COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

ON TRANSLATIONS IN QUASIDISTRIBUTIVE SEMILATTICES

BY

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1. Introduction and definitions. In papers [1] and [3]-[5] there are given some properties and implications of translations in lattices and semilattices. The purpose of this paper is to illuminate some structural properties of a join-semilattice having a translation. This will be done in terms of congruence relations of join-semilattices. In particular, we shall show when a dual ideal of a join-semilattice L generates a translation on L, and prove that congruence relations generated by a translation on L constitute a sublattice of the lattice $\theta(L)$ of congruence relations of L if L is quasidistributive. Moreover, any translation of a quasidistributive join-semilattice L is the meet of a set of specified translations on L.

A join-semilattice L is called *quasidistributive* if, for any non-comparable two elements of L, there is no common lower bound in L.

In all that follows, L is a join-semilattice. A mapping l on a semilattice L is called a *translation* if it satisfies the following identity:

$$l(x)\cup y = l(x\cup y), \qquad x, y\in L.$$

A subset J of L is called a *dual ideal* of L if $a\in J$ and $x\in L$ imply $a\cup x\in J$. The fix-elements of L with respect to l in L form a dual ideal J_l of L as shown in [3], Theorem 2.

In [1] Kolibiar constructed a join-congruence R_l of a lattice, which was based on a given translation l of this lattice. As one can easily see, the facts on which his construction was based remain true for semilattices, and hence we can directly write a copy of Theorem 1 of [1] in the case of semilattices:

THEOREM 1. *For each semilattice L there is a one-to-one mapping between translations l and congruence relations θ_l having the following property*:

(i) *In L there is a dual ideal J_l such that every rest class modulo θ_l contains exactly one element of J_l.*

The congruence θ_l related to translation l and the translation l_θ related to congruence θ_l are characterized, respectively, by

(ii) $x\theta_l y$ *if and only if* $l(x) = l(y)$, $x, y \in L$,

(iii) $l_\theta(x) = x' \in J_l$ *for which* $x\theta_l x'$.

2. On dual ideals and translations. First we consider the properties of a dual ideal J generating a congruence relation of L related to a translation l of L, briefly – a translation congruence.

THEOREM 2. *Let J be a dual ideal of L. J generates a translation congruence of L if and only if for any $x \in L-J$ there exists an element $a \in J$ such that, if $b \in J$ and $b \geqslant x$, then $b \geqslant a$.*

Proof. Let J generate a translation l_J, and hence a translation congruence θ_{lJ}, and let $a = l_J(x)$. As $l_J(x)$ is a closure operation on L ([3], Theorem 3), $x \leqslant a$ and, according to Theorem 1, $x\theta_{lJ}a$. If in J there were an element b such that $b \geqslant x$ and b non-comparable with a, then $x\theta_{lJ}a$ would imply $b\theta_{lJ}a \cup b$, where $a \neq a \cup b$. Thus θ_{lJ} would collapse two distinct elements of J, which is a contradiction, as θ_{lJ} was a translation congruence.

Conversely, let J be a dual ideal of L with the properties of the theorem. We define a mapping l_J as follows: if $x \in L-J$, then $l_J(x) = a$, and if $x \in J$, then $x = l(x)$. Clearly, l_J is a translation on L ([3], Theorem 2), and the theorem follows.

Let J and K be two dual ideals of L, and let $\wedge$ and $\vee$ denote the set-theoretical intersection and union operations, respectively. We define the meet and join of dual ideals as follows: $J \cap K = J \wedge K$ and $J \cup K = J \vee K$; they are trivially dual ideals of L. If J and K generate a translation congruence on L, the following lemma shows when $J \cap K$ and $J \cup K$ do so:

LEMMA 1. *Let L be a quasidistributive semilattice and M the lattice of dual ideals of L generating translation congruences on L. For every $J \in M$, let θ_{lJ} be a translation congruence generated by J. The map θ_l: $J \mapsto \theta_{lJ}$ is an anti-homomorphism of M into the lattice $\theta(L)$ of all congruences on L, i.e. $\theta_{lJ} \cap \theta_{lK} = \theta_{l(J \cup K)}$ and $\theta_{lJ} \cup \theta_{lK} = \theta_{l(J \cap K)}$ for all $J, K \in M$.*

Proof. Let L be a quasidistributive semilattice. We prove first that $\theta_{lJ} \cap \theta_{lK}$ and $\theta_{lJ} \cup \theta_{lK}$ are translation congruences on L.

