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EXISTENCE OF SIDON SETS OF FIRST KIND IN LCA GROUPS

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Let G be a locally compact abelian group, Γ its dual, and Λ a subset of Γ. We shall apply a multiplicative notation for Γ. The following definitions have been introduced by Déchamps-Gondim.

DEFINITION 1 ([2] and [3]). Let $K \subset G$ and $C > 0$. A pair (K, C) is said to be *associated with* Λ if every function Φ on Λ, $|\Phi|_\infty \leqslant 1$, is the restriction to Λ of the Fourier–Stieltjes transform of a Radon measure μ on G with $\operatorname{supp}\mu \subset K$ and $\|\mu\| \leqslant C$.

DEFINITION 2 ([3]). Λ is called a *Sidon set of first kind* if there exists a constant $C > 0$ such that for every compact neighbourhood $K \subset G$ there exists a finite set $\Lambda_K \subset \Lambda$ for which the pair (K, C) is associated with $\Lambda \setminus \Lambda_K$.

Obviously, every infinite Sidon set (also a Sidon set of first kind) is unbounded (i.e., it is not contained in any compact set).

It is well known that every infinite set of an abelian discrete group contains an infinite Sidon set with constant 2 (see [5]). In this paper we will prove that if G is metrizable, then every unbounded subset of Γ contains an infinite Sidon set of first kind. This result is known for the group of real numbers ([6], Chapter VI, Proposition 1).

LEMMA. *Let P be an unbounded subset of Γ and $C > 2$. If the set $\{\gamma^2 \in \Gamma\colon \gamma \in P\}$ is unbounded, then for every compact neighbourhood K of G there exists an infinite set $\Lambda \subset P$ with the following property*:

every function Φ on Λ, $|\Phi|_\infty \leqslant 1$, is the restriction to Λ of the Fourier–Stieltjes transform of a positive Radon measure μ on G such that $\operatorname{supp}\mu \subset K$ *and* $\|\mu\| \leqslant C$.

Proof. Let K be a compact neighbourhood in G and $\varepsilon > 0$. Put

$$f = \frac{1}{|K|}\chi_K,$$

where $|\cdot|$ denotes the Haar measure and χ_K is the characteristic function of K.

Since $\hat{f} \in C_0(\Gamma)$, the sets

$$K_n = \left\{\gamma \in \Gamma\colon\ |\hat{f}(\gamma)| \geq \frac{\varepsilon}{2\cdot 6^n}\right\}$$

are compact ($n = 1, 2, \ldots$). We shall define $\{\gamma_n\}_{n=1}^{\infty} \subset P$ by induction in the following way:

(i) $\gamma_1,\ \gamma_1^2 \notin K_1$;

(ii) $\gamma_{n+1},\ \gamma_{n+1}^2 \notin K_{n+1}\cdot E_n^2$, where $E_n = \prod_{i=1}^{n} \{\gamma_i, \gamma_i^{-1}, e\}$.

Since f is positive, K_n are symmetric, and therefore $\gamma_{n+1}^{-1} \notin K_{n+1}\cdot E_n^2$. We shall show that $\Lambda = \{\gamma_n\}_{n=1}^{\infty}$ has the desired property. Let Φ be a hermitian function on $\Lambda \cup \Lambda^{-1}$ (i.e., $\overline{\Phi(\lambda)} = \Phi(\lambda^{-1})$ for $\lambda \in \Lambda$), $|\Phi|_\infty \leq \frac{1}{2}$, and $\Lambda_n = \{\gamma_1, \ldots, \gamma_n\}$. By (i) and (ii), Λ_n is asymmetric and has no element of rank 2. We form the Riesz product

$$R_n(x) = \prod_{\gamma\in\Lambda_n} \left(1 + \Phi(\gamma)\gamma(x) + \overline{\Phi(\gamma)\gamma(x)}\right)$$

