A POISSON KERNEL ON HEISENBERG TYPE NILPOTENT GROUPS

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The classical nilpotent Lie groups which appear in the Iwasawa decomposition of semisimple Lie groups are embedded in a larger class of so-called groups of type H (see [8] and [12]). This class has been introduced by Kaplan in [7] and has recently attracted considerable attention (cf., e.g., [1], [2], [12], and [13]). If such a group N is extended by the one-parameter group of dilations A, the group $S = NA$ may be regarded as an analogy of a rank one symmetric space, N being the boundary; cf. [2], where the Riemannian structure of S which generalizes that of the symmetric space has been investigated.

J. Cygan has proposed to study the harmonic functions on S with respect to the Laplace–Beltrami operator on S and has written a formula for what should be the Poisson kernel P_a, $a \in A$. In this paper we verify that in fact this formula is correct. We prove that the function $P(n, a) = P_a(n)$ is harmonic on S, and consequently so is the function $\varphi * P_a(n)$ for every $\varphi \in L^p(N)$, $1 \leqslant p \leqslant \infty$. We also show that

$$\lim_{a \to 0} \varphi * P_a(n) = \varphi(n) \quad \text{a.e.}$$

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1. Introduction. Let $\mathfrak{n}$ be a nilpotent 2-step algebra with an inner product. Denote by V the orthogonal complement to its center Z. Then, for every $v \in V$, ad_v maps V into Z and we have the orthogonal decomposition of V given by

$$V = \operatorname{Ker} \mathrm{ad}_v \oplus D(v).$$

n is said to be of *Heisenberg type* (shortly, of *type H*) if for every unit vector $v \in V$ the mapping ad_v: $D(v) \to Z$ is a surjective isometry.

Every Lie algebra of type H arises as follows [7]. Let U and V be vector spaces with positive definite quadratic forms $|\cdot|^2$. By definition, a composition of these quadratic forms is a bilinear map μ: $U \times V \to V$ which satisfies

$$|\mu(u, v)| = |u| |v|, \quad u \in U, v \in V,$$

and for a u_0

$$\mu(u_0, v) = v, \quad v \in V.$$

Let Z be the orthogonal complement to Ru_0 and let π: $U \to Z$ be the orthogonal projection. Define a bilinear map Φ: $V \times V \to U$ by

$$\langle u, \Phi(v, v') \rangle = \langle \mu(u, v), v' \rangle.$$

Then $\pi\Phi$ is skew-symmetric [7], and $\mathfrak{n} = V \times Z$ with the bracket

$$[(v, z), (v', z')] = (0, \pi\Phi(v, v')),$$

and the inner product

$$\langle (v, z), (v', z') \rangle = \langle v, v' \rangle + \langle z, z' \rangle$$

is an algebra of type H.

Let N be a connected and simple connected Lie group whose Lie algebra is $\mathfrak{n}$. If we identify N with $\mathfrak{n}$ by the exponential map, the multiplication in N is given by

$$(v, z)(v', z') = \left(v + v', z + z' + \tfrac{1}{2}\pi\Phi(v, v')\right).$$

We denote by A the multiplicative group of R^+. Let

(1) $S = NA$

be a semi-direct product of N and A, A acting on N as dilations $\delta_a(v, z)$ $= (av, a^2 z)$. Thus we identify S with $V \times Z \times R^+$ and

$$(v, z, a)(v', z', a') = \left(v + av', z + a^2 z' + \tfrac{1}{2}a\pi\Phi(v, v'), aa'\right).$$

S has a Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus R$ with the bracket

$$[(v, z, \log a), (v', z', \log a')]$$

$$= \left((\log a)v' - (\log a')v, 2(\log a)z' - 2(\log a')z + \pi\Phi(v, v'), 0\right).$$

In the Lie algebra $\mathfrak{s}$ we select an inner product

$$\langle (v, z, \log a), (v', z', \log a') \rangle_s = \langle v, v' \rangle + \langle z, z' \rangle + 4(\log a)(\log a')$$

and the left-invariant metric it defines we denote also by $\langle \cdot, \cdot \rangle_s$.

The above construction includes the noncompact rank one symmetric spaces as particular cases, which of course has been the reason to study

groups of type H (cf. [9]). Somewhat more about this inclusion is in [2].

Let M be a rank one noncompact symmetric space. The group of isometries of M is semisimple. Denote by G its connected component. Let $\mathfrak{g}$ be the Lie algebra of G, B the Killing form of $\mathfrak{g}$, θ the Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition [15]. Denote by $\mathfrak{a}$ a maximal Abelian subalgebra of $\mathfrak{p}$, which by assumption is one-dimensional. The positive part of the corresponding system of restricted roots is equal to $\{\alpha, 2\alpha\}$ or $\{\alpha\}$ for a root α. Then

$$\mathfrak{n} = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$$

is a nilpotent type H subalgebra of $\mathfrak{g}$ with the center $\mathfrak{g}^{2\alpha}$. For the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$$

we write N, A, and K for the corresponding subgroups of G. The map

$$N \times A \times K \ni (n, a, k) \to nak \in G$$

is a diffeomorphism and $S = NA$ is a closed solvable subgroup of G diffeomorphic to M.

