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THE SET FUNCTION T AND HOMOTOPIES

BY

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A *space* means a topological Hausdorff space. A *continuum* means a compact connected space. We denote by $[\alpha, \beta]$ the closed segment of reals from α to β and we put I for $[0, 1]$. A *mapping* means a continuous transformation. A mapping $H: X \times I \to Y$ is called a *homotopy*. If $X \subset Y$ and if $H(x, 0) = x$ for each $x \in X$, then the homotopy H is called a *deformation* of X in Y. Furthermore, if for each $x \in X$ the point $H(x, 1)$ is all the same, then the deformation H is called a *contraction* of X in Y. If a contraction of X in $Y = X$ exists, then X is said to be *contractible* (in itself).

Given a compact space Y and a set $A \subset Y$, we define $T(A)$ as the set of all points y of Y such that every subcontinuum of Y which contains y in its interior must intersect A (see [6]). It is known (see [3], Corollary 1, p. 373) that if Y is a continuum and A is a subcontinuum of Y, then $T(A)$ is a subcontinuum of Y.

A *dendroid* means an arcwise connected and hereditarily unicoherent metric continuum.

Using a result of [1], Bellamy and the author have proved the following proposition (see [2], Corollary 1) from which a theorem of Bennett (see below; cf. [4] and [2], Theorem 1) follows as a corollary.

PROPOSITION 1. *If X is a continuum, and A and B are closed subsets of X such that*

$$(*) \qquad A \cap T(B) = \emptyset = B \cap T(A) \quad and \quad T(A) \cap T(B) \neq \emptyset,$$

then X is not contractible.

THEOREM 1 (Bennett). *If a dendroid X contains two points p and q such that $p \in X \setminus T(q)$, $q \in X \setminus T(p)$, and $T(p) \cap T(q) \neq \emptyset$, then X is not contractible.*

The aim of this paper is to prove a more general result the proof of which is based on some ideas of Lelek [7]. To this end we need one more definition. A mapping $f: X \to Y$ from a space X into a space Y

is said to be *interior at a point* $x_0 \in X$ provided that, for every open set U in X containing x_0, $f(x_0)$ is an interior point of $f(U)$ (see [8], p. 149).

The main result of this paper is the following

THEOREM 2. *Let X be a compact space, Y a continuum, A and B closed subsets of Y which satisfy* (*) *and let $f\colon X \to Y$ be a mapping of X into Y which is interior at a point $x_0 \in f^{-1}(T(A) \cap T(B))$. If $H\colon X \times I \to Y$ is a homotopy with $H(x, 0) = f(x)$ for each $x \in X$, then*

$$H(\{x_0\} \times I) \subset T(A) \cap T(B).$$

Proof. Suppose that the theorem is false. It means that there are a homotopy $H\colon X \times I \to Y$ with $H(x, 0) = f(x)$ for each $x \in X$ and a number $t_0 \in I$ such that

$$H(x_0, t_0) \in Y \setminus (T(A) \cap T(B)).$$

Thus $t_0 > 0$. We claim that there is a number $t_1 \in (0, t_0]$ such that

$$H(x_0, t_1) \in Y \setminus (T(A) \cap T(B)) \quad \text{and} \quad H(\{x_0\} \times [0, t_1]) \subset Y \setminus (A \cup B).$$

Indeed, if

$$H(\{x_0\} \times I) \cap (A \cup B) = \emptyset,$$

we put $t_1 = t_0$. If

$$H(\{x_0\} \times I) \cap (A \cup B) \neq \emptyset,$$

then let,

$$t' = \min\{t \in I\colon H(x_0, t) \in A \cup B\}.$$

Without loss of generality we may assume $H(x_0, t') \in A$. It follows from the continuity of H and from the definition of t' that there is a sufficiently small number $\delta > 0$ such that $t_1 = t' - \delta$ satisfies the required conditions. Thus the claim is proved and let some $t_1 \in I$ be as in the claim. Hence

$$H(x_0, t_1) \in (Y \setminus T(A)) \cup (Y \setminus T(B))$$

and we may assume that $H(x_0, t_1) \in Y \setminus T(A)$.

