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THE EMBEDDING OF A LINEAR DISCRETE FLOW IN A CONTINUOUS FLOW

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1. Introduction. Let X be a topological space and let R denote the reals. A *continuous flow* is a mapping $F(x, r)$ of $X \times R$ onto X such that $F(x, r)$ is continuous; for each $r \in R$, $F(x, r)$ is a homeomorphism of X onto X and such that for all $x \in X$ and all $r_1, r_2 \in R$, $F(F(x, r_1), r_2) = F(x, r_1 + r_2)$. A *discrete flow* is a homeomorphism $T(X) = X$ and its integral iterates.

The restricted embedding problem for continuous flows is that of finding a continuous flow $F(x, r)$ on X corresponding to a given X and homeomorphism $T(X) = X$ such that $F(x, 1) \equiv T(x)$ on X. In Section 2 of this paper we give the history of this problem.

Fine and Schweigert (jointly) [2] and Fort [5] solved the restricted embedding problem on a connected subset of the line in 1955 (the results of Fine and Schweigert seem to have been announced in 1950). They have shown that if T is an order preserving homeomorphism of an interval onto itself, then it is possible to embed T in a continuous flow on the same space. It is easily seen that the condition of order preservation is necessary. From this theorem one can easily secure the same theorem for an arc. In Section 3 we outline an elementary constructive proof of the embedding theorem for the line and in Sections 4 and 5, the details of this proof are given.

2. The unrestricted embedding problem, that is, the embedding problem that permits an enlargement of the space X is easily solved [4], [6], [7] for any topological space X and self-homeomorphism T. The restricted problem, with which this paper is concerned, is only partially solved. Hereafter, the results described are for the restricted problem.

In addition to the theorem cited above, Fort [5] has shown that, if T is a homeomorphism of a half open interval $(a, b]$ onto itself, T has a continuous derivative on $(a, b]$, $T(x) > x$ for $a < x < b$, $T'(x) > 0$ for $a < x \leqslant b$ and T' is monotone non-increasing on $(a, b]$, then there exists a unique flow $F(x, r)$ on $(a, b]$ such that $F(x, 1) = T$ and such that $F(x, r)$ is continuously differentiable on $(a, b]$.

Although he did not state his conclusions in this setting, Hadamard [8], in reporting results of E. Jabotinsky, gave another approach to the embedding problem on an interval.

Foland and the author [4] have shown that if T is an orientation preserving self-homeomorphism of a circle X, then it is possible to embed T in a continuous flow on the circle if, and only if, either (a) X contains a fixed point under T, or (b) T is periodic on X, or (c) T is transitive on X.

Foland [3] has shown that if T is an almost periodic, orientation preserving self-homeomorphism of a closed 2-cell, then T can be embedded in a continuous flow on the 2-cell. Moreover, if T is almost periodic but not periodic, then the embedding is unique.

Andrea [1] has given sufficient conditions that a fixed-point free homeomorphism of the Euclidean plane can be embedded in a continuous flow.

3. Results on iteration, of which there are many, often come close to the embedding problem (which might be thought of as the problem of continuous roots). In particular, the work of Michel [9], [10], is relevant.

We will give an elementary and constructive proof of the embedding theorem on a line with the help of a construction of Ward [11] for the iteration of certain real functions. The real functions considered by Ward are defined only on a ray $a \leqslant c < \infty$, they are not to have a fixed point, their graphs lie above the graph of $y = x$ and Ward only considers the positive iterates. It is because of these restrictions that the results of Ward must be modified to apply in the present paper.

In this section we outline the proof of the theorem. The longer details are reserved for later sections.

Let X be the Euclidean line and let $T(X) = X$ be an order preserving homeomorphism. Let M denote the fixed points, if any, of T on X. Since M is a closed set, $X - M$ consists of combinations of one or more of the following: the entire space X (in case M is null), bounded open intervals, and open rays $a < x < \infty$, $-\infty < x < a$.

The homeomorphism T is strictly monotone increasing and leaves M and each of its complementary sets invariant. The continuous flow in which T is to be embedded will be fixed at each point of M. That is, if $x \in M$, $F(x, r) = x$ for all $r \in R$.