The Poisson kernel for M in terms of NA is a family P_a of convolution kernels on N (see [5]):

$$P_a(n) = \frac{c\varepsilon^d}{\left[(\varepsilon^2 + (4b)^{-1}|X_\alpha|^2)^2 + b^{-1}|X_{2\alpha}|^2 \right]^{d/2}},$$

where

$$n = \exp(X_\alpha + X_{2\alpha}), \qquad X_\alpha \in \mathfrak{g}^{\alpha}, \; X_{2\alpha} \in \mathfrak{g}^{2\alpha},$$

$$\varepsilon = e^{\alpha(\log a)}, \qquad |X|^2 = -B(X, \theta X),$$

$$d = \dim \mathfrak{g}^{\alpha} + 2 \dim \mathfrak{g}^{2\alpha}, \qquad b = \dim \mathfrak{g}^{\alpha} + 4 \dim \mathfrak{g}^{2\alpha},$$

and c is such that

$$\int_N P_a(n) = 1.$$

Now, if N is a group of type H and S is given by (1), then the inner product

$$\langle (v, z, \log a), (v', z', \log a') \rangle_\theta = b\big(\langle v, v' \rangle + \langle z, z' \rangle + 2(\log a)(\log a') \big)$$

on S corresponds to the product $-B(\cdot, \theta \cdot)$ (cf., e.g., [2]). So following a suggestion of J. Cygan we write

$$P_a(n) = \frac{ca^d}{\left[(a^2 + \frac{1}{4}|v|^2)^2 + |z|^2 \right]^{d/2}},$$

where $n = \exp(v, z)$, $d = \dim V + 2 \dim Z$, and c is such that $\int_N P_a(n) = 1$. The conjecture that $P_a(n)$ is the Poisson kernel for S can be also somewhat metamathematically supported by the fact that the fundamental solution for the sublaplacian on the nilpotent groups, which appear in the Iwasawa decomposition of the rank one symmetric spaces, has been rewritten for the groups of type H in a similar way and in this shape is the fundamental solution for the corresponding sublaplacian (see [7] and [9]).

2. The Laplace–Beltrami operator. Let $\{e_i\}$, $i = 1, \ldots, \dim V$, and $\{e_r\}$, $r = \dim V + 1, \ldots, \dim V + \dim Z$, be orthonormal bases in V and in Z, respectively, and e_0 a vector in R such that $\langle e_0, e_0 \rangle = \frac{1}{4} \cdot \{e_i, e_r, e_0\}$ is a $\langle \cdot, \cdot \rangle_s$-orthonormal basis of $\mathfrak{s}$. Denote by E_β the left-invariant vector field on S corresponding to e_β for $\beta = 0, 1, \ldots, \dim V + \dim Z$.

THEOREM 2.1. *The Laplace–Beltrami operator Δ is of the form*

$$\Delta = \sum_\beta E_\beta^2 - \frac{d}{2} E_0.$$

Proof. The proof is based on the following facts:

1. If ∇ is a Riemannian connection on M, then $(\operatorname{div} X)_p$ is equal to the trace of $T_p M \ni w \to \nabla_w X \in T_p M$ (cf. [4] and [10]).

2. We have $\Delta g = \operatorname{div}(\operatorname{grad} g)$ for $g \in C^\infty(M)$.

3. The Riemannian connection on S expressed in terms of E_β (cf. [2]) is given by

$$\nabla_{E_i} E_i = \tfrac{1}{2} E_0, \qquad \nabla_{E_r} E_r = E_0, \qquad \nabla_{E_0} E_0 = 0.$$

Now

$$\operatorname{div}(\operatorname{grad} g) = \sum_\beta \langle E_\beta, \nabla_{E_\beta}(\operatorname{grad} g) \rangle_s$$

$$= \sum_\beta E_\beta \langle E_\beta, (\operatorname{grad} g) \rangle_s - \sum_\beta \langle \nabla_{E_\beta} E_\beta, (\operatorname{grad} g) \rangle_s$$

$$= \sum_\beta E_\beta^2 g - \frac{d}{2} \langle E_0, (\operatorname{grad} g) \rangle_s = \sum_\beta E_\beta^2 g - \frac{d}{2} E_0 g.$$

