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ON THE STRUCTURE OF INDECOMPOSABLE MODULES OVER ARTIN ALGEBRAS

BY

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When one studies the finitely generated modules over an Artin ring, one examines basically the indecomposable ones. In the last few years there were many new insights into the structure of the indecomposable modules over Artinian rings (see [2], [3], [6]-[9], [11], and especially [10] for fuller bibliography). In particular, the classes of indecomposable modules, called *modules with waists, cores, cocores,* and *s-indecomposable modules,* were introduced and studied in [2], [10], [11]. In this paper we introduce new classes of indecomposable modules which extend essentially a hierarchy of those defined in [11]. Namely, for a given positive integer s, we define a class of indecomposable modules which we call $(s+\frac{1}{2})$-*indecomposable modules.* Dually, we define also classes of $-(s+\frac{1}{2})$-indecomposable modules (see Section 1). The classes of $\frac{3}{2}$-indecomposable modules and $-\frac{3}{2}$-indecomposable modules are larger than previously known fundamental classes of indecomposable ones. One of the aims of this paper is to prove that if every indecomposable injective module over a left Artin ring has finite length and every indecomposable module of finite length is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable, then the ring is of *finite representation type,* i.e., it has only a finite number of nonisomorphic finitely generated indecomposable modules. Moreover, we give a complete classification of radical squared zero Artin algebras such that every indecomposable module is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable. Recall that an *Artin algebra* is an Artin ring which is a finitely generated module over its center, being a commutative Artin ring.

The paper is divided into four sections. In Section 1, for a given positive integer s we define the $(s+\frac{1}{2})$-indecomposability and $-(s+\frac{1}{2})$-indecomposability of modules and we show how these concepts are related with the previously known ones. Moreover, we give examples showing that the classes of indecomposable modules introduced here and in [11] are in fact distinct. In Section 2 we show that if indecomposable injective modules over a left Artin ring have finite length, then the lengths of $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable modules are bounded. Hence we obtain the result stated above. Section 3 is devoted to the study of $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable modules over Artin algebras with radical square zero. Applying the technique of Auslander-Reiten sequences [3] we prove some criteria for the existence of such modules. Section 4 contains a classification of radical squared zero Artin algebras such that every indecomposable module is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable. Some consequences of this classification are also stated.

Throughout the paper, rings are assumed to have an identity element, modules are unitary left modules, and indecomposable modules are nonzero. Moreover, for a ring R, R^{op} denotes the opposite ring and mod R denotes the category of finitely generated R-modules.

1. Classes of indecomposable modules. We begin by recalling a hierarchy of classes of indecomposable modules due to Green [11].

Let R be a ring and let M be an R-module. A submodule N of M is called *nonsuperfluous* if there exists a proper submodule N' of M such that $N+N' = M$ (see [5]). Dually, a submodule Q of M is called *nonessential* if there exists a proper submodule Q' of M such that $Q \cap Q' = 0$. Let s be a positive integer. We say that M is *s-indecomposable* if the intersection of any its $s+1$ nonsuperfluous submodules $N_1, \ldots, N_{s+1}$ such that

$$\sum_{i=1}^{s+1} N_i = M$$

is nonzero. We say that M is *∞-indecomposable* if M is s-indecomposable for all $s \geqslant 1$. Moreover, we say that M has a *core* if the intersection of all nonsuperfluous submodules of M is nonzero. These definitions have the following duals which, in general, lead to a distinct hierarchy of classes of indecomposable modules. We say that M is *$-s$-indecomposable* if the sum of any its $s+1$ nonessential submodules $Q_1, \ldots, Q_{s+1}$ such that

$$\bigcap_{i=1}^{s+1} Q_i = 0$$

is not M. We say that M is *$-\infty$-indecomposable* if M is $-s$-indecomposable for all $s \geqslant 1$. Finally, we say that M has a *cocore* if the sum of all nonessential submodules of M is not M. Modules with cores and cocores were investigated in [10].

We introduce new classes of indecomposable modules which, in general, extend essentially the above hierarchy.

Definition 1.1. Let R be a ring and let M be an R-module. Let s be a positive integer. We say that M is *$(s+\frac{1}{2})$-indecomposable* if the intersection of any its $s+1$ nonsuperfluous submodules is nonzero. Dually, we say that M is

$-(s+\frac{1}{2})$-*indecomposable* if the sum of any its $s+1$ nonessential submodules is not M.

From the definitions we obtain easily the following proposition which shows how these concepts and the above ones are interrelated.

PROPOSITION 1.1. *Let R be a ring and let M be an R-module. Then:*

(i) *M is indecomposable iff M is 1-indecomposable.*

(ii) *If M is $(s+1)$-indecomposable, $s \geqslant 1$, then M is $(s+\frac{1}{2})$-indecomposable.*

(iii) *If M is $(s+\frac{1}{2})$-indecomposable, $s \geqslant 1$, then M is s-indecomposable.*

(iv) *If M has a core, then M is ∞-indecomposable.*

(v) *M is indecomposable iff M is -1-indecomposable.*

(vi) *If M is $-(s+1)$-indecomposable, $s \geqslant 1$, then M is $-(s+\frac{1}{2})$-indecomposable.*

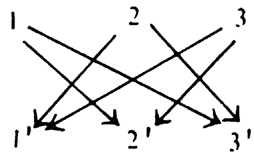
(vii) *If M is $-(s+\frac{1}{2})$-indecomposable, $s \geqslant 1$, then M is $-s$-indecomposable.*

(viii) *If M has a cocore, then M is $-\infty$-indecomposable.*

We end this section by showing that the above classes of indecomposable modules are distinct.