As shown in [5], Theorem 3,

$$l_J(x) \cup l_K(x) = l_J[l_K(x)] = l_J l_K(x) = l_K l_J(x),$$

where $l_J l_K = l_K l_J = t$ is a translation of L. At first we prove that $\theta_{lJ} \cup \theta_{lK} = \theta_t$, where $t = l_J l_K$, by showing that

(1) $\theta_{lJ} \cup \theta_{lK} \leqslant \theta_t$ and

(2) $\theta_t \leqslant \theta_{lJ} \cup \theta_{lK}$.

(1) Let $x\theta_{lJ}y$. Then $l_J(x) = l_J(y)$, and thus $l_K l_J(x) = l_K l_J(y)$, from which it follows that $x\theta_t y$. Hence $\theta_{lJ} \leqslant \theta_t$. Similarly, $\theta_{lK} \leqslant \theta_t$, and so $\theta_{lJ} \cup \theta_{lK} \leqslant \theta_t$.

(2) As $x\theta y$ is equivalent to the relations $x\theta x \cup y$ and $x \cup y \theta y$ ([2], Lemma 3), it is sufficient to consider the case $x \leqslant y$. Let $x\theta_t y$, and so $t(x) = t(y)$ implying

$$l_J(x) \cup l_K(x) = l_J(y) \cup l_K(y).$$

As L is quasidistributive and x is a common lower bound, the elements $t(x)$, $l_K(y)$, $l_J(y)$, $l_K(x)$, $l_J(x)$, y and x form a chain in L. As $y \geqslant x$, $l_K(y) \geqslant l_K(x)$. According to Theorem 2 and quasidistributivity of L, $l_K(y) = l_K(x)$. Similarly, $l_J(x) = l_J(y)$, whence $x\theta_{lK}y$ and $x\theta_{lJ}y$. So $\theta_t \leqslant \theta_{lJ} \cup \theta_{lK}$.

Consider now the congruence relation $\theta_{lJ} \cap \theta_{lK}$. Let $x \in L$. As L is quasidistributive and x is a common lower bound of $l_K(x)$ and $l_J(x)$, $l_J(x)$, $l_K(x)$ and x form a chain for any x of L. Thus we can define a meet $l_J(x) \cap l_K(x)$ for any $x \in L$. Consider the expression $\big(l_J(x) \cap l_K(x)\big) \cup y$. As $l_J(x)$, $l_K(x)$ and x form a chain in L,

$$\big(l_J(x) \cap l_K(x)\big) \cup y = \big(l_K(x) \cup y\big) \cap \big(l_J(x) \cup y\big) = l_J(x \cup y) \cap l_K(x \cup y),$$

whence $l_J \cap l_K = p$ is a translation of L for any $x \in L$. For completing the proof, we prove that, in $\theta(L)$, $\theta_{lJ} \cap \theta_{lK} = \theta_p$, where $l_J(x) \cap l_K(x) = p(x)$ for any $x \in L$, by showing that

(3) $\theta_{lJ} \cap \theta_{lK} \leqslant \theta_p$ and

(4) $\theta_{lJ} \cap \theta_{lK} \geqslant \theta_p$.

(3) Let $x(\theta_{lJ} \cap \theta_{lK})y$. Thus $x\theta_{lJ}y$ and $x\theta_{lK}y$, whence $l_J(x) = l_J(y)$ and $l_K(x) = l_K(y)$. But as $l_J(x)$, $l_K(x)$, x and $l_J(y)$, $l_K(y)$, y form chains,

$$l_J(x) \cap l_K(x) = l_J(y) \cap l_K(y),$$

whence $x\theta_p y$, and so $\theta_{lJ} \cap \theta_{lK} \leqslant \theta_p$.

