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ON COMPLETE ORBIT SPACES OF SL(2) ACTIONS

BY

ANDRZEJ BIAŁYNICKI-BIRULA AND JOANNA ŚWIĘCICKA (WARSZAWA)

Let G be an algebraic reductive group acting on a smooth projective variety X. It follows from geometric invariant theory of D. Mumford that if for some ample G-linearized sheaf L over X and a maximal torus $T \subset G$ the sets of stable and semi-stable points with respect to the induced (by restriction) action of T on X coincide and are equal to $U \subset X$, then U is $N(T)$-*invariant subset such that the geometric quotient* $U \to U/T$ *exists, where* U/T *is* Chapter 1, Section 4). Moreover, in this case the set of all stable points of X for the action of G on X (with respect to L) is equal to $\bigcap_{g \in G} gU$ ([6], Theorem 2.1), the geometric quotient

$$\bigcap_{g \in G} gU \to \bigcap_{g \in G} gU/G$$

exists and $\bigcap_{g \in G} gU/G$ is a projective algebraic variety.

The above results may lead to the following

CONJECTURE. *Let G and X be as above. Let $U \subset X$ be an open $N(T)$-invariant subset such that the geometric quotient $U \to U/T$ exists, where U/T is a complete algebraic variety. Then $\bigcap_{g \in G} gU$ is an open G-invariant subset of X, the geometric quotient*

$$\bigcap_{g \in G} gU \to \bigcap_{g \in G} gU/G$$

exists, where $\bigcap_{g \in G} gU/G$ is a complete algebraic space.

In this paper we show that the conjecture is true if $G = \mathrm{SL}(2)$ (and the ground field is of characteristic 0).

We are going to use terminology introduced in [1]–[3]. Now we quote the definitions needed in the sequel and fix notation.

For a given action of a one-dimensional torus $T = K^*$ on a smooth complete variety X we denote by X^T the fixed point subvariety of the action. Moreover, $X_1 \cup \ldots \cup X_r = X^T$ is the decomposition into irreducible components and, for $i = 1, \ldots, r$,

$$X_i^+ = \{x \in X;\ \lim_{t \to 0} tx \in X_i\}, \qquad X_i^- = \{x \in X;\ \lim_{t \to \infty} tx \in X_i\}.$$

We denote by $\varkappa_j$ the morphism $X_i^+ \to X_i$ defined by

$$\varkappa_j(x) = \lim_{t \to 0} tx \quad \text{for any } x \in X_i^+.$$

We say that X_i is *less than* X_j and write $X_i \prec X_j$ if there exists a finite sequence of points $x_1, \ldots, x_m \in X - X^T$ such that

(a) $\lim_{t \to 0} tx_1 \in X_i$,

(b) $\lim_{t \to \gamma} tx_n \in X_j$,

(c) for $k = 1, \ldots, m-1$, $\lim_{t \to \infty} tx_k$ and $\lim_{t \to 0} tx_{k+1}$ belong to the same irreducible component of X^T.

We say that X_i and X_j are not *comparable* if neither $X_i \prec X_j$ nor $X_j \prec X_i$.

By a *section* of $\{X_1, \ldots, X_r\}$ we mean a division of the set into two non-empty subsets A^- and A^+ satisfying the following condition:

if $X_i \in A^-$ and $X_j \prec X_i$, then $X_j \in A^-$.

Every section (A^-, A^+) determines an open and T-invariant subset U of X defined in the following way:

$$U = X - \Big(\bigcup_{X_j \in A^-} X_j^- \cup \bigcup_{X_j \in A^+} X_j^+\Big).$$

The set U is called the *sectional set* determined by the section (A^-, A^+).

By a *semi-section* of $\{X_1, \ldots, X_r\}$ we mean a division of $\{X_1, \ldots, X_r\}$ into three subsets A^-, A^0, A^+ satisfying the following condition:

$A^- \neq \emptyset \neq A^+$ and if $X_i \in A^- \cup A^0$, $X_j \prec X_i$, then $X_j \in A^-$.

Any semi-section (A^+, A^0, A^-) determines an open T-invariant subset

$$U = X - \Big(\bigcup_{X_j \in A^-} X_j^- \cup \bigcup_{X_j \in A^+} X_j^+\Big).$$

It has been proved (see [3]) that if U is a subset determined by a semi-section, then there exists a categorical quotient φ: $U \to U/T$, where U/T is a complete algebraic variety and φ is an affine morphism. Moreover, if U is a sectional set, then φ: $U \to U/T$ is a geometric quotient.

Let X be a smooth complete algebraic variety with a non-trivial action of $\mathrm{SL}(2)$, all defined over an algebraically closed field K. Let $T \subset \mathrm{SL}(2)$ be a fixed one-dimensional subtorus and let $N(T)$ be its normalizer in $\mathrm{SL}(2)$. Denote by B_+ and B_- two Borel subgroups of $\mathrm{SL}(2)$ containing T. We assume that

$$T = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right\},$$

$$B_+ = \left\{ \begin{bmatrix} t & \lambda \\ 0 & t^{-1} \end{bmatrix} \right\}, \quad B_- = \left\{ \begin{bmatrix} t & 0 \\ \lambda & t^{-1} \end{bmatrix} \right\},$$

$$N(T) = T \cup \tau T,$$

where

$$t \in K^*, \quad \lambda \in K, \quad \tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For any $Y \subset X$ and $H \subset \mathrm{SL}(2)$,

$$Y^H = \{y \in Y;\ \text{for any } h \in H,\ hy = y\}.$$

The Weyl group $W = N(T)/T$ acts on the set $\{X_i;\ i = 1, \dots, r\}$ by involution denoted by w.