Example 1.1. *Let s be a positive integer. There is a radical squared zero Artin algebra having an indecomposable module which is $(s+\frac{1}{2})$-indecomposable and $-(s+\frac{1}{2})$-indecomposable but neither $(s+1)$-indecomposable nor $-(s+1)$-indecomposable.*

Proof. Consider the quiver Q (in the sense of [8], [9]) with $2s+4$ vertices, numbered $1, \ldots, s+2, 1', \ldots, (s+2)'$, having exactly one arrow from vertex i to vertex j' provided $i \neq j$ (cf. [11], p. 376). For example, for $s = 1$, Q is of the form



Denote by R the associated tensor k-algebra for Q, where k is a fixed commutative field. We know [6] that R is a hereditary Artin algebra with radical square zero and there exists an isomorphism of the category of R-modules and the category of representations of Q. Let us consider the representation V of Q obtained by putting the same one-dimensional vector k-space at each vertex and letting each morphism be the identity. Let M be the R-module associated with the representation V and let P_i (resp. P_i') be the projective R-module associated with the i-th (resp. i'-th) vertex. It is not difficult to see that

$$M = \sum_{i=1}^{s+2} N_i,$$

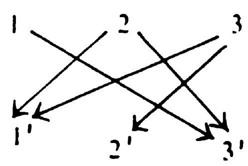
where N_i is isomorphic to P_i. Hence the N_i are nonsuperfluous submodules of M and it is easy to verify that each nonsuperfluous submodule of M contains some N_i. But

$$N_i \cap N_j = \bigoplus_{t \neq i,j} P'_t \quad \text{for } i \neq j.$$

Consequently, $\bigcap_{i=1}^{s+2} N_i = 0$ and we see that M satisfies the required conditions.

Example 1.2. *Let s be a positive integer. There is a radical squared zero Artin algebra having an indecomposable module which is s-indecomposable and $-s$-indecomposable but neither $(s+\frac{1}{2})$-indecomposable nor $-(s+\frac{1}{2})$-indecomposable.*

Proof. Consider the quiver S which arises from the quiver Q by the omission of the arrow from 1 to 2′. For example, for $s = 1$, Q is of the form



Let k be a commutative field. Let R be the associated k-algebra for S and let L be the R-module associated with the representation W of S obtained by putting the same one-dimensional vector k-space at each vertex and letting each morphism be the identity. Further, let P_i (resp. P'_i) be the projective R-module associated with the i-th (resp. i'-th) vertex of S. We claim that L has the required properties. First observe that

$$L = \sum_{i=1}^{s+2} L_i,$$

where L_i is isomorphic to P_i, and each nonsuperfluous submodule of L contains some module L_i. Observe also that $P'_i \cap L_i = 0$, $i = 1, \ldots, s+2$, so L_i is nonessential in L. It is easy to see that every nonessential submodule of L is contained in $L_1 + L_2$ or in some L_i, $i = 3, \ldots, s+2$. Moreover,

$$L_i \cap L_j = \bigoplus_{t \neq 1,2,j} P'_t \quad \text{for } j \neq 1, 2$$

and

$$L_i \cap L_j = \bigoplus_{r \neq i,j} P'_r \quad \text{for } i, j \geqslant 2 \text{ and } i \neq j.$$

Hence

$$\bigcap_{i \neq 2} L_i = 0,$$

and if $U_1, \ldots, U_{s+1}$ are nonsuperfluous submodules of L such that

$$\sum_{i=1}^{s+1} U_i = L,$$

then L_2 is contained in some U_i and, consequently,

$$\bigcap_{i=1}^{s+1} U_i \neq 0.$$

Thus L is s-indecomposable but not $(s+\frac{1}{2})$-indecomposable. On the other hand,

$$(L_1+L_2)\cap P_2' = 0, \quad (L_1+L_2)+\sum_{i=3}^{s+2} L_i = L,$$

and

$$\sum_{i \neq j} L_i \neq L \quad \text{for any } j = 1, \ldots, s+2,$$

and we see that L is $-s$-indecomposable but not $-(s+\frac{1}{2})$-indecomposable.

2. Rings of finite representation type. First, we fix some notation. Let R be an Artin ring with radical r and let M be an R-module. We denote the length of M by $l(M)$, the projective cover by $P(M)$, and the injective envelope by $E(M)$. The top of M, denoted by top M, is M/rM. Further, we denote the socle of M by soc M. If R is an Artin algebra, then there is a duality D between left and right finitely generated R-modules defined as follows. If C is the center of R and E is the C-injective envelope of top C, then $D(X) = \mathrm{Hom}_C(X, E)$, X being either a left or right R-module [3].

PROPOSITION 2.1. *Let R be a left Artin ring and let M be an R-module. Assume that each indecomposable injective R-module has finite length. Then*

(i) *if M is $\frac{3}{2}$-indecomposable, then*

$$l(M) \leqslant \max\{l(E)\} \max\{l(P)\}^2,$$

(ii) *if M is $-\frac{3}{2}$-indecomposable, then*

$$l(M) \leqslant \max\{l(P)\} \max\{l(E)\}^2,$$

where the maximum is taken over the indecomposable injective (resp. projective) R-modules E (resp. P).

Proof. Assume that M is $\frac{3}{2}$-indecomposable and let $P(M) \xrightarrow{f} M$ be the projective cover for M. Further, choose an indecomposable summand Q of $P(M)$ and put $N = f(Q)$, $S = \mathrm{soc}\, N$. Then there exists a map $g: M \to E(S)$ such that the diagram

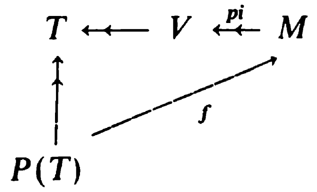
$$\begin{array}{ccccc} S & \subset\!\!\longrightarrow & N & \subset\!\!\longrightarrow & M \\ \cap\!\!\downarrow & & & \swarrow_{g} & \\ E(S) & & & & \end{array}$$

is commutative. Since N is nonsuperfluous in M, and S is essential in N, the $\frac{3}{2}$-indecomposability of M implies that the intersections of nonsuperfluous submodules of M with S are nonzero. Consequently, the kernel of g is superfluous in M and, by [10], Corollary 2.2 and Lemma 7.1, M and $X = g(M)$ have isomorphic tops. Hence

$$\begin{aligned} l(M) &\leqslant l(P(M)) \leqslant l(P(X)) \leqslant l(E(S)) \max \{l(P)\} \\ &\leqslant \max \{l(E)\} \max \{l(P)\} \, l(Q) \leqslant \max \{l(E)\} \max \{l(P)\}^2, \end{aligned}$$

which proves (i).