(4) Let $x\theta_p y$, $x < y$, and assume that $l_J(x) < l_J(y)$. Then necessarily $l_K(x) = l_K(y)$ and $l_J(y) > l_J(x) \geqslant l_K(x) = l_K(y)$. Further,

$$l_K(x) \cup l_J(x) = l_J(x) = l_K(y) \cup l_J(x) = l_J\big(l_K(y)\big)$$

and, as $l_K(y) \geqslant y$, we have $l_J\big(l_K(y)\big) \geqslant l_J(y) > l_J(x)$, which is a contradiction Hence $l_J(x) = l_J(y)$ and $l_K(x) = l_K(y)$ which together imply the relations $x\theta_{lJ}y$, $x\theta_{lK}y$ and $\theta_{lJ} \cap \theta_{lK} \geqslant \theta_p$.

The definitions $t(x) = l_J(x) \cup l_K(x)$ and $p(x) = l_J(x) \cap l_K(x)$ for any $x \in L$ imply directly that the dual ideals generating translations t and p on L are $J \cap K$ and $J \cup K$, respectively, and the lemma follows.

From Lemma 1 we obtain immediately

COROLLARY 1. *Let L be a quasidistributive semilattice. The translation congruences on L form a sublattice of $\theta(L)$.*

If L is a quasidistributive semilattice, the minimum congruence relation θ_{ab} on L collapsing two elements $a, b \in L$ has the set-theoretical union of the intervals $[a, a \cup b]$ and $[b, a \cup b]$ as its only non-trivial congruence class ([2], Theorem 6). From this fact and (i) in Theorem 1 it follows that the anti-homomorphism of Lemma 1 is not an anti-isomorphism even for chains of three elements. Trivially, in a chain of two elements, each congruence relation is a translation congruence. So we can conclude that the anti-homomorphism of Lemma 1 is an anti-isomorphism only if L is a quasidistributive semilattice where each chain consists of two elements.

COROLLARY 2. *The anti-homomorphism of Lemma* 1 *is a one-to-one mapping, i.e. an anti-isomorphism, only if in L each chain consists of two elements.*

The semilattice $L = \{1, a, b, e\}$, where $1 > a > e$ and $1 > b > e$, shows that also in a non-quasidistributive semilattice L the translation congruences may constitute a sublattice of $\theta(L)$. Namely, the only dual ideals generating a translation on this semilattice are: $\{1\}$, $\{1, a\}$, $\{1, b\}$, and $\{1, a, b, e\}$.

If c is a fixed element of L, then the mapping $x \to x \cup c$ is a translation of L. This kind of translation is called by Szász *specified* and denoted by c_S. The following result illuminates the role of specified translations on a quasidistributive semilattice.

THEOREM 3. *Any translation congruence of a quasidistributive semilattice L is the meet of a set of specified translations of L.*

Proof. Let l be a translation of L and let J be a dual ideal in L related to l. We shall show that

$$\bigcap_{j \in J} \theta_{j_S} = \theta^0$$

is equal to θ_l.

As well known, the congruence lattice $\theta(L)$ of L is complete, and hence a congruence relation θ^0 exists. Let $x\theta_l y$. Without loss of generality we can assume that $x > y$ (the case $x = y$ is trivial). As $x\theta_l y$ holds, $a = l(x) = l(y)$. Then, for any $j \in J$, $j \geqslant a$, we have $x\theta_{j_S} y$, and if $j \in J$ is non-comparable with a, then, according to quasidistributivity of L, $j \cup x = j \cup y$, and hence $x\theta_{j_S} y$ for any $j \in J$. So $x\theta^0 y$ and $\theta^0 \geqslant \theta_l$.

On the other hand, let $x\theta^0 y$, $x > y$, and $j \in J$. If $j \geqslant x$, then $x\theta_{j_S} y$ as $j \cup x = j \cup y = j$, and if j is non-comparable with x, then it is non--comparable also with y according to quasidistributivity of L. By applying the quasidistributivity once again, $j \cup x = j \cup y$, and thus $x\theta_{j_S} y$ for any $j \in J$. Hence $\theta^0 \leqslant \theta_l$, and the theorem follows.