and set $g_n = R_n \cdot f$. Since

$$R_n(x) = \sum_{F\subset \Lambda_n\cup\Lambda_n^{-1}}^{\mathrm{as}} \prod_{\lambda\in F} \Phi(\lambda)\lambda(x),$$

where the summation is taken over asymmetric sets, we have

$$\hat{g}_n(\gamma) = (R_n\cdot f)\hat{\ }(\gamma) = \sum_{F\subset \Lambda_n\cup\Lambda_n^{-1}}^{\mathrm{as}} \prod_{\lambda\in F} \Phi(\lambda)\hat{f}\Big(\gamma\big(\prod_{\lambda\in F}\lambda\big)^{-1}\Big)$$

(for $F = \emptyset$ we set $\prod_{\lambda\in\emptyset}\Phi(\lambda) = 1$ and $\prod_{\lambda\in\emptyset}\lambda = e$). We shall prove the following inequalities:

(a) $$|1-\hat{g}_{n+1}(e)| \leq |1-\hat{g}_n(e)| + \frac{\varepsilon}{6}\left(\frac{1}{2}\right)^n \ (n \geq 1), \qquad |1-\hat{g}_1(e)| \leq \frac{\varepsilon}{6};$$

(b) $$|\Phi(\gamma_{n+1}) - \hat{g}_{n+1}(\gamma_{n+1})| \leq |1-\hat{g}_n(e)| + \frac{\varepsilon}{6}\left(\frac{1}{2}\right)^n \quad (n) \geq 1,$$

$$|\Phi(\gamma_1) - \hat{g}_1(\gamma_1)| \leq \frac{\varepsilon}{6};$$

(c) $$\max\{|\Phi(\gamma) - \hat{g}_{n+1}(\gamma)|\colon\ \gamma\in\Lambda_n\}$$
$$\leq \max\{|\Phi(\gamma) - \hat{g}_n(\gamma)|\colon\ \gamma\in\Lambda_n\} + \frac{\varepsilon}{6}\left(\frac{1}{2}\right)^n \quad (n \geq 1).$$

(a) We compute

$$|1-\hat{g}_{n+1}(e)| = \Big|1- \sum^{\text{as}}_{F\subset \Lambda_{n+1}\cup\Lambda_{n+1}^{-1}} \prod_{\lambda\in F}\Phi(\lambda)\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\big)\Big|$$

$$\leqslant |1-\hat{g}_n(e)| + \sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \big|\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1}^{-1}\big)\big|$$

$$+ \sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \big|\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1}\big)\big|$$

$$\leqslant \frac{\varepsilon}{6}\Big(\frac{1}{2}\Big)^n + |1-\hat{g}_n(e)|,$$

where the last inequality follows from (ii) and the fact that there are at most 3^n asymmetric subsets of $\Lambda_n\cup\Lambda_n^{-1}$. By (i) we have

$$|1-\hat{g}_1(e)| = |1-\Phi(\gamma_1)\hat{f}(\gamma_1^{-1})-\Phi(\gamma_1^{-1})\hat{f}(\gamma_1)-\hat{f}(e)|$$

$$\leqslant |\hat{f}(\gamma_1^{-1})|+|\hat{f}(\gamma_1)| \leqslant \frac{\varepsilon}{6}.$$

(b) To prove the first inequality we write

$$|\Phi(\gamma_{n+1})-\hat{g}_{n+1}(\gamma_{n+1})|$$

$$= \Big|\Phi(\gamma_{n+1}) - \sum^{\text{as}}_{F\subset \Lambda_{n+1}\cup\Lambda_{n+1}^{-1}} \prod_{\lambda\in F}\Phi(\lambda)\hat{f}\big(\gamma_{n+1}\big(\prod_{\lambda\in F}\lambda^{-1}\big)\big)\Big|$$

$$\leqslant \Big|\Phi(\gamma_{n+1})-\Phi(\gamma_{n+1})\sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \prod_{\lambda\in F}\Phi(\lambda)\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\big)\Big|$$

