, there exists a component C of $f^{-1}(K)$ such that $f(C) = K$. But $f_0^{-1}(K) = f^{-1}(K)$ and $f_0(C) = f(C)$. Thus f_0 is weakly confluent.

Assume now that f is locally confluent and let $V \subset Y$ be a closed subset such that the mapping $g = f|f^{-1}(V)$ is confluent. Put $V_0 = B \cap V$ and $g_0 = f_0|f_0^{-1}(V_0)$. Given a continuum $K \subset V_0$ and a component C of $g_0^{-1}(K)$, we see that

$$C \subset f_0^{-1}(V_0) = f^{-1}(B) \cap f^{-1}(V_0) = f^{-1}(B \cap V_0) = f^{-1}(V_0) \subset f^{-1}(V),$$

whence $f(C) = g(C)$. On the other hand, we have

$$g_0^{-1}(K) = f_0^{-1}(V_0) \cap f_0^{-1}(K) = f_0^{-1}(K) = f^{-1}(K) = f^{-1}(V) \cap f^{-1}(K) = g^{-1}(K),$$

which implies $g(C) = K$ since g is confluent. Thus $g_0(C) = f(C) = K$, and the mapping g_0 is confluent. Consequently, the mapping f_0 is locally confluent.

5. Mappings onto some special spaces. Let us recall that a space X is said to be *connected im kleinen at a point $x \in X$* provided each neighborhood of x contains a connected closed neighborhood of x in X. A space is locally connected if and only if it is connected im kleinen at each of its points. Some theorems (see [1], p. 215, and [9], p. 14; cf. also [7], p. 140, and [8], p. 153) are known to insure that all confluent mappings of locally connected continua must be OM-mappings. In the theorem and the corollary which follow, the latter statement is to an extent localized and generalized.

5.1. THEOREM. *If $f\colon X\to Y$ is locally confluent at $y\in Y$ and Y is connected im kleinen at y, then f is quasi-interior at y.*

Proof. Suppose, on the contrary, that f is not quasi-interior at y, and let $U\subset X$ be an open set such that a component C of $f^{-1}(y)$ is contained in U and $y\notin \operatorname{Int} f(U)$. Furthermore, let V be a closed neighborhood of y in Y such that the mapping $g=f|f^{-1}(V)$ is confluent. Since Y is connected im kleinen at y, there exist continua $K_n\subset Y$ such that $K_{n+1}\subset K_n\subset V$ for $n=1,2,\ldots$ and

$$\bigcap_{n=1}^{\infty} K_n=\bigcap_{n=1}^{\infty}\operatorname{Int}(K_n)=\{y\},$$

whence $K_n\setminus f(U)\neq\emptyset$ for $n=1,2,\ldots$ We have $C\subset f^{-1}(y)\subset f^{-1}(K_n)$ and let us denote by C_n the component of $f^{-1}(K_n)$ which contains C. It follows from the inclusion $f^{-1}(K_{n+1})\subset f^{-1}(K_n)$ that $C_{n+1}\subset C_n$ for $n=1, 2,\ldots$ Consequently, the common part C' of the decreasing sequence of continua C_n is itself a continuum and $C\subset C'$. Moreover, we get

$$C'=\bigcap_{n=1}^{\infty} C_n\subset\bigcap_{n=1}^{\infty} f^{-1}(K_n)=f^{-1}\Big(\bigcap_{n=1}^{\infty} K_n\Big)=f^{-1}(y),$$

which yields $C'=C$. On the other hand, we have $f^{-1}(K_n)=g^{-1}(K_n)$, which implies $g(C_n)=K_n$ since g is confluent. Thus $f(C_n)=g(C_n)=K_n$, and

$$f(C_n\setminus U)\supset f(C_n)\setminus f(U)=K_n\setminus f(U)\neq\emptyset,$$

whence $C_n\setminus U\neq\emptyset$ for $n=1,2,\ldots$ The non-empty compact sets $C_n\setminus U$ also form a decreasing sequence so that their common part is non-empty; we conclude that

$$C\setminus U=C'\setminus U=\bigcap_{n=1}^{\infty}(C_n\setminus U)\neq\emptyset,$$

which contradicts the assumption that $C\subset U$, and the proof of 5.1 is complete.