LEMMA 1. *If $E(X) = X$ is an order preserving homeomorphism of* (A) *the Euclidean line,* (B) *a closed interval of* (C) *a closed ray $a \leqslant x < \infty$ or $-\infty < x \leqslant a$ onto itself, then E can be embedded in a continuous flow on X.*

The constructive proof of the three parts of this lemma will be deferred to later sections.

LEMMA 2. *If $F(x, r)$ is a continuous flow on X and $H(X) = Y$ is a homeomorphism, then $HF(H^{-1}(y), r)$ is a continuous flow on Y.*

Proof. Let $G(y, r) = HF(H^{-1}(y), r)$. Clearly G is continuous on $Y \times R$ and for each $r \in R$, $G(y, r)$ is a homeomorphism of Y onto Y. Consider $r_1, r_2 \in R$. Then

$$G(G(y, r_1), r_2) = HF\big(H^{-1}(G(y, r_1)), r_2\big) = HF\big(H^{-1}HF(H^{-1}(y), r_1), r_2\big)$$
$$= HF(H^{-1}(y), r_1 + r_2) = G(y, r_1 + r_2),$$

and the lemma is proved.

If, as in case (B) of Lemma 1, T is an order preserving homeomorphism of an interval $[a, b]$ onto itself, then one may use Lemma 2 to reduce the proof of (B) to the case X is the interval $[-1, 1]$. Similarly, part (C) of Lemma 1 may be restricted to rays $[-1, \infty)$ and $(-\infty, 1]$.

Once Lemma 1 is proved we have a constructive proof of the following theorem of Fort, Fine and Schweigert:

THEOREM 1. *If T is an order preserving homeomorphism of the Euclidean line onto itself, then T can be embedded in a continuous flow on the Euclidean line.*

The required flow is secured by patching together the flows on $X - M$ guaranteed by Lemma 1.

The next two sections are devoted to a proof of Lemma 1.

4. Proof of case (A). We assume that X is the entire line, that $E(X) = X$ is an order preserving homeomorphism, hence is strictly monotone increasing, and that E has no fixed points on X.

Either $E(x) > x$ for all $x \in X$ or $E(x) < x$ for all $x \in X$. We first consider the case $E(x) > x$ for all $x \in X$. Since E is an onto mapping,

$$\lim_{x \to -\infty} E(x) = -\infty,$$

and since $E(x) > x$,

$$\lim_{x \to \infty} E(x) = \infty.$$

It is clear that E has an inverse, E_{-1}, which is a monotone increasing homeomorphism of X onto X. In general, for each positive integer n, let E_n denote the n iterations of E (E_0 will denote the identity). Then E_{-n}, n iterations of E_{-1}, is the inverse of E_n. Thus E_i is defined for all integers i and $E_i(E_j(x)) = E_{i+j}(x)$ for all integers i, j. Since $E(x) > x$ for all $x \in X$, $E_i(x) > x$ if $i \geqslant 1$, $E_i(x) < x$ if $i \leqslant -1$.

To embed E in a continuous flow we will construct a function $\varphi(x)$ such that φ is a monotone increasing homeomorphism of X onto X such that

$$\varphi(x+1) = E(\varphi(x)) \tag{1}$$

for all $x \in X$. If φ satisfies (1), then it is easily seen that

$$\varphi(x+n) = E_n(\varphi(x))$$

for any integer n and any $x \in X$. If φ^{-1} denotes the inverse of φ, then

$$E_n(x) = \varphi(\varphi^{-1}(x)+n) \tag{2}$$

for any $x \in X$ and any integer n.

The continuous flow is now defined as

$$F(x, r) = \varphi(\varphi^{-1}(x)+r) \tag{3}$$

for all $x \in X, r \in R$. Clearly the function $F(x, r)$ is continuous on $X \times R$ and for each $r \in R$, $F(x, r)$ is a homeomorphism.

$$\begin{aligned} F(F(x, r_1), r_2) &= \varphi\big(\varphi^{-1}(F(x, r_1))+r_2\big) \\ &= \varphi\big(\varphi^{-1}\varphi(\varphi^{-1}(x)+r_1)+r_2\big) \\ &= \varphi(\varphi^{-1}(x)+r_1+r_2) \\ &= F(x, r_1+r_2) \end{aligned}$$

to verify that (3) defines a flow on X. Since $F(x, 1) = E_1 = E$, by (2), the flow is an embedding of E. It only remains to define φ.