THEOREM 1. *Let $U \subset X$ be an $N(T)$-invariant open subset such that the geometric quotient $U \to U/T$ exists and U/T is projective. Then X is projective and there exists an ample* SL(2)*-linearized linear vector bundle L on X such that*

$$X^s(L) = X^{ss}(L) = \bigcap_{g \in \mathrm{SL}(2)} gU.$$

Hence $\bigcap_{g \in \mathrm{SL}(2)} gU$ is open and SL(2)*-invariant, the geometric quotient*

$$\bigcap_{g \in \mathrm{SL}(2)} gU \to \bigcap_{g \in \mathrm{SL}(2)} gU/\mathrm{SL}(2)$$

exists and $\bigcap_{g \in \mathrm{SL}(2)} gU/\mathrm{SL}(2)$ is a projective normal variety.

Proof. Since the geometric quotient $U \to U/T$ exists and U/T is normal and projective, the geometric quotient $U \to U/N(T)$ also exists and $U/N(T)$ is projective. Hence there exists an $N(T)$-linearized linear bundle L on X such that U is the set of stable points with respect to L, L is very ample on U and there exist sections $s_1, \dots, s_l \in \Gamma(X, L)$ which separate points and tangent vectors in U. In fact, it follows from [8] that we may apply [6], Lemma 0.5 and then Proposition 0.7, to conclude that the quotient morphism $U \to U/T$ is affine. Therefore, the morphism $U \to U/N(T)$ is also affine and we may use [6], 1.12, 1.13 and Section 4 (2).

Now, since SL(2) has no non-trivial character, we can assume that L is SL(2)-linearized ([6], 1.5) (we replace L by $L^{\otimes n}$ for some positive integer n, if necessary). It follows from the proof of 1.4 in [6] that the T-linearization of L

induced by the SL(2)-linearization coincides with the T-linearization obtained (by restricting) from the $N(T)$-linearization given previously.

It follows from the above properties of L that L determines an SL(2)-equivariant birational map

$$\psi\colon X \to P^k,$$

where k is a positive integer and P^k is a k-dimensional projective space. Moreover, $\psi|U$ is an isomorphism onto $\psi(U) \subset P^k$. We know (see [3]) that U is a sectional set; hence $X - U = X' \cup X''$, where X' and X'' are connected disjoint and closed subsets of X. Since, for any $x \in X$, $\mathrm{SL}(2)x$ is $N(T)$-invariant and $\tau(X') = X''$,

$$\mathrm{SL}(2)x \cap X' \neq \emptyset \quad \text{if and only if} \quad \mathrm{SL}(2)x \cap X'' \neq \emptyset.$$

Since, for any $x \in X$, the orbit $\mathrm{SL}(2)x$ is connected,

$$\mathrm{SL}(2)x \cap U \neq \emptyset;$$

thus

$$\bigcup_{g \in \mathrm{SL}(2)} gU = X.$$

Since ψ is defined on U and is SL(2)-equivariant, ψ is defined on

$$\bigcup_{g \in \mathrm{SL}(2)} gU = X.$$

If, for $x_1, x_2 \in X$, $\psi(x_1) = \psi(x_2)$, then, for any $g \in \mathrm{SL}(2)$, $\psi(gx_1) = \psi(gx_2)$. Since $\mathrm{SL}(2)x_1 \cap U$ and $\mathrm{SL}(2)x_2 \cap U$ are non-empty open subsets of $\mathrm{SL}(2)x_1$ and $\mathrm{SL}(2)x_2$, respectively, there exists $g_0 \in \mathrm{SL}(2)$ such that $g_0 x_1, g_0 x_0 \in U$. But $\psi | U$ is injective, hence $g_0 x_1 = g_0 x_2$ and $x_1 = x_2$. Moreover, for any $x \in U$ the differential $d\psi_x$ is injective, so it is injective for any $x \in X$. Thus ψ is an embedding and L is very ample on X.

Consider $X^{ss}(L)$ and $X^s(L)$ with respect to L with given SL(2)-linearization. By [6], 2.1,

$$X^{ss}(L) = X^s(L) = \bigcap_{g \in \mathrm{SL}(2)} gU.$$

Hence $\bigcap_{g \in \mathrm{SL}(2)} gU$ is open and the geometric quotient

$$\bigcap_{g \in \mathrm{SL}(2)} gU \to \bigcap_{g \in \mathrm{SL}(2)} gU/\mathrm{SL}(2)$$

exists and $\bigcap_{g \in \mathrm{SL}(2)} gU/\mathrm{SL}(2)$ is projective.

In the sequel we shall assume that the ground field K is of characteristic 0.