5.2. COROLLARY. *All locally confluent mappings onto locally connected spaces are OM-mappings.*

We say that a space is *hereditarily arcwise connected* provided each of its subcontinua is arcwise connected.

5.3. *All locally confluent mappings onto hereditarily arcwise connected spaces are confluent.*

Proof. Let $f\colon X \to Y$ be such a mapping. Given a continuum $K \subset Y$ and a component C of $f^{-1}(K)$, let us select a point $x_0 \in C$. For each point $y \in K$, there exists an arc $A \subset K$ joining $f(x_0)$ and y. By 4.7, the mapping $f_0 = f|f^{-1}(A)$ is locally confluent. Hence, by 4.3 and 5.2, f_0 is confluent. Let C_0 be the component of $f_0^{-1}(A)$ which contains x_0. Then $C_0 \subset f^{-1}(A) \subset f^{-1}(K)$ and $f(C_0) = f_0(C_0) = A$. It follows that $C_0 \subset C$ and $y \in A \subset f(C)$. Thus $K \subset f(C)$, whence $f(C) = K$, and the proof of 5.3 is complete.

We say that a space is *hereditarily unicoherent* provided the common part of each two of its subcontinua is connected.

5.4. THEOREM. *If $f\colon X \to Y$ is a mapping, Y is hereditarily unicoherent and $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are continua such that the mappings $f|f^{-1}(Y_1)$ and $f|f^{-1}(Y_2)$ are confluent, then f is confluent.*

Proof. Write $f_i = f|f^{-1}(Y_i)$ $(i = 1, 2)$ and, given a continuum $K \subset Y$ and a component C of $f^{-1}(K)$, let us take a point $x_1 \in C$. Without loss of generality, we can assume that $f(x_1) \in Y_1$. Then $K \cap Y_1$ is a continuum containing $f_1(x_1)$ and since f_1 is confluent, the component C_1 of $f_1^{-1}(K \cap Y_1)$ containing x_1 satisfies $f_1(C_1) = K \cap Y_1$. We have $C_1 \subset f^{-1}(K)$, whence $C_1 \subset C$. If $K \subset Y_1$, we get

$$K = K \cap Y_1 = f_1(C_1) = f(C_1) \subset f(C),$$

which implies $f(C) = K$. If $K \setminus Y_1 \neq \emptyset$, there exists a point $y \in K \cap Y_1 \cap Y_2$. Let $x_2 \in C_1$ be a point such that $f_1(x_2) = y$. Then $K \cap Y_2$ is a continuum containing $y = f(x_2) = f_2(x_2)$ and since f_2 is confluent, the component C_2 of $f_2^{-1}(K \cap Y_2)$ containing x_2 satisfies $f_2(C_2) = K \cap Y_2$. We have $C_2 \subset C$, whence

$$K = (K \cap Y_1) \cup (K \cap Y_2) = f_1(C_1) \cup f_2(C_2) = f(C_1) \cup f(C_2) \subset f(C),$$

and so $f(C) = K$ again. Thus f is confluent.

5.5. COROLLARY. *If $f\colon X \to Y$ is a mapping, Y is hereditarily unicoherent and $Y = Y_1 \cup \ldots \cup Y_n$, where Y_i are continua such that the mappings $f|f^{-1}(Y_i)$ $(i = 1, \ldots, n)$ are confluent, then f is confluent.*

Remarks. Because of the existence of non-confluent locally confluent mappings onto hereditarily unicoherent continua, according to 4.2, the connectedness of the sets Y_i cannot be removed from 5.4 and 5.5. Also, the hereditary unicoherence of Y is an essential hypothesis in 5.4, by 5.6 of this paper. However, no condition of the connectedness or unicoherence is needed in a theorem being an analogue of 5.4 for OM-mappings (see [6], Theorem 5).