THEOREM 2.2. *We have $\Delta P = 0$.*

Proof. We write $h = a^2 + \frac{1}{4} |v|^2$ and $f = h^2 + |z|^2$. Then $P = a^d f^{-d/2}$. To calculate $E_\beta g$ for $g \in C^\infty(S)$ we use the formula

$$E_\beta g(v, z, a) = \frac{d}{dt} g((v, z, a)(\exp t E_\beta))|_{t=0}.$$

Hence by a simple calculation we obtain

$$E_0 h = a^2, \qquad E_r h = 0, \qquad E_i h = \tfrac{1}{2} a v_i,$$

$$E_0 f = 2ha^2, \qquad E_r f = 2a^2 z_r, \qquad E_i f = a\big(hv_i + \langle \mu(z, v), e_i \rangle\big),$$

where

$$v = \sum_i v_i e_i, \qquad z = \sum_r z_r e_r.$$

Moreover,

$$\sum_\beta (E_\beta f)^2 = 4h^2 a^4 + 4a^4 |z|^2 + a^2 h^2 |v|^2 +$$

$$+ 2ah \sum_i v_i \langle \mu(z, v), e_i \rangle + a^2 \sum_i \langle \mu(z, v), e_i \rangle^2.$$

But

$$\sum_i v_i \langle \mu(z, v), e_i \rangle = \langle \mu(z, v), v \rangle = \langle z, \pi\Phi(v, v) \rangle = 0$$

and

$$\sum_i \langle \mu(z, v), e_i \rangle^2 = |\mu(z, v)|^2 = |z|^2 |v|^2.$$

Hence, finally,

$$\sum_\beta (E_\beta f)^2 = 4ha^2 f.$$

On the other hand, it is easy to see that

$$E_i^2 |z|^2 = \tfrac{1}{2} a^2 |\pi\Phi(v, e_i)|^2$$

and

$$\Delta |z|^2 = 2(\dim Z) a^4 + \tfrac{1}{2} a^2 \sum_i |\pi\Phi(v, e_i)|^2$$

$$= 2(\dim Z) a^4 + \tfrac{1}{2} a^2 \sum_r \big(\sum_i \langle \mu(e_r, v), e_i \rangle^2\big)$$

$$= 2(\dim Z) a^4 + \tfrac{1}{2} a^2 \sum_r |\mu(e_r, v)|^2 = 2(\dim Z) a^2 h,$$

$$\Delta |v|^2 = 2(\dim V) a^2, \qquad \Delta a = \frac{1-d}{4} a, \qquad \Delta a^d = 0,$$

$$\Delta h = a^2 (1 + \tfrac{1}{2} \dim V - \tfrac{1}{2} d), \qquad \Delta f = 4ha^2.$$

Finally we have

$$\Delta P = a^d \Delta f^{-d/2} + 2 \sum_\beta (E_\beta a^d)(E_\beta f^{-d/2})$$

$$= a^d \Delta f^{-d/2} + 2 (E_0 a^d)(E_0 f^{-d/2})$$

$$= a^d \left(\left(-\frac{d}{2} \right) \left(-\frac{d}{2} - 1 \right) f^{-d/2 - 2} \sum_\beta (E_\beta f)^2 + \right.$$

$$\left. + \left(-\frac{d}{2} \right) f^{-d/2 - 1} \Delta f + d \left(-\frac{d}{2} \right) f^{-d/2 - 1} E_0 f \right)$$

$$= a^d \left(-\frac{d}{2} \right) f^{-d/2 - 1} \left(\left(-\frac{d}{2} - 1 \right) 4ha^2 + 4ha_2^2 + 2dha^2 \right) = 0.$$

3. Fatou's theorem. We define a gauge by

$$\| (v, z) \| = \left(\frac{1}{16} |v|^4 + |z|^2 \right)^{1/4},$$

which is homogeneous of degree 1, i.e., for dilations δ_a

$$\| \delta_a (v, z) \| = a \| (v, z) \|$$

(see [3]). N with $\|\cdot\|$ is a space of homogeneous type. Moreover, $\|\cdot\|$ is subadditive, i.e.,

$$\| nn_1 \| \leqslant \| n \| + \| n_1 \|, \qquad n, n_1 \in N$$

(see [1]). We have the following estimate for $P_a(n)$:

$$(2) \qquad\qquad P_a(n) = \frac{ca^d}{(a^4 + \| n \|^4)^{d/2}}.$$

Consequently, $P_a(n) \in L^p(N)$, $1 \leqslant p \leqslant \infty$. Moreover, for every left-invariant differential operator ∂ on N we have

$$(3) \qquad\qquad \partial P_a \in L^p(N), \qquad 1 \leqslant p \leqslant \infty.$$

Indeed, it can be easily seen that $(E_{\beta_1} \dots E_{\beta_k} f) f^{-1}$ is bounded, where as above $f = (a^2 + \frac{1}{4} |v|^2)^2 + |z|^2$. It follows from (3) that for every $\varphi \in L^p(N)$ and every left-invariant differential operator ∂ we have

$$\partial(\varphi * P_a) = \varphi * \partial P_a.$$

Now we are ready to prove an analogy of the classical Fatou's theorem (cf. [11] and [16]).