Put $J = [0, t_1]$. The set $H(\{x_0\} \times J)$ represents a path in Y whose initial point is $f(x_0)$ and which is non-degenerate by assumption. Thus $\{x_0\} \times J$ is contained in $H^{-1}(Y \setminus (A \cup B))$ which is open in $X \times I$. Since J is compact, there exists an open subset U of X such that

$$x_0 \in U \quad \text{and} \quad U \times J \subset H^{-1}(Y \setminus (A \cup B)).$$

Since $H(x_0, t_1) \in Y \setminus T(A)$, by the definition of $T(A)$ there exists a continuum $W \subset Y$ with $H(x_0, t_1) \in \operatorname{Int} W$ and $W \cap A = \emptyset$. By the continuity of H there exists an open set V in X such that

$$x_0 \in V \subset \overline{V} \subset U \quad \text{and} \quad H(V \times \{t_1\}) \subset \operatorname{Int} W.$$

Put $K = H(\overline{V}\times J)\cup W$. Thus K is a subcontinuum of Y which misses A. Since $x_0 \in V$ and f is interior at x_0, we have

$$H(x_0, 0) = f(x_0) \in \operatorname{Int} f(V) = \operatorname{Int} H(\overline{V}\times\{0\}) \subset K.$$

Thus K is a continuum containing $f(x_0)$ in its interior and missing A, and hence $f(x_0)$ is not in $T(A)$, a contradiction. Thus the proof is complete.

Recall that a mapping $f\colon X\to Y$ from a space X into a space Y is said to be *open* if it maps open sets in X onto open sets in Y. It is obvious that f is open if and only if it is interior at each point x of X. Thus the following theorem is an immediate consequence of Theorem 2:

THEOREM 3. *Let X be a compact space, Y a continuum, A and B closed subsets of Y which satisfy* (*), *and let $f\colon X\to Y$ be an open mapping of X into Y. If $H\colon X\times I\to Y$ is a homotopy with $H(x,0) = f(x)$ for each $x\in X$, then*

$$H\big(f^{-1}(T(A)\cap T(B))\times I\big) \subset T(A)\cap T(B).$$

A non-empty subset A of a space X is said to be *homotopically fixed* (see [5], Definition 2) if for every deformation $H\colon X\times I\to X$ we have $H(A\times I)\subset A$. The following proposition is proved in [5] for metric spaces, but the argument in the non-metric case is very similar.

PROPOSITION 2. *If a space X contains a proper subset A which is homotopically fixed, then X is not contractible.*

Taking in Theorem 3 a continuum $X = Y$, the identity mapping for f and applying Proposition 2, we get the following corollary which is a slightly stronger form of Proposition 1.

COROLLARY 1. *If X is a continuum, and A and B are closed subsets of X which satisfy* (*), *then $T(A)\cap T(B)$ is a homotopically fixed proper subset of X, and thus X is not contractible.*

In particular, if X is a dendroid, and A and B are one-point sets, we get, as a consequence of Corollary 1, Theorem 1 (of Bennett).

Note that the assumption on the mapping f (that f is interior at a point x_0 of $f^{-1}(T(A)\cap T(B))$) cannot be omitted in Theorem 2. This can be seen from the following example:

Example 1. There exist dendroids X and Y, two points p and q in Y such that $p\in Y\setminus T(q)$, $q\in Y\setminus T(p)$ and $T(p)\cap T(q)\neq\emptyset$, a monotone surjection $f\colon X\to Y$ that is interior at no point of $f^{-1}\big(T(p)\cap T(q)\big)$, and a homotopy $H\colon X\times I\to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = p$ for each $x\in X$.

Define X as the disjoint union of two harmonic fans A and B with the unique accumulation points a and b, respectively, on their bases joined by an arc ab, otherwise missing the fans: $X = A\cup ab\cup B$. Define Y as the union of two harmonic fans P and Q with vertices p and q, respectively, having the unique accumulation point y_0 on their bases as the only common point. Thus $T(p) = py_0$ and $T(q) = qy_0$. Define $f\colon X \to Y$ as follows: $f|A$ and $f|B$ are natural homeomorphisms between A and P and between B and Q, respectively, $f|ab$ is a constant mapping which maps the whole arc ab into the point y_0. Thus $f^{-1}(y)$ is a one-point set for every y in $Y \setminus \{y_0\}$, and $f^{-1}(y_0) = ab$. Thus f is monotone. Finally, define $H\colon X \times I \to Y$ in the following way: if $t \in [0, 1/2)$, then $H|A \times \{t\}$ and $H|B \times \{t\}$ are embeddings of A and B into P and Q, respectively, with $H(x, 0) = f(x)$; $H|ab \times \{0\}$ shrinks ab to the point y_0, and $H|ab \times \{t\}$ is an embedding of ab into pq. All this can be arranged in such a way that for $t = 1/2$ we have

$$H(A \times \{1/2\}) = \{p\}, \qquad H(B \times \{1/2\}) = \{q\}$$

and $H|ab \times \{1/2\}$ is a homeomorphism of the arc ab onto the arc pq which maps a into p and b into q. Let $C\colon pq \times I \to pq$ be an arbitrary contraction which contracts pq in itself to the point p. For $t \in (1/2, 1]$ write

$$H(x, t) = C\big(H(x, 1/2), 2t-1\big).$$

Thus H satisfies the desired conditions.

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