5.6. Example. *There exists a mapping $f\colon X \to Y = Y_1 \cup Y_2$, where X, Y, Y_1, Y_2 and $Y_1 \cap Y_2$ are hereditarily arcwise connected continua such* *that the mappings* $f|f^{-1}(Y_1)$ *and* $f|f^{-1}(Y_2)$ *are confluent and* f *is neither weakly confluent nor locally confluent.*

Proof. If p and q are points of the plane, we denote by $\overline{pq}$ the straight-line segment with end-points p and q. Let $p_0 = (1, 0)$, $q_0 = (0, 1)$ and $p_n = (1+n^{-1}, 0)$ for $n = 1, 2, \ldots$ We define the continuum M by the formula

$$M = \overline{p_0 q_0} \cup \bigcup_{n=1}^{\infty} \overline{p_n q_0},$$

and let φ be the symmetry of the plane defined by $\varphi[(x, y)] = (-x, y)$.

Let M_1 and M_2 be the continua

$$M_1 = M \cup \bigcup_{n=1}^{\infty} \overline{p_{2n-1} p_{2n}} \quad \text{and} \quad M_2 = M \cup \bigcup_{n=1}^{\infty} \overline{p_{2n} p_{2n+1}},$$

and let us put $X = M_1 \cup M_2 \cup \varphi(M_1 \cup M_2)$. Now, let R be the equivalence relation in X defined by the following formula:

$$R = \{(p, \varphi(p)) \colon p \in \overline{p_0 q_0}\} \cup \{(\varphi(p), p) \colon p \in \overline{p_0 q_0}\} \cup \{(p, p) \colon p \in X\}.$$

A rather routine argument shows that both X and the quotient space $Y = X/R$ are hereditarily arcwise connected continua. We define f to be the natural projection and observe that the continua $Y_i = f[M_i \cup \cup \varphi(M_i)]$ $(i = 1, 2)$ also are hereditarily arcwise connected as well as their intersection. Moreover, if a continuum $K \subset Y_i$ meets the limit arc $A = f(\overline{p_0 q_0})$ and K is not contained in A, then K contains the junction point $f(q_0)$ and, consequently, the set $f^{-1}(K)$ is a continuum. But the mapping $f|f^{-1}(A)$ is open and the mapping $f|f^{-1}(Y_i \setminus A)$ is a homeomorphism, which implies that $f|f^{-1}(Y_i)$ $(i = 1, 2)$ is confluent. The segments $\overline{p_0 p_n}$ and $\varphi(\overline{p_0 p_n})$ are components of the set $f^{-1}(K_n)$, where

$$K_n = f(\overline{p_0 p_n}) \cup f\varphi(\overline{p_0 p_n}) \qquad (n = 1, 2, \ldots),$$

and none of them is mapped onto K_n by f. Since K_n are arcs converging to the point $f(p_0)$, we see that f is not weakly confluent on the inverse image of any neighborhood of $f(p_0)$ in Y.

5.7. *A continuum* Y *is hereditarily indecomposable if and only if each continuous mapping of a continuum onto* Y *is confluent.*

Proof. If Y is hereditarily indecomposable, the condition from 5.7 is satisfied (see [2], p. 243). Assume it is satisfied and suppose, by the way of contradiction, that Y is not hereditarily indecomposable. This means there exist subcontinua K and L of Y such that the sets $K \setminus L$, $L \setminus K$ and $K \cap L$ are non-empty. Let $p_0 \in L \setminus K$ be a point and let us take a topological copy L' of L such that L' is disjoint with Y. Consider a homeomorphism $h: L \to L'$ and the space $Y \cup L'$ in which one pair of points, namely p_0 and $h(p_0)$, have been identified. Denote by X the resulting quotient space. Putting

$$f(x) = \begin{cases} x & \text{for } x \in Y, \\ h^{-1}(x) & \text{for } x \in L', \end{cases}$$

we define a continuous mapping $f: X \to Y$ of the continuum X onto Y. Let $q \in K \cap L$ and let C be the component of $f^{-1}(K)$ containing $h(q)$. Since $C \subset X \setminus f^{-1}(p_0)$, we get $C \subset L'$, whence

$$K \setminus f(C) = K \setminus h^{-1}(C) \supset K \setminus L \neq \emptyset,$$

which contradicts the assumption that f is confluent. Thus Y is hereditarily indecomposable.

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