Now assume that M is $-\frac{3}{2}$-indecomposable and let $M \overset{i}{\hookrightarrow} E(M)$ be the injective envelope for M. Let L be an indecomposable summand of $E(M)$ and put $V = pi(M)$, $T = \operatorname{top} V$, where p denotes the canonical projection $E(M) \to L$. Then there exists a map $f\colon P(T) \to M$ such that the diagram



is commutative. We claim that $W = f(P(T))$ is an essential submodule of M. Indeed, since $W + \ker pi = M$ and $\ker pi$ is nonessential in M, the $-\frac{3}{2}$-indecomposability of M implies that W is essential in M. Then

$$\begin{aligned} l(M) &\leqslant l(E(M)) \leqslant l(E(W)) \leqslant l(P(T)) \max \{l(E)\} \\ &\leqslant \max \{l(P)\} \, l(T) \max \{l(E)\} \leqslant \max \{l(P)\} \max \{l(E)\}^2. \end{aligned}$$

Thus the proof of the proposition is complete.

We get the following consequences of this result:

Corollary 2.1. *If every indecomposable injective module over a left Artin ring has finite length and every indecomposable module of finite length is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable, then the ring is of finite representation type.*

Proof. By Proposition 2.1 there is a bound on the length of indecomposable modules of finite length. Thus the result follows from [1], Theorem 3.1.

Corollary 2.2. *If R is an Artin algebra such that every indecomposable module of finite length is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable, then R is of finite representation type.*

Proof. The result follows from Corollary 2.1 and from the fact that the duality D preserves length, and if E is indecomposable left injective, then $D(E)$ is indecomposable right projective.

COROLLARY 2.3. *If every indecomposable module over a left Artin ring of left pure semisimple type* ([13], [14]) *is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable, then the ring is of finite representation type.*

3. Auslander-Reiten sequences. In this section we restrict our attention to the study of $\frac{3}{2}$-indecomposable modules and $-\frac{3}{2}$-indecomposable modules over Artin algebras with radical square zero. Unless otherwise stated, R will denote an Artin algebra and r its radical.

Recall that a nonsplit exact sequence of finitely generated R-modules $0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$ is called an *Auslander-Reiten sequence* if A and C are indecomposable and, given any morphism $h: X \to C$ which is not a splittable epimorphism, there is some $g: X \to B$ such that $pg = h$ (see [3]). In [3], Proposition 4.3, the existence and uniqueness (up to isomorphism) of an Auslander-Reiten sequence $0 \to A \to B \to C \to 0$ are shown for a given indecomposable finitely generated nonprojective R-module C or an indecomposable finitely generated noninjective R-module A.

We begin with some general observations. One may easily check that if M is a finitely generated R-module, then M is $\frac{3}{2}$-indecomposable if and only if $D(M)$ is a $-\frac{3}{2}$-indecomposable R^{op}-module. Moreover, it is clear that modules with simple tops and modules with simple socles are $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable.

LEMMA 3.1. *Let R be a left Artin ring. Then every indecomposable R-module of length at most* 4 *is $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable.*

Proof. Let M be an indecomposable R-module and let $l(M) \leqslant 4$. By the above remark it suffices to prove the required properties for M with nonsimple top and socle. In this case, $l(\text{top } M) = 2$ and $l(\text{soc } M) = 2$. Then, since M is indecomposable, nonsuperfluous submodules of M are nonsimple and, consequently, M is $\frac{3}{2}$-indecomposable. On the other hand, it is clear that nonessential submodules of M have length at most 2 and from the indecomposability of M we conclude its $-\frac{3}{2}$-indecomposability.

LEMMA 3.2. *Let R be a left Artin ring and let*

$$0 \to M \xrightarrow{j} E_1 \oplus E_2 \xrightarrow{p} S \to 0$$

be a nonsplit exact sequence in mod R *with a simple module S. Suppose that E_1 and E_2 have nonsimple tops. Then M is not $\frac{3}{2}$-indecomposable.*

Proof. Observe that the maps $p_i: E_i \to S$, $i = 1, 2$, induced by p, are proper epimorphisms. Then $V_i = \ker p_i$ is a maximal submodule of E_i. Since E_i has a nonsimple top, there exists a maximal submodule B_i of E_i different from V_i. Put

$$N_1 = j^{-1}(V_1 \oplus 0), \quad N_2 = j^{-1}(0 \oplus V_2),$$
$$L_1 = j^{-1}(B_1 \oplus E_2), \quad L_2 = j^{-1}(E_1 \oplus B_2).$$

It is clear that N_1, N_2, L_1, and L_2 are proper submodules of M. But $V_i + B_i$

$= E_i$, the modules $V_1 \oplus 0$ and $0 \oplus V_2$ are contained in the image of j, so $L_i + N_i = M$ for $i = 1, 2$. Consequently, N_1 and N_2 are nonsuperfluous submodules of M. Since $N_1 \cap N_2 = 0$, M is not $\frac{3}{2}$-indecomposable.