THEOREM 2. *Suppose that X is projective and let $U \subset X$ be an $N(T)$-invariant open subset of X such that the geometric quotient $U \to U/T$ exists* *and U/T is a complete algebraic variety. Then the geometric quotient*

$$\bigcap_{g\in SL(2)} gU \to \bigcap_{g\in SL(2)} gU/SL(2)$$

exists and $\bigcap_{g\in SL(2)} gU/SL(2)$ *is a complete normal algebraic space.*

Proof. The existence of this quotient in the category of algebraic spaces follows from [2], Theorem 4.1. (Notice that the assumption stated in Theorem 4.1 of [2] that the stabilizer groups are finite was not used in the proof of the implication (b) $\Rightarrow$ (a) presented in the paper.) The proof of the completeness of $\bigcap_{g\in SL(2)} gU/SL(2)$ is based on the following lemmas:

LEMMA 1. *Let X be complete (not necessarily projective). Let X_j be a connected component of X^T. Then X_j^+ is B_+-invariant, $X_j^+ -(B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))$ is open in X_j^+ and the geometric quotient*

$$X_j^+ -(B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+})) \to X_j^+ -(B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$$

exists. Moreover, the morphism

$$X_j^+ -(B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+ \to X_j - X_j^{B+}$$

induced by $\varkappa_j$ is proper. In particular, if $X_j^{B+} = \emptyset$, then $(X_j^+ - B_+ X_j)/B_+$ is complete.

Proof. It follows from [7] that X_j^+ is B_+-invariant and the canonical retraction $\varkappa_j: X_j^+ \to X_j$ is B_+-equivariant (with respect to the trivial action of B_+ on X_j). Hence if $X_j^+ -(B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$ exists, then $\varkappa_j$ induces a morphism

$$X_j^+ -((B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+ \to X_j - X_j^{B+}.$$

Moreover, for any $x_0 \in X_j$ there exists a neighbourhood U of x_0 in X_j such that

$$\varkappa_j^{-1}(U) \approx U \times K^d,$$

where d is an integer (see [8] and [1], Corollary 4.1) and there exists an SL(2)-equivariant embedding

$$\psi\colon SL(2)\varkappa_j^{-1}(U) \to P^m$$

(for some integer m). In fact, since X is smooth, there is an open SL(2)-invariant quasi-projective neighbourhood U_1 of x_0 (see [8]). It follows from [4] or [6], Corollary 1.6, that U_1 can be SL(2)-equivariantly embedded into a projective space P^m. Since, for any $x \in \varkappa_j^{-1}(U_1 \cap X_j)$,

$$\overline{Tx} \cap (U_1 \cap X_j) \neq \emptyset$$

and U_1 is open T-invariant, we infer that

$$\varkappa_j^{-1}(U_1 \cap X_j) \subset U_1.$$

Therefore $SL(2)\varkappa_j^{-1}(U_1 \cap X_j) \subset U_1$. Taking $U = U_1 \cap X_j$ we obtain the desired result. This allows us to reduce the proof of the lemma to the case where $X = P^m$.

In fact, it follows from the validity of the lemma for $X = P^m$ and from the above considerations that if X satisfies the assumptions of the lemma, then $X_j - X_j^{B+}$ can be covered by a family of open subsets U_i such that, for any i, $U_i^+ - B_+ U_i$ is open in X_j^+ and the geometric quotient

$$U_i^+ - B_+ U_i \to U_i^+ - B_+ U_i/B_+$$

exists and the morphism $U_i^+ - B_+ U_i/B_+ \to U_i$ induced by $\varkappa_j$ is proper (and hence separated). Hence $X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))$ is open in X_j^+ and the geometric quotient

$$X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+})) \to X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$$

exists, where $X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$ is an algebraic prevariety and the morphism

$$X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+ \to X_j - X_j^{B+}$$

induced by $\varkappa_j$ is proper (and hence separated). Now it suffices to show that $X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$ is a variety. This follows from the facts that $X_j - X_j^{B+}$ and the described above morphism

$$X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+ \to X_j - X_j^{B+}$$

are separated.

Therefore, in the rest of the proof of the lemma we may assume that $X = P^m$.

Any action of $SL(2)$ on P^m can be lifted to a linear representation of $SL(2)$ in the linear space A^{m+1}. On the other hand, any linear representation of $SL(2)$ in A^{m+1} is a direct sum $\bigoplus_{i=1}^{n} F^i$, where F^i is the space of forms of degree d_i in two variables x, y, with the action of $SL(2)$ induced by the natural representation of $SL(2)$ on linear forms. Any point $x_0 \in X_j$ lifts to a line generated by a non-zero vector $\bar{x}_0 \in \bigoplus_{i=1}^{n} F^i$ with zero coefficient at each monomial $x^{s_i} y^{t_i} \in F^i$ such that $s_i - t_i \neq s(j)$, where $s(j)$ is an integer determined uniquely by the component X_j. Moreover, $x_0 \in X_j - X_j^{B+}$ if and only if $\bar{x}_0$ depends on y, i.e., if and only if $\bar{x}_0$ is not a sum of forms of degree $s(j)$.

