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EPICS IN THE CATEGORY OF T_2 k-GROUPS NEED NOT HAVE DENSE RANGE

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Introduction. In a category C, an *epic* is a morphism $g\colon X \to Y$ such that for each pair of morphisms $\alpha, \beta\colon Y \to Z$ in C, whenever $\alpha g = \beta g$, then $\alpha = \beta$. In the category of sets epics are surjective functions; in topological spaces they are continuous surjections; and in T_2 topological spaces they are continuous functions with dense range. Also, in both groups and abelian groups, as well as topological groups and abelian topological groups, morphisms which are surjective on the underlying sets are precisely the epics. In T_2 abelian topological groups, the epics have dense range, and, as shown in [1], this is also the case for both the category of weakly Hausdorff k-groups and the category of weakly Hausdorff abelian k-groups. In [3] Poguntke proved that in the category of compact groups epics must be surjective, which here is equivalent to their having dense range.

For T_2 topological groups, the question is still unanswered. And from all the evidence, one would suspect, as above, that epics here must also have dense range. Unfortunately, none of the techniques used to identify epics in these categories have proved fruitful when applied to T_2 topological groups. Indeed, in this instance epics may not have dense range; and, depending on one's point of view, the example provided in this paper, namely, that epics in the category of T_2 k-groups need not have dense range, lends credence to this belief. However, one must not infer too much from this result on the situation of T_2 topological groups, since the same example with only slight modification shows that epics in the category of T_2 abelian k-groups need not have dense range — hardly analogous with the situation of T_2 abelian topological groups.

Preliminaries. For a topological space X, $U \subset X$ is said to be *k-open* if for every continuous $f\colon T \to X$ with T compact (including T_2), $f^{-1}(U)$ is open. We let kX denote the space with the same underlying set as X and the topology consisting of all k-open subsets.

Clearly, the identity $kX \to X$ is continuous, and, if this is an isomorphism, X is called a *k-space*. For X being T_2, U is k-open if and only if $U \cap C$ is open in C for each compact C in X. For information concerning T_2 k-spaces see [4]; for information on non-separated k-spaces as well see [1]. Common examples of k-spaces include spaces that are first countable or locally compact.

Given two k-spaces X and Y, the k-product is given by $X \times Y = k[X \times_c Y]$, where $X \times_c Y$ is the usual topological product. In general, the k-product and the topological product are not topologically isomorphic; however, they are the same when either X or Y is locally compact or when both X and Y are first countable.

In line with X being T_2 when the diagonal is closed in $X \times_c X$, a k-space is said to be *weakly Hausdorff* (t_2) when the diagonal is closed in $X \times X$ (see [1] and [2]). As with T_2 k-spaces, U is open in a t_2 k-space if $U \cap C$ is open in C for each compact subset C.

A *k-group* is a group provided with a k-topology such that inversion is continuous and such that multiplication is continuous on the k-product (as compared with the topological product for topological groups). If H is a topological group, its k-refinement kH is a k-group; however, every k-group is not the k-refinement of some topological group. By a straightforward verification, one has

PROPOSITION 1. *A k-group G is T_0 if and only if G is t_2.*

Unlike topological groups, though, a T_0 k-group need not be T_2 [1]. The following is Theorem 2.12 of [1]:

THEOREM 1. *For a k-space X, the free group FX can be provided with a k-group topology, such that when X is t_2, FX is t_2 and X is topologically embedded in FX as a closed subspace.*

A space X is said to be *functionally* T_2 if the continuous functions into the unit interval separate points. A k-space X is said to be *diagonally separable* if for any (x, y) not in the diagonal $\Delta(X)$, there exist disjoint open sets U and V in $X \times X$ with $(x, y) \in U$ and $\Delta(X) \subset V$. A k-space is diagonally separable if it is functionally T_2, but the converse is not valid. By 2.26 of [1] we have

THEOREM 2. *A k-space X is functionally T_2 if and only if FX is functionally T_2.*

Certainly functionally T_2 spaces are T_2, with the converse being false. However, it is still open whether or not these two concepts coincide for k-groups. By a simple observation we have

PROPOSITION 2. *A k-group G is T_2 if and only if G is diagonally separable.*

Thus the category of T_2 k-groups is the same as the category of diagonally separable k-groups. And showing that epics need not have dense range in the category of T_2 k-groups is equivalent to showing it in the category of diagonally separable k-groups, which we shall do. The following is 2.48 of [1]:

THEOREM 3. *Epics in the category of t_2 k-groups are precisely those morphisms with dense range.*

Regarding epics in t_2 k-spaces and their preservation and reflection by the free functor we have the following two propositions:

PROPOSITION 3. *Epics in the category of t_2 k-spaces are precisely those morphisms with dense range.*

Proof. Clearly, continuous functions with dense range are epics. Suppose, now, that $g\colon X \to Y$ is an epic. If $\overline{g(X)} \neq Y$, let Z be the quotient of Y with $\overline{g(X)}$ collapsed to a point z_0. One checks that Z is a t_2 k-space. If $\alpha\colon Y \to Z$ is the canonical quotient and $\beta\colon Y \to Z$ is the constant map with $\beta(Y) = z_0$, then $\alpha \circ g = \beta \circ g$. This, however, is impossible since $\alpha \neq \beta$. Therefore $\overline{g(X)} = Y$.

PROPOSITION 4. *A continuous $g\colon X \to Y$ is an epic in the category of t_2 k-spaces if and only if $Fg\colon FX \to FY$ is an epic in the category of t_2 k-groups.*

Proof. Since the functor F is a left adjoint, it preserves epics. Conversely, suppose that $Fg\colon FX \to FY$ is an epic. If $\alpha, \beta\colon Y \to Z$ in t_2 k-spaces with $\alpha \circ g = \beta \circ g$, then $F\alpha \circ Fg = F(\alpha \circ g) = F(\beta \circ g) = F\beta \circ Fg$; therefore, since Fg is an epic, $F\beta = F\alpha$. But this requires that $\alpha = \beta$, since $\alpha = F\alpha | Y$ and $\beta = F\beta | Y$ for $F\beta | Y$ denoting the restriction of $F\beta$ to Y.

