COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

# THE LEAST ELEMENT MAP*

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Capel and Strother ([3], p. 42) have shown that *if a compact Hausdorff space is provided with a closed partial order, the function which maps each closed set with least element into its least element is continuous* (¹). This theorem has a long history (Michael [8], theorem 1.9, Eilenberg [5], proofs of theorems 5 and 14, and probably others), and several applications to selections ([8], [6], proposition 3.1.2), fixed points ([3]), and the factoring of maps ([5]).

It is our purpose to set this result in its most natural context, remove the superfluous hypotheses, simplify the proof, and provide such a converse as is available.

If $R \subset X \times X$ is any relation on a set $X$ and $A \subset X$, $x$ is said to be the *least element of* $A$ (with respect to the relation $R$) iff

(i) $x \in A$,

(ii) if $y \in A$, then $xRy$,

and

(iii) if $y \in A$ and $yRx$, then $y = x$.

Now suppose that $X$ is provided with a Hausdorff topology, that $\Sigma$ is the family of all compact subsets of $X$ having a least element, and that $f$ is the function that maps each $K \in \Sigma$ into its least element $f(K) \in X$. $\Sigma$ and $f$ determine a relation $R'$ on $X$ defined by

(iv) $xR'y$ iff for some $K \in \Sigma, x = f(K)$ and $y \in K$.

Clearly $R' \subset R$ and, when $R$ is antisymmetric, they are equal (¹). It is also easy to see that $R = R'$ implies that $f(\Sigma) = RX = \{x \mid xRy$ for some $y \in X\}$.

The relationship between $R$, $R'$, $\Sigma$ and $f$ is exhibited by the following

THEOREM. (a) *When $R$ is closed, $\Sigma$ is closed and $f$ is continuous* (²).

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* The authors thankfully acknowledge the support of NSF grants GP 2080 and GP 3623.

(¹) See *Added in proof*, p. 221.

(²) i.e., continuous with respect to the Vietoris topology (see [8]).

(b) *When $\Sigma$ is closed, $f$ is proper* (3) *and $R = R'$, then $R$ is closed*(1).

Some remarks are in order before proceeding to the proof. (a) shows that the hypotheses of order (i.e., transitivity) on $R$ and of compactness on $X$ are extraneous, and also gives us some new information about $\Sigma$.

(b) provides a converse of (a) in the compact case since $f$ will be proper when $X$ is compact, $\Sigma$ is closed, and $f$ is continuous.

Some condition such as $R = R'$ is needed in (b) since $\Sigma$ and $f$ could be empty, and hence closed, without $R$ being closed. But $R = R'$ is not a necessary condition for $R$ to be closed, since $\Sigma$ and $f$ can also be empty with $R$ closed.

It is tempting to try to strengthen (a) into a converse of (b) by showing that $f$ is proper when $R$ is closed. Unfortunately, this is not in general true when $X$ is not compact as an example will show. Let $X$ be the non-negative reals and $R = \{(x, y) \mid x, y \in X \text{ and } x = y\} \cup \{(x, y) \mid x, y \in X \text{ and } y = x + 1/x\}$. Let $\Phi = \{A \in \Sigma \mid A \cap [1, \infty) \neq \emptyset\}$. Then $\Phi$ is a closed subset of $\Sigma$, but $0 \in Clf(\Phi) \setminus f(\Phi)$.

Proof of the theorem. (a) Suppose $\{K_a\}$ is a net in $\Sigma$ converging to a compact subset $K_0$ of $X$. If the net $\{f(K_a)\}$ fails to converge to any point of $K$, it is eventually outside of some open neighborhood $U$ of $K_0$. But this contradicts the fact that $\{K_a\}$ is eventually in $\{K \subset U \mid K \text{ is compact}\}$. Hence suppose $\{f(K_a)\} \to a_0 \in K_0$. If $x \in K_0$ and $(a_0, x) \notin R$, there are disjoint open neighborhoods $U'$ and $V'$ of $a_0$ and $x$ such that $(U' \times V') \cap R = \emptyset$. But $K_0$ is compact and disjoint from $V'$, and so eventually $K_a \cap V' \neq \emptyset$ which is a contradiction. Hence $K_0 \in \Sigma$ and $f(K_0) = a_0$, and $\Sigma$ is closed.