THEOREM 3.1. *If* $\varphi \in L^p(N)$, *then*

$$\lim_{a \to 0} \varphi * P_a(n) = \varphi(n) \quad a.e.$$

Proof. The proof goes by a routine way. First we show
(a) If $\varphi \in C_c(N)$, then

$$\lim_{a \to 0} \varphi * P_a(n) = \varphi(n).$$

Then we prove

(b) The operator T defined by

$$T\varphi(n) = \sup_{a>0} |\varphi * P_a(n)|$$

is of weak type $(1, 1)$ (see [14]).

From (a) and (b) we infer that Theorem 3.1 holds for $\varphi \in L^1(N)$ (cf., e.g., [14], p. 60).

Since

$$P_a(n) = a^{-d} P_1(\delta_{a^{-1}}(n)) \quad \text{and} \quad \int P_1(n)\,dn = 1,$$

$P_a(n)$ is an approximate identity and (a) follows.

To prove (b), let $B_r(n) = \{n_1 : \|nn_1^{-1}\| < r\}$. Denote by m the standard Lebesgue measure on N. Since N with $\|\cdot\|$ is a space of homogeneous type, the operator which assigns to each function $\varphi \in L^1(N)$ its maximal function

$$\varphi^*(n) = \sup_{r>0} \frac{1}{m(B_r(n))} \int_{B_r(n)} |\varphi(n_1)|\,dn_1 = \sup_{r>0} \frac{1}{m(B_r(0))} \int_{B_r(0)} |\varphi(nn_1^{-1})|\,dn_1$$

is of weak type $(1, 1)$.

Now it is sufficient to show that there is a constant D such that for every $\varphi \in L^1(N)$

$$T\varphi(n) \leq D\,\varphi^*(n).$$

Notice that

$$P_a(n) \leq \frac{ca^d}{[a^4 + \|n\|^4]^{d/2}} \leq ca^{-d},$$

and if $\|n\| \geq a$, then

$$\frac{ca^d}{[a^4 + \|n\|^4]^{d/2}} \leq \frac{ca\|n\|^{d-1}}{\|n\|^{2d}} = ca\|n\|^{-d-1}.$$

By a standard method [6], since $m(B_r(0)) = r^d m(B(0))$, we have

$$\left| \int_N \varphi(nn_1^{-1}) P_a(n_1)\,dn_1 \right|$$

$$\leq \int_{\|n_1\| < a} |\varphi(nn_1^{-1})|\, P_a(n_1)\,dn_1 + \sum_{k=1}^{\infty} \int_{2^{k-1}a \leq \|n_1\| < 2^k a} |\varphi(nn_1^{-1})|\, P_a(n_1)\,dn_1$$

$$\leq ca^{-d} \int_{\|n_1\| < a} |\varphi(nn_1^{-1})|\,dn_1 + \sum_{k=1}^{\infty} ca(2^{k-1}a)^{-d-1} \int_{B_{2^k a}(0)} |\varphi(nn_1^{-1})|\,dn_1$$

$$\leq c\varphi^*(n) + c \cdot 2^{d+1} \left(\sum_{k=1}^{\infty} 2^{-k} \right) \varphi^*(n)$$

$$= (c + c \cdot 2^{d+1})\,\varphi^*(n).$$

To prove Theorem 3.1 for φ in $L^p(N)$, $p > 1$, we write for every $r > 0$

$$\varphi = \psi_r + \xi_r,$$

where $\psi_r \in L^1$ and ξ_r is bounded and vanishes in the ball $B_r(0)$. Thus for $n \in B_{r/2}(0)$ we have

$$\varphi * P_a(n) = \psi_r * P_a(n) + \xi_r * P_a(n)$$

$$= \psi_r * P_a(n) + \int_{\|n_1\| > r/2} \xi_r(nn_1^{-1}) P_a(n_1) \, dn_1.$$

Since by (2) we get

$$\lim_{a \to 0} \left| \int_{\|n_1\| > r/2} \xi_r(nn_1^{-1}) P_a(n_1) \, dn_1 \right| \leqslant \lim_{a \to 0} \|\xi_r\|_{L^\infty} \int_{\|n_1\| > r/2} P_a(n_1) \, dn_1 = 0,$$

we have

$$\lim_{a \to 0} \varphi * P_a(n) = \varphi(n) \quad \text{a.e. for } n \in B_{r/2}(0),$$

and since r is arbitrary, the equality holds for all n in N.

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