LEMMA 3.3. *Let R be a left Artin ring and let*

$$0 \to M \xrightarrow{j} E_1 \oplus E_2 \oplus E_3 \xrightarrow{p} S \to 0$$

be a nonsplit exact sequence in mod R *with a simple module S. Then M is not $-\frac{3}{2}$-indecomposable.*

Proof. Since p is not splittable, the maps $p_i: E_i \to S$, $i = 1, 2, 3$, induced by p, are proper epimorphisms. Let

$$L = j^{-1}(E_1 \oplus E_2 \oplus 0), \qquad L' = j^{-1}(0 \oplus 0 \oplus E_3),$$
$$N = j^{-1}(0 \oplus E_2 \oplus E_3), \qquad N' = j^{-1}(E_1 \oplus 0 \oplus 0).$$

By the above remark, L, L', N, N' are proper submodules of M and, clearly, $L \cap L' = 0$ and $N \cap N' = 0$. Hence L and N are nonessential in M. Since the p_i are epimorphisms, $L + N = M$. Consequently, M is not $-\frac{3}{2}$-indecomposable.

The following theorem plays an important role in the classification of radical squared zero Artin algebras such that indecomposable modules are either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable.

THEOREM 3.1. *Let R be a radical squared zero Artin algebra and let*

$$0 \to M \xrightarrow{j} E \xrightarrow{p} S \to 0$$

be an Auslander-Reiten sequence in mod R *with a simple module S. Suppose that*

$$E = \bigoplus_{i=1}^{n} E_i$$

is a representation of E as a direct sum of indecomposable modules. Then

(i) *M is $\frac{3}{2}$-indecomposable if and only if one of the following conditions is satisfied:*

(1) *$n = 3$ and E_1, E_2, E_3 have simple tops,*

(2) *$n = 2$ and either E_1 or E_2 has a simple top,*

(3) *$n = 1$;*

(ii) *M is $-\frac{3}{2}$-indecomposable if and only if $n \leq 2$.*

Proof. From Proposition 5.3 (a) in [3] and the result dual to Theorem 5.5 and Proposition 5.7 in [3] it follows that $E = E(P(S))$ (see also [10], Lemma 10.2). Consequently, $E_1, \ldots, E_n$ are indecomposable injective modules.

(i) If M is $\frac{3}{2}$-indecomposable, then by Lemma 3.2 one of the conditions (1)-(3) holds. Therefore, for (i), it remains to show that M is $\frac{3}{2}$-indecomposable if one of the conditions (1)-(3) is satisfied. Assume that (1) is satisfied. Since E_1, E_2, E_3 have simple tops and socles, $l(M) = 5$. Thus, for our aim, it is enough to prove that $l(N) \geqslant 3$ for any nonsuperfluous submodule N of M. Suppose that there exists a nonsuperfluous submodule X of M of length less than or equal to 2. Since M is indecomposable, X is not semisimple, so X has a simple socle and length 2. On the other hand, the modules M and E have the same socle and we conclude that X is injective. But this contradicts the indecomposability of M and we are done. Now suppose that (2) holds and assume that E_1 has a simple top. If E_1 and E_2 are isomorphic, then $l(M) = 3$ and M is $\frac{3}{2}$-indecomposable by Lemma 3.1. Assume that E_1 is not isomorphic to E_2 and put $S_i = \operatorname{soc} E_i$ and $T_i = j^{-1}(S_i)$, $i = 1, 2$. For the $\frac{3}{2}$-indecomposability of M it suffices to show that every nonsuperfluous submodule of M contains T_2. Let N be a nonsuperfluous submodule of M which contains no T_2. Then N contains T_1 and, since semisimple submodules of indecomposable modules are not nonsuperfluous, $l(N) \geqslant 2$. But $l(E_1) = 2$, and then N is not contained in $j^{-1}(E_1 \oplus 0)$. Hence N contains T_2 and we get a contradiction with our assumption. If $n = 1$, then M has a simple socle and is $\frac{3}{2}$-indecomposable.

(ii) If M is $-\frac{3}{2}$-indecomposable, then $n \leqslant 2$ by Lemma 3.3. If $n = 1$, then M is of course $-\frac{3}{2}$-indecomposable. Assume $n = 2$. We will use the duality D. Suppose, for the moment, that M is not $-\frac{3}{2}$-indecomposable. Then $D(M)$ is an indecomposable R^{op}-module which is not $\frac{3}{2}$-indecomposable, and so there exist nonsuperfluous submodules X and Y of $D(M)$ such that $X \cap Y = 0$. Since $\operatorname{top} D(M) \cong D(\operatorname{soc} M)$, we get $l(\operatorname{top} D(M)) = 2$. Further, from the indecomposability of $D(M)$ it follows that X and Y are not semisimple. Hence

$$l(X) = l(\operatorname{soc} X) + 1 \quad \text{and} \quad l(Y) = l(\operatorname{soc} Y) + 1.$$

On the other hand, $X \cap Y = 0$, so there exists a semisimple submodule T of $D(M)$ such that

$$\operatorname{soc} D(M) = (\operatorname{soc} X) \oplus (\operatorname{soc} Y) \oplus T.$$

Then $T \cap (X \oplus Y) = 0$ and we obtain

$$l(X \oplus Y \oplus T) = 2 + l(\operatorname{soc} X) + l(\operatorname{soc} Y) + l(T) = 2 + l(\operatorname{soc} D(M)) = l(D(M)).$$

Consequently, $D(M) = X \oplus Y \oplus T$, and this contradicts the indecomposability of M. The proof of the theorem is completed.