$$+ \sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \big|\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1}\big)\big| + \sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \big|\hat{f}\big(\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1}^2\big)\big|$$

$$\leqslant |1-\hat{g}_n(e)| + \frac{\varepsilon}{6}\Big(\frac{1}{2}\Big)^n.$$

The second inequality is obtained as follows:

$$|\Phi(\gamma_1)-\hat{g}_1(\gamma_1)| = |\Phi(\gamma_1)-\Phi(\gamma_1)\hat{f}(\gamma_1\gamma_1^{-1})-\Phi(\gamma_1^{-1})\hat{f}(\gamma_1^2)-\hat{f}(\gamma_1)|$$

$$\leqslant |\hat{f}(\gamma_1^2)|+|\hat{f}(\gamma_1)| \leqslant \frac{\varepsilon}{6}.$$

(c) Analogously we find

$$\max\{|\Phi(\gamma)-\hat{g}_{n+1}(\gamma)|:\ \gamma\in\Lambda_n\}$$

$$= \max\Big\{\Big|\Phi(\gamma) - \sum^{\text{as}}_{F\subset \Lambda_{n+1}\cup\Lambda_{n+1}^{-1}} \prod_{\lambda\in F}\Phi(\lambda)\hat{f}\big(\gamma\prod_{\lambda\in F}\lambda^{-1}\big)\Big|:\ \gamma\in\Lambda_n\Big\}$$

$$\leqslant \max\Big\{\Big|\Phi(\gamma) - \sum^{\text{as}}_{F\subset \Lambda_n\cup\Lambda_n^{-1}} \prod_{\lambda\in F}\Phi(\lambda)\hat{f}\big(\gamma\prod_{\lambda\in F}\lambda^{-1}\big)\Big|:\ \gamma\in\Lambda_n\Big\}$$

$$+\max\Big\{\sum^{\mathrm{as}}_{F\subset\Lambda_n\cup\Lambda_n^{-1}}|\hat{f}(\gamma\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1})|:\ \gamma\in\Lambda_n\Big\}$$

$$+\max\Big\{\sum^{\mathrm{as}}_{F\subset\Lambda_n\cup\Lambda_n^{-1}}|\hat{f}(\gamma\prod_{\lambda\in F}\lambda^{-1}\gamma_{n+1}^{-1})|:\ \gamma\in\Lambda_n\Big\}$$

$$\leqslant \max\{|\Phi(\gamma)-\hat{g}_n(\gamma)|:\ \gamma\in\Lambda_n\}+\frac{\varepsilon}{6}\left(\frac{1}{2}\right)^n.$$

From conditions (a)–(c) we infer that

$$\|g_n\|_1=|\hat{g}_n(e)|\leqslant 1+|1-\hat{g}_n(e)|\leqslant 1+\frac{\varepsilon}{6}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=1+\frac{\varepsilon}{3},$$

$$\max\{|\Phi(\gamma)-\hat{g}_n(\gamma)|:\ \gamma\in\Lambda_n\}\leqslant\frac{\varepsilon}{6}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\varepsilon}{3}.$$

Let g be a *-weak accumulation point of the measures $g_n\in L^1_+(G)$. Then $g\in M_+(G)$, $\operatorname{supp} g\subset K$, $\|g\|\leqslant 1+\varepsilon/3$, and $|\hat{g}(\gamma)-\Phi(\gamma)|\leqslant\varepsilon/3$ for $\gamma\in\Lambda$. Thus for any $\psi\in L^\infty(\Lambda)$ there exists a positive measure μ such that

$$|\hat{\mu}-\psi|_{L^\infty(\Lambda)}\leqslant\frac{2\varepsilon}{3}|\psi|_{L^\infty(\Lambda)},\qquad \|\mu\|\leqslant 2\left(1+\frac{\varepsilon}{3}\right)|\psi|_\infty,\qquad\text{and}\qquad \operatorname{supp}\mu\subset K.$$