A set $I \subset L$ is an ideal of L if

(1) $a \in I$, $x \in L$, $x \leqslant a$ imply $x \in I$, and
(2) $a \in I$, $b \in I$ imply $a \cup b \in I$.

As usually, by $(a]$ we denote the principal ideal of L generated by the element a of L, and by $\theta[I]$ the least congruence relation in $\theta(L)$ having I as a congruence class. As a simple consequence of results in [2], Theorems 2 and 3, we see that the congruence relations θ_{a_S} on L generated by a specified translation a_S on L are equal to the congruence relations $\theta\big[(a]\big]$ if L is a distributive or modular semilattice.

Let I be an ideal of L. As shown in [2], Lemma 8, in a quasidistributive semilattice L, $x\theta[I]y$, $x \neq y$, if and only if $x, y \in I$. Hence, in general, if L is quasidistributive, $\theta_{a_S} \neq \theta\big[(a]\big]$.

If L is quasidistributive and finite, a representation for a translation congruence θ_l can be found. Let θ_{ab} denote the least congruence relation on L having the closed interval $[a, b]$ of L, $a < b$, as a congruence class. In a quasidistributive semilattice L, $c\theta_{ab}d$, $c \neq d$, only if $c, d \in [a, b]$ (see [2], Theorem 6). An interval $[a, b]$, $a < b$, is called *prime* if $c \in [a, b]$ implies $c = a$ or $c = b$.

Let L be a finite and quasidistributive semilattice and let J be a dual ideal generating a translation on L. We denote by A the set of all maximal elements in $L - J$, and by Q the set of all prime intervals $[a, b]$ of L, where $a \in A$ and $b \in J$.

LEMMA 2. *Let L be a finite quasidistributive semilattice, l a translation on L, and J the dual ideal of L corresponding to l. Then*

$$\theta_l = \Big(\bigcup_{a \in A} \theta\big[(a]\big]\Big) \cup \Big(\bigcup_{q \in Q} \theta_q\Big).$$

Proof. Denote by θ^0 the congruence relation

$$\Big(\bigcup_{a \in A} \theta\big[(a]\big]\Big) \cup \Big(\bigcup_{q \in Q} \theta_q\Big).$$

As L is quasidistributive and as l determines a closure operation on L, $\theta_l \geqslant \theta^0$. Let $x\theta^0 y$ and $x > y$. According to the definition of θ^0, x and y cannot be simultaneously contained in J. If $x, y \notin J$, then, as $x > y$, L is quasidistributive and l is a closure operation on L, $l(x) = l(y) = c \in J$, and hence $x\theta_l y$. If $x \in I$ (and $y \notin J$), then $l(x) = l(y) = x$, for if $l(y) = x$, then $x\theta^0 y$ would imply $l(y)\,\theta^0 x$, where $l(y)$, $x \in J$, a contradiction. Hence $\theta^0 \geqslant \theta_l$ and the lemma follows.

The following corollary to Theorem 3 illuminates the relation between congruences θ_{a_S} and $\theta\big[(a]\big]$ in a quasidistributive semilattice L.

COROLLARY 3. *Let L be a quasidistributive semilattice. Then $\theta\big[(a]\big]$ is a translation congruence on L for any $a \in L$. Moreover, if $B = \{L - (a]\} \vee a$, then*

$$\theta\big[(a]\big] = \bigcap_{b \in B} \theta_{b_S}.$$

Proof. As $x\theta\left[(a]\right]y$, $x \neq y$, if and only if $x, y \in (a]$, then, if $\theta\left[(a]\right]$ is a translation congruence, B is the corresponding dual ideal, and the representation of $\theta\left[(a]\right]$ as the meet of specified relations θ_{bS} follows from Theorem 3.

As L is quasidistributive, B is a dual ideal of L, and as $x\theta\left[(a]\right]y$, $x \neq y$, only if $x, y \in (a]$, each congruence class modulo $\theta\left[(a]\right]$ contains exactly one element of B. Hence $\theta\left[(a]\right]$ is a translation congruence on L.

The author wishes to express his sincere thanks to the referee for his remarks and suggestions in this paper.

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Reçu par la Rédaction le 29. 9. 1974;
en version modifiée le 30. 12. 1974