Example. To produce an example of an epic in T_2 k-groups (i.e., in diagonally separable k-groups) with non-dense range, we proceed as follows: We first find an epic $g\colon X \to Y$ in both the category of diagonally separable k-spaces and the category of functionally T_2 k-spaces which has non-dense range. We next note that this prohibits $Fg\colon FX \to FY$ from being an epic in t_2 k-groups (Propositions 3 and 4) and, therefore, requires that Fg has non-dense range (Theorem 3). We conclude by showing that Fg is an epic in diagonally separable k-groups (hence in functionally T_2 k-groups as well).

Proceeding with this, let R denote the reals with their usual topology, and let Y denote the reals with the topology in which each point r in Y has basic open neighborhoods of the form

$$N_\varepsilon(r) = \{r\} \cup \{y \mid y \text{ is irrational and } |y - r| < \varepsilon\}, \qquad \varepsilon > 0.$$

Clearly, the identity $Y \to R$ is continuous. Therefore, since R is functionally T_2, Y is functionally T_2; also, Y is first countable, consequently a k-space. Letting X denote the rationals, X is closed in Y, and we have

PROPOSITION 5. *The inclusion $X \to Y$ is an epic in the category of diagonally separable k-spaces and, consequently, also in functionally T_2 k-spaces.*

Proof. Suppose that α, β: $Y \to Z$ in the category of diagonally separable k-spaces with $\alpha|X = \beta|X$. Define φ: $Y \to Z \times Z$ by $\varphi(y) = (\alpha(y), \beta(y))$; then φ is continuous and $\varphi(X) \subset \Delta(Z)$. If $\alpha(y) \neq \beta(y)$ for some $y \in Y$, then $\varphi(y) \notin \Delta(Z)$ and there exist disjoint open sets U and V in $Z \times Z$ with $\varphi(y) \in U$ and $\varphi(X) \subset V$; consequently, $y \in \varphi^{-1}(U)$ and $X \subset \varphi^{-1}(V)$. This, however, is impossible, since the only open set in Y containing X is Y; thus, $\alpha = \beta$.

PROPOSITION 6. *The image of FX is non-dense in FY.*

Proof. Since X is non-dense in Y, $X \to Y$ is not an epic in t_2 k-spaces (Proposition 3); therefore, $FX \to FY$ is not an epic in t_2 k-groups (Proposition 4). Invoking Theorem 3 we are finished.

All that remains in establishing the example is the following proposition:

PROPOSITION 7. *The morphism $FX \to FY$ is an epic in both the category of diagonally separable k-groups and the category of functionally T_2 k-groups.*

Proof. Note that FX and FY are functionally T_2 since X and Y are. Assume now that $\alpha, \beta: FY \to G$ are k-group morphisms with G a diagonally separable k-group such that $\alpha|FX = \beta|FX$ (where FX is continuously injected into FY). If $\alpha' = \alpha|Y$ and $\beta' = \beta|Y$, then α', β': $Y \to G$ with $\alpha'|X = \beta'|X$, which, since $X \to Y$ is an epic in diagonally separable k-spaces, implies $\alpha' = \beta'$. But $\alpha|Y = \beta|Y$ requires that $\alpha = \beta$ since Y algebraically generates FY; therefore, $FX \to FY$ is an epic in diagonally separable k-groups. That it is also an epic in functionally T_2 k-groups follows immediately from the fact that any functionally T_2 k-space is also diagonally separable.

Using the free abelian k-group instead of the free k-group, one produces in complete analogy an epic in the category of T_2 abelian k-groups with non-dense range. This, of course, is in marked contrast with T_2 abelian topological groups where epics trivially have dense range. But to one familiar with "k-objects", this is not surprising, since t_2 k-groups, instead of T_2 k-groups, appear to be the proper categorical equivalent of T_2 topological groups.

In concluding we should note why the method used to produce this example will not carry over to T_2 topological groups. For any functionally T_2 space X, the free topological group $F_T X$ generated by X is T_2 (however X is only embedded as a closed subspace when X is completely regular). Thus for any epic g: $X \to Y$ in functionally T_2 spaces, since the functor F_T is a left adjoint, $F_T g$: $F_T X \to F_T Y$ is an epic in the category of T_2 topological groups. Denote by H the image of $F_T X$ in $F_T Y$. If $\bar{H} \neq F_T Y$, then there exists y_0 in Y but not in $\bar{H}$. But $F_T Y$ is completely regular; therefore, there is a continuous f: $F_T Y \to R$ with $f(\bar{H}) = 0$ and $f(y_0) = 1$. Letting α: $Y \to R$ be the restriction of f to Y and letting β: $Y \to R$ be the constant map with $\beta(Y) = 0$, we have $\alpha \circ g = \beta \circ g$, which contradicts g being an epic in functionally T_2 spaces. Consequently, $\bar{H} = F_T Y$.

REFERENCES

[1] W. F. LaMartin, *On the foundations of k-group theory*, Dissertationes Mathematicae (to appear).

[2] M. McCord, *Classifying spaces and infinite symmetric products*, Transactions of the American Mathematical Society 146 (1970), p. 273-298.

[3] D. Poguntke, *Epimorphisms of compact groups are onto*, Proceedings of the American Mathematical Society 26 (1969), p. 503-504.

[4] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Mathematical Journal 14 (1967), p. 133-152.

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