Let $u \in (X_j - X_j^{B+})^+$. Then for any lifting $\bar{u}$ of u in A^{m+1} the component of $\bar{u}$ in F^i is of the form

$$(*) \qquad a_0 x^{r+s(j)} y^r + a_1 x^{r+s(j)+1} y^{r-1} + \ldots + a_r x^{r+s(j)+r},$$

where $r \geqslant 0$, $2r+s(j) = d_i$ and $a_0, \ldots, a_r \in K$.

Let $x_0 \in X_j - X_j^{B+}$. Then it follows from the above that $\bar{x}_0$ has a non-zero component in F^i for some i, with $d_i = 2r+s(j)$, where $r > 0$. We may assume that $i = 1$. Let V be the open subset of $X_j - X_j^{B+}$ composed of all points which have lifts in A^{m+1} with non-zero component in F^1. Then any $u \in V^+$ has a unique lifting $\bar{u} \in A^{m+1}$ with its component in F^1 of the form

$$(**) \qquad x^{r+s(j)} y^r + a_1 x^{r+s(j)+1} y^{r-1} + \ldots + a_r x^{r+s(j)+r},$$

where $r > 0$, $2r+s(j) = d_1$ and $a_1, \ldots, a_r \in K$.

Let

$$G_a = \left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \right\}, \quad \text{where } \lambda \in K.$$

The G_a-orbit of u is composed of vectors in A^{m+1} with component in F^1 equal to

$$x^{r+s(j)} y^r + a_1(\lambda) x^{r+s(j)+1} y^{r-1} + \ldots + a_r(\lambda) x^{r+s(j)+r},$$

where $a_1(\lambda), \ldots, a_r(\lambda)$ are polynomial functions of λ with coefficients in K and $a_1(\lambda) = a_1 + \lambda$. Hence for exactly one $\lambda \in K$ (namely, for $\lambda = -a_1$) the coefficient at $x^{r+s(j)+1} y^{r-1}$ in the component is equal to zero. It follows that the geometric quotient

$$\varphi_a \colon V^+ \to V^+/G_a$$

exists with V^+/G_a being the subset of A^{m+1} composed of all points v satisfying the following conditions:

(i) for $i \neq 1$ the component of v in F^i is of the form $(*)$:

(ii) the component of v in F^1 is of the form $(**)$ with $a_1 = 0$.

The action of T on V^+ induces an action of T on V^+/G_a. The action can be described as follows: for $v \in V^+/G_a$ and $t \in T$, the value of the map corresponding to t at v is equal to the product $t^{-s(j)}$ times the value of the linear transformation corresponding to t at v in the representation space A^{m+1}. Then one can check that the geometric quotient

$$V^+/G_a - (V^+/G_a)^T \to (V^+/G_a - (V^+/G_a)^T)/T$$

exists. Moreover, $\varphi_a^{-1}(V^+/G_a - (V^+/G_a)^T) = V^+ - B_+ V$ (hence $V^+ - B_+ V$ is open in X_j^+) and $\varkappa_j | V^+ \colon V^+ \to V$ induces a projective map

$$(V^+/G_a - (V^+/G_a)^T)/T \to V$$

with weighted projective spaces as fibres.

The composition of geometric quotients

$$(V^+ - B_+ V) \to V^+ - B_+ V/G_a = V^+/G_a - (V^+/G_a)^T$$
$$\to (V^+/G_a - (V^+/G_a)^T)/T$$

is the geometric quotient

$$(V^+ - B_+ V) \to (V^+ - B_+ V)/B_+$$

and the map $V^+ - B_+ V/B_+ \to V$ induced by $\varkappa_j$ is projective.

It follows from the above that the geometric quotient

$$X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+})) \to X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+$$

exists and the map

$$X_j^+ - (B_+ X_j \cup \varkappa_j^{-1}(X_j^{B+}))/B_+ \to X_j - X_j^{B+}$$

induced by $\varkappa_j$ is proper. This completes the proof of the lemma.

We shall now consider the following situation:

Let $U_1, U_2 \subset X$ be two different $N(T)$-invariant sectional sets defined by sections (A_1^-, A_1^+) and (A_2^-, A_2^+), respectively. We say that U_1 is an *elementary transform* of U_2 if there exists a minimal element X_{i_0} in A_1^+ such that

$$A_2^+ = (A_1^+ \cup \{w(X_{i_0})\}) - \{X_{i_0}\}.$$

It follows that in this case $w(X_{i_0})$ is a maximal element in A_1^-,

$$A_2^- = (A_1 \cup \{X_{i_0}\}) - \{w(X_{i_0})\}$$

and X_{i_0} is not comparable with $w(X_{i_0})$. Therefore $X_{i_0}^{B+} = \emptyset$. In fact, let $x_0 \in X_{i_0}^{B+}$. Then the orbit $SL(2) x_0$ contains a dense T-orbit. Since $SL(2) x_0$ contains both x_0 and $\tau(x_0)$, we infer that X_{i_0} and $w(X_{i_0})$ are comparable, a contradiction.