Applying the duality D we obtain the following result from Theorem 3.1:

THEOREM 3.2. *Let R be a radical squared zero Artin algebra and let*

$$0 \to S \to P \to M \to 0$$

be an Auslander-Reiten sequence in mod R *with a simple module* S. *Suppose that*

$$P = \bigoplus_{i=1}^{n} P_i$$

is a representation of P *as a direct sum of indecomposable modules. Then*

(i) M *is* $\frac{3}{2}$-*indecomposable if and only if* $n \leqslant 2$;

(ii) M *is* $-\frac{3}{2}$-*indecomposable if and only if one of the following conditions is satisfied*:

(1) $n = 3$ *and* P_1, P_2, P_3 *have simple socles,*

(2) $n = 2$ *and either* P_1 *or* P_2 *has a simple socle,*

(3) $n = 1$.

We get the following consequence of Theorems 3.1 and 3.2:

COROLLARY 3.1. *Let* R *be an Artin algebra with radical square zero. Assume that one of the following conditions is satisfied*:

(i) $l(\text{soc } E(P)) \geqslant 3$ *and* $l(\text{top } E(P)) \geqslant 4$ *for some indecomposable projective* R-*module* P;

(ii) $l(\text{top } P(E)) \geqslant 3$ *and* $l(\text{soc } P(E)) \geqslant 4$ *for some indecomposable injective* R-*module* E.

Then there is a finitely generated indecomposable R-*module of length greater than or equal to* 6 *which is neither* $\frac{3}{2}$-*indecomposable nor* $-\frac{3}{2}$-*indecomposable.*

4. Rings with $\frac{3}{2}$-indecomposable modules. In this section we give a classification of radical squared zero Artin algebras such that every indecomposable module is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable.

Throughout this section R will denote a radical squared zero Artin algebra and r its radical. It is well known how to analyze the R-modules: one constructs the hereditary Artin algebra

$$R' = \begin{pmatrix} R/r & 0 \\ r & R/r \end{pmatrix}$$

and two functors α, β: mod $R \to$ mod R' such that

$$\alpha(M) = (M/rM) \oplus rM \qquad \text{and} \qquad \beta(M) = (M/\text{soc } M) \oplus \text{soc } M$$

for any R-module M. Then the finitely generated R'-modules X which are isomorphic to $\alpha(M)$ for some M in mod R are the ones with no simple projective summands. Similarly, the objects in mod R' which are isomorphic to $\beta(M)$ for some M in mod R are the finitely generated R'-modules with no simple injective summands. For details, see [1], [4], and [10]. It is not difficult to see that α and β are isomorphic on the full subcategory of mod R whose objects have no simple summands. Consequently, α induces a one-to-one correspondence between isomorphism classes of indecomposable nonsimple R-modules and isomorphism classes of indecomposable nonsimple R'-modules such that projective modules correspond to projective modules and injective modules correspond to injective modules (see [10], Lemma 15.6). Of course, α and β preserve the length of modules.

In order to state the main theorem of this section, we need the notion of the separated quiver ([6], [9], [10]. [12]) of a radical squared zero Artin algebra. We use the notation of [10], Section 14.

Let $P_1, \ldots, P_n$ be a full set of nonisomorphic indecomposable projective left R-modules. Then $P_1^*, \ldots, P_n^*$, where $P_i^* = \mathrm{Hom}(P_i, R)$, constitute a full set of nonisomorphic projective right R-modules. Let a_{ij} be the number of times the simple module P_j/rP_j occurs as a composition factor of the semisimple module rP_i and let a_{ij}^* be the number of times $P_j^*/P_j^* r$ occurs as a composition factor of $P_i^* r$. The *left quiver* $Q(R)$ of R consists of the n points $1, \ldots, n$ together with a_{ij} arrows from the point i to the point j. The *right quiver* $Q^*(R)$ of R is the quiver with n points $1, \ldots, n$ and a_{ij}^* arrows from i to j. Then the *quiver* $\Gamma(R)$ of R is the ordered pair $\Gamma(R) = (Q(R), Q^*(R))$. Further, let $S_1, \ldots, S_n$ be the full set of nonisomorphic simple R-modules such that $S_i = \mathrm{top}\, P_i$ and put $E_i = E(S_i)$. Then from the proof of Lemma 9.7 in [10] we have $P_i^* \cong D(E_i)$ and, consequently, a_{ij}^* is the number of times rE_j occurs as a composition factor of E_i/rE_i. The definition of the quiver of R is left-right symmetric in the sense that if we think of $Q^*(R)$ as of the right quiver of R, then, with the obvious meaning, $Q^{**}(R) = Q(R)$. Put $Q(R^{\mathrm{op}}) = Q^*(R)$ and $\Gamma(R^{\mathrm{op}}) = (Q^*(R), Q(R))$. The *separated quiver* $\Gamma'(R)$ of R is the quiver of R'.

Our main aim in this section is to prove the following

THEOREM 4.1. *If R is a radical squared zero Artin algebra, then the following statements are equivalent:*

(i) *Every indecomposable module of finite length is either $\frac{3}{2}$-indecomposable or $-\frac{3}{2}$-indecomposable.*

(ii) *Either every indecomposable module of finite length is $\frac{3}{2}$-indecomposable or else every indecomposable module of finite length is $-\frac{3}{2}$-indecomposable.*

(iii) *Every indecomposable module has length at most 5.*

(iv) *The separated quiver of R is a disjoint union of quivers of the following types:*

A_1: $1, 1$,

A_2: $1 \leftarrow 2,\ 1 \rightarrow 2$,

A_3: $1 \leftarrow 2 \rightarrow 3,\ 1 \rightarrow 2 \leftarrow 3$,

A_3^*: $1 \rightarrow 2 \leftarrow 3,\ 1 \leftarrow 2 \rightarrow 3$,

A_4: $1 \leftarrow 2 \rightarrow 3 \leftarrow 4,\ 1 \rightarrow 2 \leftarrow 3 \rightarrow 4$,

A_5: $1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5,\ 1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$,

A_5^*: $1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$, $1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5$,

