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ON METRIZABLE ABELIAN GROUPS WITH A COMPLETENESS-TYPE PROPERTY

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We are concerned with the following completeness-type property of a metrizable Abelian group X considered in [5]:

(K) Every sequence (x_n) in X with $x_n \to 0$ contains a subsequence (x_{n_k}) such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is convergent.

As is well known, (K) holds if X is complete. In fact, (x_{n_k}) can then be chosen in such a way that the series $\sum_{k=1}^{\infty} x_{n_k}$ is subseries convergent. The existence of a noncomplete metrizable topological linear space with property (K) was first proved, under the continuum hypothesis, by the second-named author ([5], Theorem 2) and next, without that hypothesis, by I. Labuda and the third-named author ([6], Theorem 2).

The present paper falls into two independent sections. Section 1 is devoted to constructing, for a given nondiscrete metrizable complete Abelian group of cardinality $2^{\aleph_0}$ satisfying some algebraic assumption, a dense proper subgroup with property (K) (Theorem 1; cf. also Theorem 1'). In Section 2 we prove that every group with property (K) is a Baire space (Theorem 2), which seems to indicate that the former property is a good substitute for completeness. On the other hand, we construct metrizable Baire linear spaces without property (K) (Theorem 3).

1. Existence of dense subgroups with property (K). We start with some purely algebraic considerations. Let Z be an uncountable Abelian group and denote by $\mathfrak{z}$ the cardinality of Z. We assume that Z satisfies the following condition:

$(*)$ There exists $\mathfrak{z}_0 < \mathfrak{z}$ such that for all $n \in N$ and $z \in Z$ with $z \neq 0$ we have $\operatorname{card}\{x \in Z\colon nx = z\} \leqslant \mathfrak{z}_0$.

Note that (*) holds if the subgroup of all elements of Z of finite order has cardinality less than $\mathfrak{z}$ or if all non-zero elements of Z are of the same order.

Given $E \subset Z$, we denote by $\langle E \rangle$ the subgroup of Z generated by E. With this notation we can formulate

LEMMA 1. *Let $(S_\alpha)_{\alpha<\gamma}$, where γ is an initial ordinal with* $\operatorname{card}\gamma \leqslant \mathfrak{z}$, *be a (transfinite) sequence of subsets of Z with* $\operatorname{card} S_\alpha = \mathfrak{z}$. *Then there exists a sequence $(Z_\alpha)_{\alpha<\gamma}$ of subgroups of Z with the following properties*:

(i) $Z_\alpha \cap \langle \bigcup_{\substack{\alpha'<\gamma \\ \alpha' \neq \alpha}} Z_{\alpha'} \rangle = \{0\}$ *for all* $\alpha < \gamma$.

(ii) $Z_\alpha \cap S_\beta \neq \emptyset$ *for all* $\alpha \leqslant \beta < \gamma$.

Proof (cf. [5], proof of Theorem 2, and [6], proof of Theorem 2). Applying (*), it is easy to construct, by transfinite induction, a double sequence $\{x_\alpha^\beta\colon \beta \leqslant \alpha < \gamma\}$ of elements of Z such that for all $\beta \leqslant \alpha < \gamma$

(i)′ $\langle x_\alpha^\beta \rangle \cap \langle \{x_{\alpha'}^{\beta'}\colon \beta' \leqslant \alpha' < \alpha \text{ or } \alpha' = \alpha \text{ and } \beta' < \beta\} \rangle = \{0\}$,

(ii)′ $x_\alpha^\beta \in S_\alpha$.

(At the first stage we choose x_0^0, next x_1^0, x_1^1, and so on.) It follows from (i)′ that for any disjoint finite subsets E, F of $\{x_\alpha^\beta\colon \beta \leqslant \alpha < \gamma\}$ we have $\langle E \rangle \cap \langle F \rangle = \{0\}$. Thus, it is enough to put $Z_\alpha = \langle \{x_{\alpha'}^\alpha\colon \alpha \leqslant \alpha' < \gamma\} \rangle$ for all $\alpha < \gamma$.