LEMMA 2. *Let U_1 and U_2 be two $N(T)$-invariant sectional sets in X. If $U_1 \neq U_2$, then there exists a chain of $N(T)$-invariant sectional sets $V_1 = U_1, V_2, \ldots, V_k = U_2$ in X such that V_{i+1} is an elementary transform of V_i for $i = 1, \ldots, k-1$.*

Proof. Let U_1 by determined by the section (A_1^-, A_1^+). For any $N(T)$-invariant sectional set V determined by (A^-, A^+), let $s(V, U_1)$ be the number $\#(A^+ - A_1^+)$ (notice that since the sections are $N(T)$-invariant, $\#A^+ = \#A_1^+ = r/2$). Suppose that the set Z of $N(T)$-invariant sectional sets U_2, which cannot be obtained be a sequence of elementary transformations starting from U_1, is not empty. Choose $U_2 \in Z$ for which $s(U_2, U_1)$ is minimal. Let U_2 be given by the section (A_2^-, A_2^+). There exists an element

$X_{i_0} \in A_2^+ - A_1^+$ which is minimal in A_2^+. It follows that

$$w(X_{i_0}) \in A_2^- - A_1^- \quad \text{and} \quad X_{i_0} \in A_2^+ \cap A_1^-;$$

in particular, X_{i_0} is not comparable with $w(X_{i_0})$. Hence we can define a new section (A^-, A^+), where

$$A^+ = \left(A_2^+ \cup \{w(X_{i_0})\}\right) - \{X_{i_0}\}, \qquad A^- = w(A^+).$$

Let W be the sectional set corresponding to (A^-, A^+). Then W is an $N(T)$-invariant sectional set and W is an elementary transform of U_2. Since $s(W, U_1) < s(U_2, U_1)$, there exists a chain of $N(T)$-invariant sectional sets $V_1 = U_1, V_2, \ldots, V_k = W$ such that V_{i+1} is an elementary transform of V_i for $i = 1, \ldots, k-1$. Thus U_2 can be obtained from U_1 by a chain of elementary transformations, which contradicts our assumption that $U_2 \in Z$.

LEMMA 3. *Let Y_1 and Y_2 be algebraic normal spaces. Suppose that Y_1 is complete. Assume that there exist non-empty complete subsets $Z_i \subset Y_i$ $(i = 1, 2)$ such that $Y_1 - Z_1$ is isomorphic to $Y_2 - Z_2$. Let, for $i = 1, 2$, n_i be the number of connected components of Z_i. Then $n_1 \leqslant n_2$ implies that Y_2 is complete.*

Proof. We are indebted to F. Bogomolov for the following simple proof of the lemma. We may assume that $K = C$, the field of complex numbers, and consider X as an analytic space with its topology induced by the natural topology of C. Since $Y_1 - Z_1$ is isomorphic to $Y_2 - Z_2$, we may identify these two spaces. Suppose that Y_2 is not complete, hence it is not compact. There exists a one-point compactification $\tilde{Y}_2$ of Y_2 (since Y_2 is locally compact). Let $\tilde{Y}_2 - Y_2 = \{y_0\}$. Since Z_2 is compact, there exist a neighbourhood U of Z_2 such that $U - Z_2$ has exactly n_2 connected components each with non-compact closure in $\tilde{Y}_2 - Z_2$ and a neighbourhood V of y_0 such that $U \cap V = \emptyset$. The set $\tilde{Y}_2 - (U \cup V)$ is compact and, for any compact subset $F \supset Y_2 - (U \cup V)$ of $Y_2 - Z_2$, $(Y_2 - Z_2) - F$ has at least $n_2 + 1$ connected components with non-compact closures in $Y_2 - Z_2$. However, for any such subset F we have

$$(Y_2 - Z_2) - F = (Y_1 - Z_1) - F$$

and $Y_1 - F$ is a neighbourhood of Z_1 in Y_1. Let $f\colon Y_1' \to X_1$ be a desingularization of X_1. Then the number of connected components of $f^{-1}(Z_1)$ is equal to n_1 (by ZMT), and hence for any neighbourhood U' of $f^{-1}(Z_1)$ in Y_1' the difference $U' - f^{-1}(Z_1)$ has at most n_1 connected components with non-compact closures in $Y_1' - f^{-1}(Z_1)$. Now it suffices to take

$$U' = f^{-1}\left((Y_1 - Z_1) - F\right) \cup f^{-1}(Z_1) = f^{-1}(Y_1 - F)$$

to conclude that the number of connected components of $(Y_1 - Z_1) - F$ with non-compact closure in $Y_1 - Z_1$ is at most equal to n_1. Thus $n_2 + 1 \leqslant n_1$. This contradiction shows that Y_2 is complete.