B_2: $1 \leftleftarrows 2$, $1 \rightarrow 2$,

B_2^*: $1 \rightarrow 2$, $1 \leftleftarrows 2$,

B_3: $1 \leftleftarrows 2 \rightarrow 3$, $1 \rightarrow 2 \leftarrow 3$,

B_3^*: $1 \rightarrow 2 \leftarrow 3$, $1 \leftleftarrows 2 \rightarrow 3$,

C_3: $1 \leftarrow 2 \rightarrow 3$, $1 \rightrightarrows 2 \leftarrow 3$,

C_3^*: $1 \rightrightarrows 2 \leftarrow 3$, $1 \leftarrow 2 \rightarrow 3$,

$$D_4:\ \begin{array}{ccccc} & & 3 & & \\ & & \uparrow & & \\ 1 & \leftarrow & 2 & \rightarrow & 4, \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & & \downarrow & & \\ 1 & \rightarrow & 2 & \leftarrow & 4, \end{array}$$

$$D_4^*:\ \begin{array}{ccccc} & & 3 & & \\ & & \downarrow & & \\ 1 & \rightarrow & 2 & \leftarrow & 4, \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & & \uparrow & & \\ 1 & \leftarrow & 2 & \rightarrow & 4, \end{array}$$

G_2: $1 \overset{\leftarrow}{\leftleftarrows} 2$, $1 \rightarrow 2$,

G_2^*: $1 \rightarrow 2$, $1 \overset{\leftarrow}{\leftleftarrows} 2$.

Moreover, if R is indecomposable and hereditary, then any of the statement (i)-(iv) *is equivalent to*

(v) *R has at most* 15 *nonisomorphic indecomposable modules.*

The theorem is a generalization of Theorem 15.1 in [10]. The letters A, B, C, D, and G indicate the Dynkin diagrams (see [6]-[8]) from which the designated separated quivers are derived. We should mention that in the nonhereditary case it is not possible to prescribe a bound on the number of nonisomorphic indecomposable modules as it is done in (v) (see [10], p. 105).

In order to prove the theorem we need the following lemma which reduces the study of $\frac{3}{2}$-indecomposable R-modules to R'-modules:

LEMMA 4.1. *Let R be a radical squared zero Artin algebra, R' the hereditary Artin algebra associated with R, and $\alpha\colon \operatorname{mod} R \rightarrow \operatorname{mod} R'$ the functor defined previously. Moreover, let X be a finitely generated indecomposable nonsimple R-module and let $M = \alpha(X)$. Then X is $\frac{3}{2}$-indecomposable if and only if M is $\frac{3}{2}$-indecomposable.*

Proof. First observe that every nonsuperfluous submodule of X (resp. M) contains a nonsuperfluous submodule of X (resp. M) with a simple top, called a *basic submodule* of X (resp. M) (cf. [10]). By Lemma 15.5 in [10], the functor α gives a one-to-one correspondence between basic submodules of X and basic submodules of M. Assume that X is $\frac{3}{2}$-indecomposable and let N and P be basic submodules of M. By the above remark, $N = \alpha(Y)$ and $P = \alpha(Z)$ for some basic submodules Y and Z of X, and there is a simple R-module S such that $S \subset Y \cap Z$. But then $T = 0 \oplus S$ is a simple R'-module contained in $N \cap P$, so M is $\frac{3}{2}$-indecomposable. Conversely, let M be $\frac{3}{2}$-indecomposable and let Y and Z be basic submodules of X. Then $\alpha(Y)$ and

$\alpha(Z)$ are basic submodules of M, so there is a simple R'-module T such that $T \subset \alpha(Y) \cap \alpha(Z)$. Since $\alpha(Y)$ and $\alpha(Z)$ have simple tops, $T = 0 \oplus S$ for a simple R-module S, and $S \subset Y \cap Z$. Hence X is $\frac{3}{2}$-indecomposable and the lemma is proved.

Proof of Theorem 4.1. First we prove that (iv) implies (ii) and (iii). By duality, it is enough to prove that if the quiver of a radical squared zero hereditary Artin algebra R is one of the quivers A_1, A_2, A_3, A_3^*, A_4, A_5, B_2, B_2^*, B_3, C_3, D_4, or G_2, then every indecomposable finitely generated R-module has length at most 4 or has length 5 and satisfies (i) of Theorem 3.1 or (ii) of Theorem 3.2. Using Theorems 14.1 and 14.5 from [10] we get the following table for the number of nonisomorphic indecomposable R-modules in every case:

A_1	A_2	A_3	A_3^*	A_4	A_5	B_2^*	B_2	B_3	C_3	D_4	G_2
1	3	6	6	10	15	4	4	9	9	12	6

We apply the same numeration of vertices of the above quivers as in the theorem and put $S_i = \text{top } P_i$ and $E_i = E(S_i)$. It is not difficult to see that if

$$\Gamma(R) = A_1,\ A_2,\ A_3,\ A_3^*,\ B_2, \text{ or } B_2^*,$$

then every indecomposable R-module has length at most 4. If $\Gamma(R) = A_4$, then the modules S_1, S_2, S_3, S_4, P_2, P_4, E_1, E_3, P_2/S_1 of length at most 3 and the module M of length 4, given by the Auslander-Reiten sequence

$$0 \to M \to E_1 \oplus E_3 \to S_2 \to 0,$$

form a full list of nonisomorphic indecomposable R-modules.

Now suppose that $\Gamma(R) = A_5$. In this case the list of nonisomorphic indecomposable R-modules is the following: S_1, S_2, S_3, S_4, S_5, P_2, P_4, E_1, E_3, E_5, P_2/S_1, P_4/S_5, M_1, M_2, N, where M_1, M_2, and N are given by the Auslander-Reiten sequences

$$0 \to M_1 \to E_1 \oplus E_3 \to S_2 \to 0,$$

$$0 \to M_2 \to E_3 \oplus E_5 \to S_4 \to 0,$$

$$0 \to S_3 \to P_2 \oplus P_4 \to N \to 0.$$

Observe that $l(M_1) = l(M_2) = 4$ and $l(N) = 5$. Then, since the first 12 indecomposable modules on our list have length at most 3 and, by Theorem 3.2, N is $\frac{3}{2}$-indecomposable, the required conditions are satisfied.