Let $E(0) = a$ and let $[y]$ denote, as usual, the largest integer not greater than the real number y. Since $a > 0$,

$$0 = E_0(0) \leqslant a(x-[x]) < E_1(0) = a.$$

Define

$$\varphi(x) = E_{[x]}(a(x-[x])).$$

(i) $\varphi(x)$ *satisfies* (1). For consider,

$$\varphi(x+1) = E_{[x]+1}(a(x-[x])) = E\big(E_{[x]}(a(x-[x]))\big) = E(\varphi(x)).$$

(ii) $\varphi(x)$ *is monotone increasing*. Consider $x' > x$. We will show that $\varphi(x') > \varphi(x)$.

Consider the case $x' \geqslant [x]+1$.

$$\varphi(x') = E_{[x']}(a(x'-[x'])) \geqslant E_{[x']}(0),$$

since $a(x'-[x']) \geqslant 0$ and $E_{[x']}$ is monotone.

$$E_{[x']}(0) = E_{[x']-1}(a) \geqslant E_{[x]}(a),$$

since $[x']-1 \geqslant [x]$ and since $E_n(x)$ is monotone in n.

$$E_{[x]}(a) > E_{[x]}\big(a(x-[x])\big) = \varphi(x)$$

because $a(x-[x]) < a$. Thus $\varphi(x') > \varphi(x)$.

Now, consider the case $[x'] = [x]$. In this case

$$\varphi(x') = E_{[x]}\big(a(x'-[x'])\big) = E_{[x]}\big(a(x'-[x])\big) > E_{[x]}\big(a(x-[x])\big) = \varphi(x).$$

(iii) *$\varphi(x)$ is continuous.* Consider any $x \in X$. If x is not an integer, let $[x] = n$. Then $\varphi(x) = E_n\big(a(x-n)\big)$ near x. Thus φ is continuous for any non-integral x.

Now, suppose $x = n$, an integer. Consider any $\varepsilon > 0$.

$$\lim_{\varepsilon\to 0}\varphi(n+\varepsilon) = \lim_{\varepsilon\to 0}E_n(a\cdot\varepsilon) = E_n(0) = \varphi(n)$$

and

$$\lim_{\varepsilon\to 0}\varphi(n-\varepsilon) = \lim_{\varepsilon\to 0}E_{n-1}\big(a(1-\varepsilon)\big) = E_{n-1}(a) = \varphi(n).$$

Thus, $\varphi(x)$ is continuous for all $x \in X$.

(iv) *φ has the entire space X as its range.* We have seen that φ is monotone increasing and continuous. With (iv) we will know that φ is a homeomorphism of X onto X. As has been remarked, E_n is monotone in n for any fixed $x = x^*$. This follows from the fact that $E(x^*) > x^*$. Further, since $E_n(x^*) > E_{n-1}(x^*)$ it follows that

$$\lim_{n\to\infty}E_n(x^*) = \infty.$$

To see this, consider the monotone increasing sequence $\{E_n(x^*)\}$. If $E_n(x^*) \to p \neq \infty$, then $E_{n+1}(x^*) \to E(p)$ hence $E(p) = p$ contrary to $E(p) > p$. Similarly, one can show that

$$\lim_{n\to-\infty}E_n(x^*) = -\infty.$$

Returning to the proof of (iv), we have $\{\varphi(n)\} = \{E_n(0)\}$. But $E_n(0) \to \pm\infty$ as $n \to \pm\infty$, respectively, hence φ is unbounded above and below to complete the proof of (iv).

Since φ is continuous and strictly monotone increasing and since the entire real axis is its range, φ has an inverse, φ^{-1}, defined for all real x.

(v) *We exhibit $\varphi^{-1}(x)$ and verify that for all real x, $\varphi^{-1}\big(\varphi(x)\big) = \varphi\big(\varphi^{-1}(x)\big)$* $= x$. Since $E_k(0) \to \infty$ as $k \to \infty$ and $E_k(0) \to -\infty$ as $k \to -\infty$, there is a unique integer k such that

$$E_k(0) \leqslant x < E_{k+1}(0) = E_k(a). \tag{4}$$

Then

$$\varphi^{-1}(x) = \frac{1}{a}E_{-k}(x)+k. \tag{5}$$

Observe that from (4) one has

$$0 \leqslant E_{-k}(x) < a.$$

Thus, from (5), one has $k = [\varphi^{-1}(x)]$. Since the inverse of φ is unique, if we verify that φ^{-1} satisfies $\varphi^{-1}(\varphi(x)) = \varphi(\varphi^{-1}(x)) = x$ for all real x we will have shown that φ^{-1} of (5) is the inverse of φ. Using (5) and the definition of φ, we have

$$\varphi(\varphi^{-1}(x)) = E_{[\varphi^{-1}(x)]}\Big(a\big(\varphi^{-1}(x) - [\varphi^{-1}(x)]\big)\Big).$$