We apply the foregoing inductively to $\psi_1=\psi$, $\psi_2=\psi_1-\hat{\mu}_1$, etc. to obtain μ_1, μ_2, etc. in $M_+(G)$ such that

$$\Big|\sum_{k=1}^{n}\hat{\mu}_k-\psi\Big|_\infty=|\hat{\mu}_n-\psi_n|_\infty\leqslant\left(\frac{2\varepsilon}{3}\right)^n|\psi|_\infty,$$

$$\Big\|\sum_{k=1}^{n}\mu_k\Big\|\leqslant\sum_{k=1}^{n}\|\mu_k\|\leqslant 2\left(1+\frac{\varepsilon}{3}\right)|\psi|_\infty\sum_{k=1}^{n}\left(\frac{2\varepsilon}{3}\right)^{k-1}.$$

The *-weak sum $\mu=\sum_{k=1}^{\infty}\mu_k$ has the desired properties.

THEOREM. *Let G be metrizable and $C>2$. Then every unbounded set $P\subset\Gamma$ contains an infinite Sidon set of first kind with constant C.*

Proof. Since G is metrizable, there exists a basis $\{K_n\}_{n\geqslant 1}$ of compact neighbourhoods of the neutral element in G. Let us fix $n\geqslant 1$. First we will find an infinite set $\Lambda_n\subset P$ such that (K_n, C) is associated with Λ_n.

Suppose that $P\subset H$, where H is a subgroup of Γ generated by a compact neighbourhood. Hence $H\simeq \boldsymbol{R}^m\times\boldsymbol{Z}^n\times F$, where F is a compact group ([4], Theorem (9.8)). Since P is not contained in any compact set, the same holds for the projection of P on $\boldsymbol{R}^m$ or for that on $\boldsymbol{Z}^n$. Hence $\{\gamma^2\in\Gamma:\ \gamma\in P\}$ is not contained in any compact set and by the Lemma there exists an infinite set $\Lambda_n\subset P$ such that (K_n, C) is associated with Λ_n.

Let us consider the second case: P is not contained in any subgroup

$H \in \Delta$, where Δ denotes the family of all subgroups of Γ generated by a compact neighbourhood. For every $H \in \Delta$, $H^{\perp}$ (the annihilator of H) is a compact subgroup of G and

$$\bigcap_{H \in \Delta} H^{\perp} = e.$$

Thus there exists $H_0 \in \Delta$ such that $H_0^{\perp} \subset K_n$. Let

$$P_0 = \{A \in \Gamma/H_0 \colon A \cap P \neq \emptyset\}.$$

Then P_0 is an infinite subset of the discrete group Γ/H_0. There exists an infinite set $S_0 \subset P_0$ such that $(\Gamma/H_0, 2)$ is associated with S_0 (see [5]). Let $\Lambda_n \subset P$ be such that $\operatorname{card}(\Lambda_n \cap A) = 1$ for $A \in S_0$ and $\Lambda_n \cap A = \emptyset$ for $A \notin S_0$. Since $H_0^{\perp} \subset K_n$ and for any $\mu \in M(H_0^{\perp})$ the transform $\hat{\mu}$ is constant on cosets of H_0, the set Λ_n is associated with (K_n, C).

Thus we obtain a sequence of sets $\{\Lambda_n\}_{n \geq 1}$ associated with (K_n, C) ($n = 1, 2, \ldots$). Without loss of generality we can assume that $\Lambda_{n+1} \subset \Lambda_n$ for $n \geq 1$. Let $S = \{\lambda_n\}_{n \geq 1}$, where $\lambda_n \in \Lambda_n$, $\lambda_i \neq \lambda_j$ for $i \neq j$. It is obvious that S is an infinite Sidon set of first kind with constant C.

Remark. For discrete groups this theorem is an immediate consequence of the following result of Bourgain: every Sidon set tending to infinity is of first kind ([1], Corollary 2).

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