To complete the proof of (a) note that (i), (ii) and (iii) may be rephrased as

(i)′ $f(A) \in A$,

(ii)′ $\{f(A)\} \times A \subset R$

and

(iii)′ $\left[\left(A \setminus \{f(A)\}\right) \times \{f(A)\}\right] \cap R = \emptyset$.

For any $K \in \Sigma$ (iii)′ implies that $\left(K \setminus \{f(K)\}\right) \times \{f(K)\} \subset (X \times X) \setminus R$ which is open in $X \times X$. If $f(K)$ belongs to an open set $W$ of $X$, then

$$\begin{aligned} K \times \{f(K)\} &= \left[\left(K \setminus \{f(K)\}\right) \times \{f(K)\}\right] \cup \left\{\left(f(K), f(K)\right)\right\} \\ &\subset [(X \times X) \setminus R] \cup [W \times W] \end{aligned}$$

which is open in $X \times X$. Since $K \times \{f(K)\}$ is compact, there are open neighborhoods $U$ and $V$ in $X$ such that

$$K \times \{f(K)\} \subset U \times V \subset [(X \times X) \setminus R] \cup [W \times W].$$

(3) i.e., $f$ is closed, continuous, and for each $x \in X$, $f^{-1}(x)$ is compact (see [2]).

Then $K \in \Lambda = \{A \in \Sigma \mid A \subset U \text{ and } A \cap V \neq \emptyset\}$ which is open in $\Sigma$. If $B \in \Lambda$, by (ii)′

$$\{f(B)\} \times (B \cap V) \subset R \cap (U \times V) \subset W \times W$$

and so $f(B) \subset W$. Thus $f$ is continuous.

(b) Let $\Theta = \{(A, b) \mid b \in A \in \Sigma\}$. Then $\Theta \subset \Sigma \times X$ and if $(A_0, b_0) \in (\Sigma \times X) \setminus \Theta$, there are disjoint open neighborhoods $U$ and $V$ of $A_0$ and $b_0$. Let $\Omega = \{A \in \Sigma \mid A \subset U\}$. Then $\Omega \times V$ is a neighborhood of $(A_0, b_0)$ in $\Sigma \times X$ disjoint from $\Theta$ and, hence $\Theta$ is closed. Hence $R = (f \times \mathrm{id}_X)(\Theta)$ is closed (2).

A *semigroup* is a non-void Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition. (Reference may be made to the excellent expository dissertation of Paalman-de Miranda [10], and, for discrete semigroups, to the books of Clifford-Preston [4] and Ljapin [7]). For simplicity of exposition we assume in the following example that *$S$ is a compact semigroup.*

EXAMPLE. Define a subset $R$ of $S \times S$ by

$$R = \{(x, y) \mid x = xy = yx\}$$

so that $R$ is closed, antisymmetric, and transitive. The closed set $A \in \Sigma$ iff there is an element $z$ in $A$ such that

$$z = zA = Az,$$

that is to say, $A$ contains a zero. Since $S$ is compact, it contains an element $e$ such that $e^2 = e$ and thus, at least, $\{e\} \in \Sigma$. Indeed, such elements constitute precisely the values of $f$ and these are also the one-element members of $\Sigma$. If $z^2 = z$, then

$$\{z \mid z = zx = xz\} \in \Sigma$$

and also is a maximal set in $\Sigma$ as well as a subsemigroup. If it is assumed that $S$ is commutative and idempotent ($x^2 = x$ for all $x \in S$), then Example 1 of [3] is herein subsumed, $R$ becomes reflexive and, indeed, $R$ is also a subsemigroup of $S \times S$.

Maintaining all of the above hypotheses on $S$, supposing that $S$ is contained in a space $X$ and that there is a retraction $r$ of $X$ onto $S$, then the multiplication may be extended to all of $X$ by the rule

$$xy = r(x)r(y).$$

With this multiplication $R$ is also extended to $X \times X$ and there obtains a situation which includes Example 2 of [3].

It may be noted that Example 3 of [3] is a special case of a very general selection theorem of Michael ([9], Theorem 8.1).

Conditions (i)′, (ii)′, (iii)′ (of the proof of the previous theorem) can be supplemented in order to insure that $R$ contains a partial order (4).