Conversely, in case $\mathfrak{z}$ is a regular cardinal, e.g., $\mathfrak{z} = \aleph_1$, the assertion of Lemma 1 implies (*). Indeed, if (*) fails for some $n_0 \in N$ and $z_0 \in Z$ with $z_0 \neq 0$, then Lemma 1 fails for $\gamma = 2$ and $S_0 = S_1 = \{x \in Z\colon n_0 x = z_0\}$.

We shall need the following version of Proposition 2 of [6].

PROPOSITION 1. *Let X be a Hausdorff Abelian group and let (x_n) be a sequence in X such that $x_n \neq 0$ and the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent. Then there exists a subsequence (x_{n_k}) with the following property*:

if $\sum_{k=1}^{\infty} \delta_k x_{n_k} = 0$ *and* $(\delta_k) \in \{-1, 0, 1\}^N$, *then* $(\delta_k) = 0$. *In particular,*

$$\operatorname{card}\Big\{\sum_{n=1}^{\infty} \varepsilon_n x_n\colon (\varepsilon_n) \in \{0, 1\}^N\Big\} = 2^{\aleph_0}.$$

Proof. Clearly, it is enough to define $n_1 < n_2 < \ldots$ so that

$$\Big\{\sum_{m=n_k+1}^{\infty} \delta_m x_m\colon \delta_m = -1, 0, 1\Big\} \subset X \setminus \{x_{n_k}\}.$$

Put $n_1 = 1$ and proceed by induction using the property that the series $\sum_{n=1}^{\infty} \delta_n x_n$ are uniformly Cauchy and $X \setminus \{x_n\}$ is a neighbourhood of 0 in X.

Remark 1. The proof above yields, in fact, a partition of N into two increasing sequences (r_k) and (s_k) such that both (x_{r_k}) and (x_{s_k}) satisfy the conclusion of Proposition 1 (cf. [6], Proposition 2).

THEOREM 1 (cf. [6], Theorem 2). *Let X be a nondiscrete metrizable complete Abelian group with* $\operatorname{card} X = 2^{\aleph_0}$ *such that the equation $nx = z$ has (at most) countably many solutions given $n \in N$ and $z \in X$ with $z \neq 0$. Then there exists a sequence $(X_\alpha)_{\alpha<\varphi}$, where φ is the initial ordinal of cardinality $2^{\aleph_0}$, of dense subgroups of X with the following properties:*

(i) $X_\alpha \cap \langle \bigcup_{\substack{\alpha' < \varphi \\ \alpha' \neq \alpha}} X_{\alpha'} \rangle = \{0\}$ *for all* $\alpha < \varphi$.

(ii) *For each sequence (x_n) in X such that $x_n \neq 0$ and the series $\sum_{n=1}^{\infty} x_n$ is subseries convergent there exists a sequence $(\varepsilon_n^\alpha) \in \{0, 1\}^N$ not eventually zero with $\sum_{n=1}^{\infty} \varepsilon_n^\alpha x_n \in X_\alpha$ for all $\alpha < \varphi$.*

In particular, X_α has property (K).

Proof. Let $\mathscr{U}$ be a base for the topology of X with $\operatorname{card} \mathscr{U} = 2^{\aleph_0}$. Denote by $\mathscr{V}$ the family of all sets

$$\Big\{\sum_{n=1}^{\infty} \varepsilon_n x_n \colon (\varepsilon_n) \in \{0, 1\}^N \text{ is not eventually zero}\Big\},$$

where (x_n) satisfies the conditions of (ii). We have

$$\operatorname{card}(\mathscr{U} \cup \mathscr{V}) = 2^{\aleph_0},$$

and so we can arrange $\mathscr{U} \cup \mathscr{V}$ into a sequence $(S_\alpha)_{\alpha<\varphi}$ with each element of $\mathscr{U} \cup \mathscr{V}$ repeated $2^{\aleph_0}$ times. As X is nondiscrete and complete, using Proposition 1, one can easily show that $\operatorname{card} S_\alpha = 2^{\aleph_0}$ for all $\alpha < \varphi$. Hence, in view of Lemma 1, there exist subgroups X_α of X satisfying (i) and $X_\alpha \cap W \neq \emptyset$ for all $W \in \mathscr{U} \cup \mathscr{V}$. It follows that X_α is dense in X and (ii) holds.