LEMMA 4 (A. J. Sommese). *Let* $U \subset X$ *be an* $N(T)$*-invariant semi-sectional set. Let* $x \in U$. *Then either* $\mathrm{SL}(2)x \subset U$ *or there exist* x^+, $x^- \in \mathrm{SL}(2)x$ *such that*

$$\mathrm{SL}(2)x - U = B_+ x^+ \cup B_- x^- .$$

Proof. Let U be determined by the semi-section (A^-, A^0, A^+). Assume that $\mathrm{SL}(2)x \not\subset U$. Then there exists a connected component X_i of X^T such that either

$$X_i \in A^- \quad \text{and} \quad X_i^- \cap \mathrm{SL}(2)x \neq \emptyset$$

or

$$X_i \in A^+ \quad \text{and} \quad X_i^+ \cap \mathrm{SL}(2)x \neq \emptyset .$$

Since $wA^- = A^+$, $\tau(X_i^-) = (wX_i)^+$; hence

$$X_i \in A^- \quad \text{and} \quad X_i^- \cap \mathrm{SL}(2)x \neq 0$$

iff

$$w(X_i) \in A^+ \quad \text{and} \quad \big(w(X_i)\big)^+ \cap \mathrm{SL}(2)x \neq \emptyset .$$

Thus we may assume that $X_i \in A^-$ and $X_i^- \cap \mathrm{SL}(2)x \neq \emptyset$. Let

$$x^- \in X_i^- \cap \mathrm{SL}(2)x .$$

It follows from [7], Theorem 7.1, or [5], Theorem 2, that

$$B_- x^- \subset X_i^- \cap \mathrm{SL}(2)x \subset \mathrm{SL}(2)x - U$$

and for $x^+ = \tau(x^-)$ we have

$$B_+ x^+ \subset \mathrm{SL}(2)x - U .$$

Therefore, it suffices to show that if $B_- x_1 \subset \mathrm{SL}(2)x - U$ for some $x_1 \in X$, then $B_- x^- = B_- x_1$. Consider the Bruhat decomposition $\mathrm{SL}(2) = B_- \cup B_- \tau B_-$ and assume that $B_- x^- \neq B_- x_1$. Then

$$B_- \tau B_- x^- \supset B_- x_1$$

and we may assume that $\tau b x^- = x_1$ for some $b \in B_-$. Thus $x_1 \in \tau(X_i^-) = (wX_i)^+$. Therefore

$$B_+ x_1 \cup B_- x_1 \subset \mathrm{SL}(2)x - U .$$

Since $B_- x_1 \cup B_+ x_1$ is connected and $X - U$ is a union of two disjoint and closed subsets $\bigcup_{X_j \in A^-} X_j^-$ and $\bigcup_{X_j \in A^+} X_j^+$, we have

$$B_- x_1 \cup B_+ x_1 \subset \bigcup_{X_j \in A^-} X_j^- \quad \text{or} \quad B_- x_1 \cup B_+ x_1 \subset \bigcup_{X_j \in A^+} X_j^+ .$$

Since (again by [7], Theorem 7.1, or [5], Theorem 2) $\bigcup_{X_j \in A^-} X_j^-$ is B_--invariant and $\bigcup_{X_j \in A^+} X_j^+$ is B_+-invariant, we infer that either

$$B_- B_+ x_1 \subset \bigcup_{X_j \in A^-} X_j^-$$

or

$$B_+ B_- x_1 \subset \bigcup_{X_j \in A^+} X_j^+ .$$

Thus

$$\mathrm{SL}(2) x_1 \subset \bigcup_{X_j \in A^+} X_j^+ \quad \text{or} \quad \mathrm{SL}(2) x_1 \subset \bigcup_{X_j \in A^-} X_j^-$$

(since $B_+ B_-$ and $B_- B_+$ are dense in $\mathrm{SL}(2)$). This leads to a contradiction since $\mathrm{SL}(2) x_i$ is τ-invariant and no τ-invariant connected non-empty set is contained in

$$\bigcup_{X_j \in A^+} X_j^+ \cup \bigcup_{X_j \in A^-} X_j^- .$$

LEMMA 5. *Let U be an $N(T)$-invariant sectional set determined by a section (A^-, A^+). Let X_{i_0} be a maximal element of A^- and let X_{i_0} be not comparable with $w(X_{i_0})$. Then $\bigcap_{g \in \mathrm{SL}(2)} gU$ is open and $\mathrm{SL}(2)(X_{i_0}^+ - B_+ X_{i_0})$ is a closed subvariety in $\bigcap_{g \in \mathrm{SL}(2)} gU$.*

Proof. In the proof we use several times without mentioning [7], Theorem 7.1 (or [5], Theorem 2). First we prove that

$$\mathrm{SL}(2)(X_{i_0}^+ - B_+ X_{i_0}) \subset \bigcap_{g \in \mathrm{SL}(2)} gU .$$

Suppose that, for some $x_0 \in X_{i_0}^+ - B_+ X_{i_0}$, $\mathrm{SL}(2) x_0$ is not contained in $\bigcap_{g \in \mathrm{SL}(2)} gU$. Then $\mathrm{SL}(2) x_0$ is not contained in U. It follows from Lemma 4 that

$$\mathrm{SL}(2) x_0 - U = B_+ g_1 x_0 \cup B_- g_2 x_0 \quad \text{for some } g_1, g_2 \in \mathrm{SL}(2).$$

Let U' be the sectional set determined by the $N(T)$-invariant section

$$\big((A^- - \{X_{i_0}\}) \cup \{w(X_{i_0})\}, (A^+ - \{w(X_{i_0})\}) \cup \{X_{i_0}\}\big).$$