THEOREM. *$R$ contains a partial order iff there is a family $\Sigma$ and a surjective $f\colon \Sigma \to X$ satisfying for all $A \in \Sigma$,* (i)′, (ii)′, (iii)′, *and*

(v) *if $f(B) \in A$, then $A \cup B \in \Sigma$.*

*If the conditions are satisfied, $R'$ is the desired partial order.*

Since $f$ is surjective, (i)′ implies that $R'$ is reflexive. If $aR'b$ and $bR'a$, since $R' \subset R$, $a = b$. (Indeed $aRb$ and $a \neq b$ imply by (iii)′, not $bRa$.) If $aR'b$ and $bR'c$, suppose $a = f(A)$, $b = f(B) \in A$ and $c \in B$. By (v), $A \cup B \in \Sigma$. If $d = f(A \cup B) \in A$ by the antisymmetry just proved, $a = d$. If not, $d \in B$ and thus $d = b$. But this again implies $a = d$ and so $aR'c$.

Conversely, if $R$ contains a partial order $P$, let $\Sigma$ be those subsets containing their $P$-infimum. $f$ is surjective since $\Sigma$ contains all singletons, and the conditions are easily verified.

Of course much smaller choices of $\Sigma$ may be made. For example, the collection of all doubletons $\{a, b\}$, where $aPb$ is canonically small.

Ralph DeMarr points out that partial orders may be axiomatized by reference to surjective $f\colon \Sigma \to X$ alone using (i)′ and

(v)′ if $f(B) \in A \in \Sigma$, then $A \cup B \in \Sigma$

and $f(A) = f(A \cup B)$.

The order is defined as $R'$ is.

A subset $A$ of $X$ is an *$R$-chain* provided that $A \times A \subset R \cup R^{-1}$ and the relation $Q$ on $X$ is *chain-equivalent* to $R$ if and only if $Q \cup Q^{-1} = R \cup R^{-1}$. From the Hausdorff Maximality Principle it follows that any $R$-chain is contained in a maximal such. Also, Bednarek [1] has shown that if $Q$ is a reflexive and transitive relation on $X$, then any maximal partial order on $X$ which is contained in $Q$ is chain-equivalent to $Q$. Again by the maximality principle, any partial order on $X$ which is contained in $Q$ is also contained in a maximal such.

**P 555**. If $R$ is antisymmetric and closed, and if $X$ is compact, then each member of $\Sigma$ is contained in a maximal member. Is the collection of maximal members closed? Is it closed if, in addition, the second projection *restricted* to $R$ is an open function?

**P 556**. If $R$ is a closed partial order on $X$ and if $X$ is compact, is the collection of maximal $R$-chains closed?

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(4) Note that these considerations do not involve Topology.

**P 557.** If $X$ is compact and if $R$ is closed, reflexive and transitive, under what conditions will there exist a closed partial order contained in $R$ which is chain-equivalent to $R$?

Added in proof. $R$ must be taken to be reflexive. The authors are indebted to A. R. Bednarek for this observation.

*REFERENCES*

[1] A. R. Bednarek, *On chain-equivalent orders*, Nieuw Archiv voor Wiskunde 12 (1964), p. 137-138.

[2] N. Bourbaki, *Topologie générale*, Paris 1961.

[3] C. E. Capel and W. L. Strother, *Multi-valued functions and partial order*, Portugaliae Mathematica 17 (1958), p. 41-47.

[4] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, American Mathematical Society Surveys 7, Rhode Island 1961.

[5] S. Eilenberg, *Sur les transformations d'espaces métriques en circonférence*, Fundamenta Mathematicae 24 (1935), p. 160-176.

[6] S. P. Franklin, *Concerning continuous relations*, U. C. L. A. Dissertation 1963.

[7] E. S. Ljapin, *Semigroups*, American Mathematical Society Translations of Mathematical Monographs 3, Rhode Island 1963.

[8] E. Michael, *Topologies on spaces of subsets*, Transactions of the American Mathematical Society 71 (1951), p. 152-158.

[9] — *Continuous selections III*, Annals of Mathematics 65 (1957), p. 375-390.

[10] A. B. Paalman-de Miranda, *Topological semigroups*, Mathematisch Centrum, Amsterdam 1964.

*Reçu par la Rédaction le 5. 9. 1965*