Assume that $\Gamma(R) = B_3$. Then we have 8 nonisomorphic indecomposable modules S_1, S_2, S_3, P_2, E_1, E_3, P_2/S_1, P_2/S_3 of length at most 4 and one indecomposable module M of length 5 given by the Auslander-Reiten sequence

$$0 \to M \to E_1 \oplus E_1 \oplus E_3 \to S_2 \to 0.$$

Since E_1 and E_3 have simple tops, M is $\frac{3}{2}$-indecomposable by Theorem 3.1.

Let $\Gamma(R) = C_3$. In this case we have 7 nonisomorphic indecomposable modules S_1, S_2, S_3, P_2, E_1, E_3, P_2/S_3 of length at most 3, and 2 indecomposable modules M and N of length 4 and 5, respectively, given by the Auslander-Reiten sequences

$$0 \to M \to E_1 \oplus E_3 \to S_2 \to 0,$$

$$0 \to S_1 \to P_2 \oplus P_2 \to N \to 0.$$

By Theorem 3.2, N is $\frac{3}{2}$-indecomposable, and the required conditions hold.

Let $\Gamma(R) = D_4$. Then we have 11 nonisomorphic indecomposable modules S_1, S_2, S_3, S_4, P_2, E_1, E_3, E_4, P_2/S_1, P_2/S_3, P_2/S_4 of length at most 4 and one indecomposable module M of length 5 given by the Auslander-Reiter sequence

$$0 \to M \to E_1 \oplus E_3 \oplus E_4 \to S_2 \to 0.$$

Since E_1, E_3, E_4 have simple tops, M is $\frac{3}{2}$-indecomposable by Theorem 3.1 (i).

Finally, let $\Gamma(R) = G_2$. Then we have 5 nonisomorphic indecomposable modules S_1, S_2, P_2, E_1, P_2/S_1 of length at most 4 and one indecomposable module M of length 5 given by the Auslander-Reiten sequence

$$0 \to M \to E_1 \oplus E_1 \oplus E_1 \to S_2 \to 0.$$

But $l(E_1) = 2$ and, by Theorem 3.1, M is $\frac{3}{2}$-indecomposable.

Now we prove simultaneously the implications (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv). By Corollary 2.2, Lemma 3.2, [1] (Theorem 3.1), [10] (Theorems 14.1 and 14.5), and duality D, for our aim it is enough to show that if one of the connected components of $\Gamma(R')$ is of the type

A_n: $1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \leftarrow 6 - \ldots - n$, $1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 - \ldots - n$, $n \geqslant 6$,

B_n: $1 \leftrightarrows 2 \rightarrow 3 \leftarrow 4 - \ldots - n$, $1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow \ldots - n$, $n \geqslant 4$,

C_n: $1 \leftarrow 2 \rightarrow 3 \leftarrow 4 - \ldots - n$, $1 \rightrightarrows 2 \leftarrow 3 \rightarrow 4 - \ldots - n$, $n \geqslant 4$,

$$D_n:\ \begin{array}{c}3\\ \uparrow\\ 1 \leftarrow 2 \rightarrow 4 \leftarrow 5 - \ldots - n\end{array},\quad \begin{array}{c}3\\ \downarrow\\ 1 \rightarrow 2 \leftarrow 4 \rightarrow 5 - \ldots - n\end{array},\quad n \geqslant 5,$$

$$E_n:\ \begin{array}{c}4\\ \downarrow\\ 1 \leftarrow 2 \rightarrow 3 \leftarrow 5 - \ldots - n\end{array},\quad \begin{array}{c}4\\ \uparrow\\ 1 \rightarrow 2 \leftarrow 3 \rightarrow 5 - \ldots - n\end{array},\quad n = 6, 7, 8,$$

F_4: $1 \leftarrow 2 \rightrightarrows 3 \leftarrow 4$, $1 \rightarrow 2 \leftarrow 3 \rightarrow 4$,

then there is a finitely generated indecomposable R'-module of length greater than or equal to 6 which is neither $\frac{3}{2}$-indecomposable nor $-\frac{3}{2}$-indecomposable. We may assume that $\Gamma(R')$ is connected.

First assume that $\Gamma(R') = A_n$, where $n \geqslant 6$. Let us consider the following exact sequence in mod R':

$$0 \to S_3 \oplus S_5 \xrightarrow{f} P_2 \oplus P_4 \oplus P_6 \xrightarrow{g} M \to 0,$$

where f is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},$$

1 denotes the canonical injections, and g is the cokernel of f. It is not difficult to verify that M is indecomposable. Observe that $g(P_2)$ and $g(P_6)$ are nonsuperfluous submodules of M and $g(P_2) \cap g(P_6) = 0$. On the other hand, $g(P_2)$ and $g(P_4 \oplus P_6)$ are nonessential submodules of M and $g(P_2) + g(P_4 \oplus P_6) = M$. Thus M satisfies the required conditions.