Then $B_+ x_0 \cap U' = \emptyset$ and $B_- \tau(x_0) \cap U' = \emptyset$. Therefore, again by Lemma 4, $B_+ g_1 x_0 \subset U'$ and $B_- g_2 x_0 \subset U'$, and thus

$$B_+ g_1 x_0 \cup B_- g_2 x_0 \subset X_{i_0}^- \cup \big(w(X_{i_0})\big)^+$$

(since $U' - U = X_{i_0}^- \cup \big(w(X_{i_0})\big)^+$). But $B_+ g_1 x_0$ is irreducible; hence

$$B_+ g_1 x_0 \subset X_{i_0}^- \quad \text{or} \quad B_+ g_1 x_0 \subset \big(w(X_{i_0})\big)^+ .$$

If $B_+ g_1 x_0 \subset X_{i_0}^-$, then $B_- B_+ g_1 x_0 \subset X_{i_0}^-$ (in fact, $X_{i_0}^-$ is B_--invariant) and $\overline{SL(2)x_0} \subset \overline{X_{i_0}^-}$ (since $\overline{B_- B_3} = SL(2)$). But this is not possible since

$$x_0 \in X_{i_0}^+ - B_+ X_{i_0} \quad \text{and} \quad (X_{i_0}^+ - X_{i_0}) \cap \overline{X_{i_0}^-} = \emptyset.$$

If $B_+ g_1 x_0 \subset (w(X_{i_0}))^+$, then $g_1 x_0 \in (w(X_{i_0}))^+$, and hence $g_1 \notin B_+$. Since $SL(2) = B_+ \cup B_+ \tau B_+$, $g_1 \in B_+ \tau B_+$, and we obtain $\tau b_1 x_0 \in (w(X_{i_0}))^+$ for some $b_1 \in B^+$. But $b_1 x_0 \in X_i^+$, and therefore $\tau b_1 x_0 \in \overline{(w(X_{i_0}))}$. Thus $\tau b_1 x_0 \in w(X_{i_0})$ and $x_0 \in B_+ X_{i_0}$, but this contradicts our assumption that $x_0 \in X_{i_0}^+ - B_+ X_{i_0}$.

The sets $\bigcap_{g \in SL(2)} gU$ and $\bigcap_{g \in SL(2)} gU'$ are open (see [2], 4.2). Moreover, since

$$SL(2)(X_{i_0}^+ - B_+ X_{i_0}) \subset \bigcap_{g \in SL(2)} gU,$$

we have

$$SL(2)(X_{i_0}^+ - B_+ X_{i_0}) = \bigcap_{g \in SL(2)} gU - \bigcap_{g \in SL(2)} gU'.$$

Thus $SL(2)(X_i^+ - B_+ X_{i_0})$ is a closed subvariety of $\bigcap_{g \in SL(2)} gU$. The proof is complete.

COROLLARY 1. *Let U be an $N(T)$-invariant sectional set determined by a section (A^-, A^+). Let X_{i_0} be not comparable with $w(X_{i_0})$ and maximal in A^-. Then the geometric quotient*

$$SL(2)(X_{i_0}^+ - B_+ X_{i_0}) \to SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2)$$

exists and

$$SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2) \approx X_{i_0}^+ - B_+ X_{i_0}/B_+.$$

Proof. The existence of the geometric quotients

$$SL(2)(X_{i_0}^+ - B_+ X_{i_0}) \to SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2)$$

and

$$X_{i_0}^+ - B_+ X_{i_0} \to (X_{i_0}^+ - B_+ X_{i_0})/B_+$$

follows from Lemma 5 (and [2], Theorem 4.1) and Lemma 1, respectively. Since the geometric quotient is categorical, we have a canonical morphism

$$\eta\colon X_{i_0}^+ - B_+ X_{i_0}/B_+ \to SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2).$$

The morphism η is surjective. Moreover, η is injective. Otherwise, we would have two different B_+-orbits in $X_{i_0}^+ - B_+ X_{i_0}$ contained in one SL(2)-orbit. However, this is not possible since then for the sectional set U' defined as in the proof of Lemma 5 the SL(2)-orbit would contain two different B_+-orbits not contained in U', and this contradicts Lemma 5. Since $SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2)$ is normal, η is an isomorphism.

Remark. If $X_i^+ - B_+ X_i = \emptyset$ and U is any $N(T)$-invariant sectional set determined by a section (A^-, A^+) such that

$$\bigcap_{g \in SL(2)} gU \neq \emptyset,$$

then $X_i \in A^+$. In fact, $X_i^+ - B_+ X_i = \emptyset$ implies $X_i^+ = B_+ X_i$. Hence

$$\dim X_i^+ = \dim B_+ X_i \leqslant \dim X_i + 1.$$

Thus $\dim X_i^- \geqslant \dim X - 1$. If $\dim X_i^- = \dim X$, then X_i is the sink, and hence $X_i \in A^+$. If $\dim X_i^- = \dim X - 1$, then either

$$\dim SL(2)\, X_i^- = \dim X - 1$$

or

$$\dim SL(2)\, X_i^- = \dim X.$$

In the first case,

$$X_i \succ w(X_i) \quad \text{and} \quad X_i \in A^+;$$

in the second case,

$$SL(2)\, X_i^- \cap \bigcap_{g \in SL(2)} gU \neq \emptyset,$$

and hence again $X_i \in A^+$.