Remarks. 2. The completeness assumption in Theorem 1 can be actually replaced by property (K). In this case one has to apply Proposition 1 in the completion of X. That $\operatorname{card} S_\alpha = 2^{\aleph_0}$ then follows by a well-known result of Sierpiński asserting the existence of $2^{\aleph_0}$ almost disjoint (infinite) subsets of N.

3. For $X = R$ the X_α's can be chosen subspaces of R, where R is considered a linear space over the rationals Q, with

$$\bigoplus_{\alpha<\varphi} X_\alpha = R.$$

In this case the proof is closer to that of Theorem 2 of [6] and is based on the equality $\dim \operatorname{lin}_Q E = \operatorname{card} E$ for every uncountable subset E of R.

Reverting to the beginning of this section, we establish a general result on existence of subgroups satisfying condition $(*)$.

LEMMA 2. *Every uncountable Abelian group Z contains an uncountable subgroup W satisfying condition $(*)$ with $\mathfrak{z}_0 = \aleph_0$. If Z is, moreover, equipped with a group topology, then W can be chosen closed.*

Proof (cf. [2], proof of Lemma 3). Put

$$Z[n] = \{x \in Z\colon\ nx = 0\}.$$

If $\operatorname{card} Z[n] \leqslant \aleph_0$ for all $n \in N$, then Z itself is as desired. In the other case, denote by n_0 the first $n \in N$ with $\operatorname{card} Z[n] > \aleph_0$. In order to prove that $W = Z[n_0]$ satisfies $(*)$ with $\mathfrak{z}_0 = \aleph_0$, fix $n \in N$ and $z \in Z$ with $z \neq 0$. Let $nx_1 = z = nx_2$, where $x_1, x_2 \in Z[n_0]$. Then $n \neq n_0$, and so we may assume that $n < n_0$. It follows that $x_1 - x_2 \in \bigcup_{k<n_0} Z[k]$ while $\operatorname{card}(\bigcup_{k<n_0} Z[k]) \leqslant \aleph_0$. This shows that W is as desired.

THEOREM 1′. *Every nondiscrete metrizable complete Abelian group X of cardinality $2^{\aleph_0}$ contains a direct sum of $2^{\aleph_0}$ nonclosed subgroups with property* (K).

Proof. We may assume that X is separable by choosing an arbitrary nondiscrete countable subset E of X and taking the closed subgroup of X generated by E. Hence the assertion follows by an application of Lemma 2 and Theorem 1.

Lemmas 1 and 2 yield the following improvement of a result recently published by Harazišvili ([2], Lemma 3).

PROPOSITION 2. *Every uncountable Abelian group contains a direct sum of $\aleph_1$ subgroups each having cardinality $\aleph_1$.*

Let us note that this last proposition can also be obtained by analyzing the proof of a related result of Scott ([7], Theorem 9).

2. Property (K) and Baire category. Recall that a topological space X is named a *Baire space* provided for every sequence (U_n) of dense open subsets of X the set $\bigcap_{n=1}^{\infty} U_n$ is dense in X (cf. [4], Theorem 1.13). As easily seen, for X being a topological group it is enough to require that the neutral element of X is in the closure of $\bigcap_{n=1}^{\infty} U_n$. In view of a classical theorem of Banach ([4], Theorem 1.6), the latter condition can further be weakened to $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$ or, in other words, to X being of second category. The result just mentioned will not be used in the sequel in its full generality although its application would simplify slightly the proof of Theorem 2 below. For X being a topological linear space, however, this result is straightforward and will be used in the proof of Theorem 3 in the sequel.