Now let $\Gamma(R') = C_n$, $n \geqslant 4$. From Propositions 1.9 and 2.6 in [7] we know that there is an indecomposable finitely generated R'-module M with

$$\text{top } M \cong S_2 \oplus S_2 \oplus S_4, \qquad \text{soc } M = S_1 \oplus S_3 \oplus S_3 \oplus mS_5,$$

where $m = 0$ for $n = 4$ and $m = 1$ for $n \geqslant 5$. Then the minimal projective presentation for M is of the form

$$0 \to S_1 \oplus S_3 \xrightarrow{f} P_2 \oplus P_2 \oplus P_4 \xrightarrow{g} M \to 0,$$

where

$$f = \begin{pmatrix} a & c \\ b & d \\ 0 & e \end{pmatrix}, \quad e \neq 0, \quad \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Assume $a \neq 0$, $c \neq 0$, and let $g = (g_1, g_2, g_3)$. Since soc M contains two copies of S_3, either $g_1(P_2) \cap g(P_4)$ or $g_2(P_2) \cap g(P_4)$ is zero, and then M is not $\frac{3}{2}$-indecomposable. By [7], Propositions 2.1, 2.4, End (P_2) and End (S_3) are division rings having the same finite dimension over a central subfield K, so the restriction map End $(P_2) \to$ End (S_3) is an isomorphism. Hence the map

$$\begin{pmatrix} c \\ d \end{pmatrix} : S_3 \to P_2 \oplus P_2$$

has an extension to $h: P_2 \to P_2 \oplus P_2$. Then the socle of $N = gh(P_2) + g_3(P_4)$ has only one copy of S_3, so N is nonessential in M. Moreover, $g_2(P_2) + N = M$, $g_2(P_2) \cap g_3(S_3) = 0$ and, consequently, M is not $-\frac{3}{2}$-indecomposable.

For the analysis of the remaining cases we use Corollary 3.1. If $\Gamma(R')$ is one of the quivers B_n $(n \geqslant 4)$, D_n $(n \geqslant 5)$, or F_4, then

$$l(\text{soc } E(P_2)) = 3 \quad \text{and} \quad l(\text{top } E(P_2)) = 4.$$

If $\Gamma(R') = E_n$ ($n = 6, 7, 8$), then

$$l(\text{top } P(E_3)) = 3 \quad \text{and} \quad l(\text{soc } P(E_3)) = 4,$$

and we are done.

Since the implication (ii) ⇒ (i) is obvious, the equivalence of (i)-(iv) is proved. If R is indecomposable and hereditary, then the equivalence of (iv) and (v) is a consequence of Theorems 14.1, 14.5, and the table on page 90 in [10]. The proof of the theorem is complete.

We state some consequences of Theorem 4.1.

THEOREM 4.2. *The following properties of a radical squared zero Artin algebra are equivalent:*

(i) *Every indecomposable module of finite length is* $\frac{3}{2}$*-indecomposable.*

(ii) *The separated quiver of* R *is a disjoint union of quivers (specified as in Theorem* 4.1). *of the types* A_1, A_2, A_3, A_3^*, A_4, A_5, B_2, B_2^*, B_3, C_3, D_4, *and* G_2.

Proof. The implication (ii) ⇒ (i) follows from the proof of (iv) ⇒ (ii) of Theorem 4.1. For (i) ⇒ (ii), by Theorem 4.1 and the lemma dual to Lemma 4.1, it suffices to show that if a connected component of $\Gamma(R')$ is equal to one of the quivers A_5^*, B_3^*, C_3^*, D_4^*, or G_2^*, then there is a finitely generated indecomposable R'-module which is not $\frac{3}{2}$-indecomposable. Without loss of generality we may assume that $\Gamma(R')$ is connected. As in the proof of Theorem 4.1 we use the notation $S_i = \text{top } P_i$ and $E_i = E(S_i)$.

If $\Gamma(R') = A_5^*$, then we have the Auslander-Reiten sequence

$$0 \to M \to E_2 \oplus E_4 \to S_3 \to 0.$$

Since E_2 and E_4 have nonsimple tops, M is not $\frac{3}{2}$-indecomposable by Theorem 3.1 (i) (2).

If $\Gamma(R') = B_3^*$, then we have the Auslander-Reiten sequence

$$0 \to S_2 \to P_1 \oplus P_1 \oplus P_3 \to M \to 0,$$

and, by Theorem 3.2 (i), M is not $\frac{3}{2}$-indecomposable.

Let $\Gamma(R') = C_3^*$. Then we have the Auslander-Reiten sequence

$$0 \to M \to E_2 \oplus E_2 \to S_1 \to 0,$$

where top $E_2 \cong S_1 \oplus S_3$, and from Theorem 3.1 (i) (2) we conclude that M is not $\frac{3}{2}$-indecomposable.

Assume $\Gamma(R') = D_4^*$. In this case we have the Auslander-Reiten sequence

$$0 \to S_2 \to P_1 \oplus P_3 \oplus P_4 \to M \to 0,$$

and from Theorem 3.2 (i) we infer that M is not $\frac{3}{2}$-indecomposable.

Finally, suppose that $\Gamma(R') = G_2^*$. Then we have the Auslander-Reiten sequence

$$0 \to S_2 \to P_1 \oplus P_1 \oplus P_1 \to M \to 0,$$

and, by Theorem 3.2 (i), M is not $\frac{3}{2}$-indecomposable. Thus the theorem is proved.

We observe that Theorem 4.2 leads immediately, by the duality D, to a classification of radical squared zero Artin algebras such that every indecomposable module of finite length is $-\frac{3}{2}$-indecomposable.

From Theorems 4.1 and 4.2, and [10] (Theorems 15.1 and 15.3) we get

COROLLARY 4.1. *If R is a radical squared zero Artin algebra, then the following statements are equivalent*:

(i) *Every indecomposable module is $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable.*

(ii) *Every indecomposable module has a core and a cocore.*

(iii) *The separated quiver of R is a disjoint union of quivers of the types A_1, A_2, A_3, A_3^*, A_4, B_2, and B_2^*.*

Concerning this result we should mention that, in general, there are modules which are simultaneously $\frac{3}{2}$-indecomposable and $-\frac{3}{2}$-indecomposable but having neither a core nor a cocore (see Example 1.1).

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