Lemma 6. *If there exists an $N(T)$-invariant sectional subset of X, then there exists an $N(T)$-invariant sectional subset U such that the geometric quotient*

$$\bigcap_{g \in SL(2)} gU \to \bigcap_{g \in SL(2)} gU/SL(2)$$

exists and $\bigcap_{g \in SL(2)} gU/SL(2)$ is a complete algebraic space.

Proof. Since X is projective and smooth, there exists an $SL(2)$-linearized very ample sheaf L. Let X^s and X^{ss} be the sets of stable and semi-stable points with respect to T-linearization of L. The set X^{ss} is semi-sectional. Let the semi-sectional set correspond to a semi-section (A^-, A^0, A^+). Since X^{ss} is $N(T)$-invariant, we have $w(A^+) = A^-$ and $w(A^0) = A^0$. Since there exists an $N(T)$-invariant sectional set, there is no component X_j of X^T such that $w(X_j) = X_j$. Thus there exists an $N(T)$-invariant section (A_1^-, A_1^+) such that $A_1^+ \supset A^+$ (and hence $A_1^- \supset A^-$). Let U be the sectional set corresponding to this section. It follows from Theorem 4.1 of [2] that the quotient

$$\varphi\colon \bigcap_{g \in SL(2)} gU \to \bigcap_{g \in SL(2)} gU/SL(2)$$

exists. It follows from Chapter 2, Section 1, of [6] that the semi-geometric quotient

$$\psi\colon \bigcap_{g\in SL(2)} gX^{ss} \to \bigcap_{g\in SL(2)} gX^{ss}/SL(2)$$

exists and $\bigcap_{g\in SL(2)} gX^{ss}/SL(2)$ is projective. Notice that $X^{ss} \supset U \supset X^s$ and there exists a surjective morphism

$$\alpha\colon \bigcap_{g\in SL(2)} gU/SL(2) \to \bigcap_{g\in SL(2)} gX^{ss}/SL(2)$$

(since φ is a categorical quotient) and α restricted to

$$\bigcap_{g\in SL(2)} gX^s/SL(2) \subset \bigcap_{g\in SL(2)} gU/SL(2)$$

is an isomorphism onto

$$\bigcap_{g\in SL(2)} gX^s/SL(2) \subset \bigcap_{g\in SL(2)} gX^{ss}/SL(2).$$

Moreover,

$$X^{ss} - U = \bigcup_{X_i\in C} \left(X_i^+ \cup \left(w(X_i)\right)^-\right),$$

$$U - X^s = \bigcup_{X_i\in C} \left(X_i^- \cup \left(w(X_i)^+\right)\right),$$

where $C = A_1^+ - A_0$. It follows from Lemma 5 that

$$\bigcap_{g\in SL(2)} gU = \bigcap_{g\in SL(2)} gX^s \cup \bigcup_{X_i\in C} SL(2)(X_i^- - B_-\, X_i)$$

and

$$\bigcap_{g\in SL(2)} gU/SL(2) = \bigcap_{g\in SL(2)} gX^s/SL(2) \cup \bigcup_{X_i\in C} SL(2)(X_i^- - B_-\, X_i)/SL(2).$$

But, by Lemma 1, for any $X_i \in C$ the geometric quotient

$$(X_i^- - B_-\, X_i) \to (X_i^- - B_-\, X_i)/B_-$$

exists and $(X_i^- - B_-\, X_i)/B_-$ is complete.

Therefore, by Corollary 1, $SL(2)(X_i^- - B_-\, X_i)/SL(2)$ is complete and we may apply Lemma 3 taking

$$Y_1 = \bigcap_{g\in SL(2)} gX^{ss}/SL(2), \qquad Y_2 = \bigcap_{g\in SL(2)} gU/SL(2),$$

$$Z_1 = \alpha(Z_2) \quad \text{and} \quad Z_2 = \bigcup_{X_i\in C} SL(2)(X_i^- - B_-\, X_i)/SL(2)$$

to conclude that $\bigcap_{g\in SL(2)} gU/SL(2)$ is complete.

Now, we proceed with the proof of the theorem. By Lemmas 6 and 2 it is enough to prove that if U_1 and U_2 are $N(T)$-invariant sectional sets such that

(i) U_1 is an elementary transform of U_2,

(ii) $\bigcap_{g\in SL(2)} gU_1/SL(2)$ is complete,

then $\bigcap_{g\in SL(2)} gU_2/SL(2)$ is complete.

It follows from (i) above, Corollary 1, the Remark following Corollary 1 and Lemma 1 that we may apply Lemma 3 for

$$Y_1 = \bigcap_{g\in SL(2)} gU_1/SL(2), \qquad Y_2 = \bigcap_{g\in SL(2)} gU_2/SL(2),$$

$$Z_1 = SL(2)(X_{i_0}^+ - B_+ X_{i_0})/SL(2),$$

$$Z_2 = SL(2)(X_{i_0}^- - B_- X_{i_0})/SL(2)$$

and conclude that $\bigcap_{g\in SL(2)} gU_2/SL(2)$ is complete.

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