Since $k = [\varphi^{-1}(x)]$, this is

$$E_k\left(a\left(\frac{1}{a}E_{-k}(x)+k-k\right)\right)$$

which simplifies to $E_k(E_{-k}(x)) = x$. Thus $\varphi(\varphi^{-1}(x)) = x$ for all real x.

From the definition of $\varphi(x)$ one has

$$E_{[x]}(0) \leqslant \varphi(x) < E_{[x]}(a).$$

Then

$$\varphi^{-1}(\varphi(x)) = \frac{1}{a}E_{-[x]}(\varphi(x))+[x] = \frac{1}{a}E_{-[x]}\Big(E_{[x]}\big(a(x-[x])\big)\Big)+[x]$$
$$= \frac{1}{a}a(x-[x])+[x] = x$$

to show that $\varphi^{-1}(\varphi(x)) = x$.

It is now possible to give an explicit formulation for the continuous flow:

$$F(x,r) = E_r(x) = E_{[r+k+1/aE_{-k}(x)]}\left(a\left(r+\frac{1}{a}E_{-k}(x)-\left[r+\frac{1}{a}E_{-k}(x)\right]\right)\right),$$

where k is determined by equation (4).

At the beginning of this section we assumed $E(x) > x$ for all real x. In case $E(x) < x$ for all real x, we consider $e(x) = E_{-1}(x)$, the inverse of E. Then $e(x) > x$ for all real x, it is an order preserving homeomorphism of the real line onto itself and has no fixed points. By the above arguments $e(x)$ can be embedded in a continuous flow $G(x,r)$ on the real line so that $G(x,1) = e(x)$. Then the given homeomorphism $E(x)$ satisfies the equation $G(x,-1) = E(x)$ since $G(x,t_1)$ is the inverse of $G(x,-t_1)$ in a continuous flow. The proof of (A) is completed by the following observation.

(vi) *If $G(x,r)$ is a continuous flow on X, then so is $F(x,r) = G(x,-r)$.* One need only verify that

$$F(F(x,r_1),r_2) = G(G(x,-r_1),-r_2) = G(x,-r_1-r_2) = F(x,r_1+r_2).$$

5. Case (B) of Lemma 1. This case, and case (C), are similar to (A). We will only point out the changes needed in Section 4 to establish (B).

We first assume $E(x) > x$ for all $x \epsilon (-1, 1), E(-1) = -1$ and $E(1) = 1$. As before, define

$$\varphi(x) = E_{[x]}\big(a(x-[x])\big)$$

for all real x. Now,

$$-1 < a < 1 \quad \text{and} \quad -1 < \varphi(x) < 1.$$

Again, equation (1) is satisfied for all real x since $|\varphi(x)| < 1$ for all real x. It is easy to show that φ is monotone increasing and continuous for all real x.

E_n is monotone increasing in n for a fixed $x^* \epsilon (-1, 1)$ and one has

$$\lim_{n\to\infty} E_n(x^*) = 1, \quad \lim_{n\to\infty} E_n(x^*) = -1.$$

Thus $\{\varphi(n)\} = \{E_n(0)\} \to \pm 1$ as $n \to \pm\infty$ and φ is a homeomorphism of the reals onto the open interval $(-1, 1)$.

Equation (4) gives a unique $k(x)$ for each $x \epsilon (-1, 1)$ with which one uses (5) to define $\varphi^{-1}(x)$. The equation $\varphi^{-1}\big(\varphi(x)\big) = x$ is valid for all real x and the equation $\varphi\big(\varphi^{-1}(x)\big) = x$ is valid for all $x \epsilon (-1, 1)$. The proof is completed as in the previous section.

The modifications necessary to adapt (B) to (C) are obvious. In particular φ becomes a homeomorphism from the reals to an open ray.

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