THEOREM 2. *Every group X with property* (K) *is a Baire space.*

Proof. Let $|\cdot|$ be a norm in X. Fix a decreasing sequence (U_n) of dense open subsets of X. We shall prove that 0 is in the closure of $\bigcap_{n=1}^{\infty} U_n$. To this end, given $\delta > 0$, we shall construct a sequence (x_n) in X and a sequence (V_n) of open subsets of X with the following properties:

(a) $|x_n| < \delta \cdot 2^{-n}$,

(b) $\bar{V}_n \subset U_n$,

(c) $\sum_{i=1}^{n} \varepsilon_i x_i \in \bigcap_{i=1}^{n} V_i^{\varepsilon_i}$ for all $\varepsilon_i = 0, 1$,

where, by definition, $A^0 = X$ and $A^1 = A$ for every subset A of X.

For $n = 1$ choose $x_1 \in U_1$ with $|x_1| < \delta \cdot 2^{-1}$. Next choose an open set V_1 such that $x_1 \in V_1$ and $\bar{V}_1 \subset U_1$. Suppose now the inductive construction has been carried out up to some n. Then

$$V = \bigcap_{\varepsilon_i = 0,1} \Big(\Big(\bigcap_{i=1}^{n} V_i^{\varepsilon_i}\Big) - \Big(\sum_{i=1}^{n} \varepsilon_i x_i\Big)\Big)$$

is, in view of (c), a neighbourhood of 0 in X. Moreover,

$$U = \bigcap_{\varepsilon_i = 0,1} \Big(U_{n+1} - \sum_{i=1}^{n} \varepsilon_i x_i\Big)$$

is a dense open subset of X. It follows that there exists $x_{n+1} \in U \cap V$ with $|x_{n+1}| < \delta \cdot 2^{-(n+1)}$. Let U' be an open subset of X with $x_{n+1} \in U'$ and $\bar{U}' \subset U \cap V$. Put

$$V_{n+1} = \bigcup_{\varepsilon_i = 0,1} \Big(U' + \sum_{i=1}^{n} \varepsilon_i x_i\Big).$$

From $\bar{U}' \subset U$ we infer that $\bar{U}' + \sum_{i=1}^{n} \varepsilon_i x_i \subset U_{n+1}$ for $\varepsilon_i = 0, 1$. Hence (b) holds for $n+1$.

It follows from $x_{n+1} \in U' \subset V$ and the definition of V that

$$\sum_{i=1}^{n} \varepsilon_i x_i + x_{n+1} \in \bigcap_{i=1}^{n} V_i^{\varepsilon_i} \cap V_{n+1}$$

for $\varepsilon_i = 0, 1$. This together with the inductive hypothesis proves (c) for $n+1$.

In view of (K) and (a), there exists a subsequence (x_{n_k}) such that the series $\sum_{k=1}^{\infty} x_{n_k}$ converges to some $x \in X$. We claim that $|x| < \delta$ and $x \in \bigcap_{n=1}^{\infty} U_n$. The first assertion follows immediately from (a). By (c), $\sum_{k=1}^{m} x_{n_k} \in V_{n_i}$ for all i $(1 \leqslant i \leqslant m)$. Hence $x \in \bigcap_{k=1}^{\infty} \bar{V}_{n_k}$, and so, in virtue of (b), $x \in \bigcap_{k=1}^{\infty} U_{n_k} = \bigcap_{n=1}^{\infty} U_n$. Since $\delta > 0$ was arbitrary, the proof is complete.

Theorem 2 and Pettis' lemma ([1], Theorem 5.1) yield, in view of [4], Proposition 1.3, the following

COROLLARY. *Let X be a metrizable Abelian group which has the Baire property in its completion. Then X has property* (K) *if and only if X is complete.*

The converse of Theorem 2 fails in the following general sense.

THEOREM 3. *Every infinite-dimensional F-space X contains a subspace without property* (K), *which is a Baire space.*

Proof. We shall construct an increasing sequence (X_m) of subspaces of X without property (K) such that

$$X = \bigcup_{m=1}^{\infty} X_m.$$

Then, by Baire's theorem, X_{m_0} is of second category in X for some m_0. Hence, in view of [4], Proposition 1.3, X_{m_0} is of second category in itself, and so it is a Baire space.

In order to construct the X_m's we shall make use of a sequence (x_n) in X with the following two properties:

(a) $\sum_{n=1}^{\infty} x_n$ is subseries convergent.

(b) For each bounded sequence (λ_n) of scalars such that $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ we have $(\lambda_n) = 0$.

(The existence of such a sequence follows from, e.g., Theorem 1 of [6].)

Let $\{A_m\colon m \in N\}$ be a partition of N into infinite sets. Put

$$Y_m = \operatorname{lin}\Big\{\sum_{n=1}^{\infty} 1_A(n)\, x_n\colon A \subset \bigcup_{i=1}^{m} A_i \text{ or } A \text{ is finite}\Big\}$$

and

$$Y = \bigcup_{m=1}^{\infty} Y_m.$$

(The definitions make sense in view of (a).) Let Z be an algebraic complement of Y in X, and put $X_m = Y_m \oplus Z$. Then, clearly, $\bigcup_{m=1}^{\infty} X_m = X$. Moreover, in view of (b) and the definition of the Y_m's, we have

$$\sum_{n=1}^{\infty} 1_A(n)\, x_n \in Y_{m+1} \setminus Y_m$$

for each infinite $A \subset A_{m+1}$. Hence, taking into account that $X_m \cap Y_{m+1} = Y_m$, we get

$$\sum_{n=1}^{\infty} 1_A(n)\, x_n \notin X_m.$$

Thus X_m does not have property (K).

In connection with Theorem 3, we note that, apparently, the first example of a noncomplete normed Baire space is due to Hausdorff ([3], p. 303).

Since a topological space which contains a dense Baire space is itself a Baire space ([4], Theorem 1.15), Theorems 2 and 3 (along with Theorem 2 of [6]) suggest the following

PROBLEM. Does every metrizable Baire linear space contain a dense linear subspace with property (K)? **(P 1279)**

Postscript. We have recently learned that property (K) had already been isolated, but not given a name, by S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires (II)*, Studia Mathematica 13 (1953), p. 137-179; 2.83 (see also A. Alexiewicz, *On sequences of operations (II)*, ibidem 11 (1950), p. 200-236; postulate (a_2'), p. 203).

For some results related to the subject of the present paper see Z. Lipecki, *On some dense subspaces of topological linear spaces*, Studia Mathematica 77, in print.

REFERENCES

[1] J. P. R. Christensen, *Topology and Borel structure*, Amsterdam 1974.

[2] A. B. Harazišvili, *Absolutely nonmeasurable sets in Abelian groups* [in Russian, with an English summary], Soobščenija Akademii Nauk Gruzinskoĭ SSR 97 (1980), p. 537-540.

[3] F. Hausdorff, *Zur Theorie der linearen metrischen Räume*, Journal für die reine und angewandte Mathematik 167 (1932), p. 294-311.

[4] R. C. Haworth and R. A. McCoy, *Baire spaces*, Dissertationes Mathematicae 141, Warszawa 1977.

[5] C. Kliś, *An example of noncomplete normed (K)-space*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 26 (1978), p. 415-420.

[6] I. Labuda and Z. Lipecki, *On subseries convergent series and m-quasi-bases in topological linear spaces*, Manuscripta Mathematica 38 (1982), p. 87-98.

[7] W. R. Scott, *Groups and cardinal numbers*, American Journal of Mathematics 74 (1952), p. 187-197.

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