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# ON BEHAVIOUR OF NON-NEGATIVE WEAK SOLUTIONS OF PARABOLIC EQUATIONS AT THE BOUNDARY

BY

J. CHABROWSKI (KATOWICE)

The well known Fatou theorem asserts that if $f$ is an analytic bounded function in the disc $|z| < 1$, then $f$ has a non-tangential limit at almost every point $e^{i\tau}$ $(0 \leqslant \tau < 2\pi)$. Several authors extended this result to harmonic functions. Some known generalizations of Fatou theorem to harmonic functions in certain kinds of domains of an $n$-dimensional Euclidean space $E_n$ are due to Brelot and Doob [5], Calderón [6], Carleson [7], Widman [14], Hunt and Wheeden [10]. In a subsequent study Kato [11] extended these results to non-negative classical solutions of the equation of heat conduction in an infinite strip $(0, T] \times E_n$. Results of Kato were extended by the author in [8] and [9] to classical solutions of special parabolic systems of equations and to non-negative weak solutions of parabolic equations in a divergence form. The present paper deals with similar problems for weak solutions of a very general class of linear parabolic equations in an infinite strip and in a bounded domain.

**1. Definitions.** By $x = (x_1, \ldots, x_n)$ we shall denote a point in the space $E_n$ with $n \geqslant 1$ and by $t$ a point of the real line. Let $T$ be a fixed positive number. In this paper $\Omega$ will always denote a bounded domain in $E_n$ and $\Sigma$ will denote a domain which is either a bounded domain $\Omega$ or the space $E_n$. We shall use notation introduced by Aronson in [3].

Let $B_1(\Sigma)$ denote a Banach space of functions defined on $\Sigma$ with a norm $|\cdot|_1$, and let $B_2(I)$ denote a Banach space of functions defined on an interval $I$ with a norm $|\cdot|_2$. A function $w = w(t, x)$ defined and measurable on $I \times \Sigma$ is said to belong to the class $B_2[I; B_1(\Sigma)]$ if $w(t, \cdot) \in B_1(\Sigma)$ for almost all $t \in I$ and $\| w|_1(t)|_2 < \infty$.

Classes $L^q[I; L^p(\Sigma)]$ will be denoted by $L^{pq}(I \times \Sigma)$, and for $w \in L^{pq}(I \times \Sigma)$ with $1 \leqslant p, q < \infty$ we define

$$\|w\|_{pq} = \left(\int_I \left(\int_\Sigma |w|^p \, dx\right)^{q/p} dt\right)^{1/q}.$$

A measurable function $w$ on $\Omega$ is said to belong to $H^{1,2}(\Omega)$ if $w$ possesses a distribution derivative $w_x$ and

$$\|w\|_{L^2(\Omega)}+\|w_x\|_{L^2(\Omega)} = \Big(\int_\Omega \Big(|w|^2+\sum_{i=1}^n |w_{x_i}|^2\Big)\,dx\Big)^{1/2} < \infty .$$

The space $H_0^{1,2}(\Omega)$ is the completion of the function space $C_0^\infty(\Omega)$ in the norm $\|w\|_{L^2(\Omega)}+\|w_x\|_{L^2(\Omega)}$, and the space $H^{1,2}(E_n)$ is the completion of the function space $C^\infty(E_n)$ in the norm $\|\varphi\|_{L^2(E_n)}+\|\varphi_x\|_{L^2(E_n)}$.

Given $(t, x)\in(0, T]\times\Sigma$, consider the second order linear differential operator

$$Lu = u_t-\{A_{ij}(t, x)u_{x_i}+A_j(t, x)u\}_{x_j}-B_j(t, x)u_{x_j}-C(t, x)u,$$

where coefficients $A_{ij}$, $A_j$, $B_j$ and $C$ are measurable in $(0, T]\times\Sigma$. We use the convention of summation over repeated Latin indices. Throughout this paper we shall assume that the operator $L$ is uniformly parabolic (i.e., there exists a $\nu>0$ such that for all $\xi\in E_n$ and for almost all $(t, x)\in(0, T]\times\Sigma$ there is $A_{ij}(t, x)\xi_i\xi_j \geqslant \nu|\xi|^2$) and coefficients $A_{ij}(t, x)$ are bounded in $(0, T]\times\Sigma$.

**2. Fatou property for solutions in** $(0, T]\times E_n$. In this section we shall deal with the non-negative solutions of $Lu = 0$ in the strip $(0, T]\times E_n$. Troughout this section it will be assumed that the coefficients of $L$ satisfy the following conditions:

I. Let $Q_0 = (0, T]\times(|x|<R_0)$. The coefficients $A_j$ and $B_j$ are bounded for $|x|\geqslant R_0$ and $t\in(0, T]$. Each of the coefficients $A_j$ and $B_j$ belongs to space $L^{pq}(Q_0)$, where $p$ and $q$ are such that

$$2<p, q\leqslant\infty \quad\text{and}\quad \frac{n}{2p}+\frac{1}{q}<\frac{1}{2}.$$

II. Let $F$ denote the family of cylinders of the form $(0, T]\times R(\sigma)$ contained in $(0, T]\times E_n$, where $R(\sigma)$ denotes an open cube in $E_n$ of edge length $\sigma$ and $\sigma = \min(1, \sqrt{T})$. The coefficient $C$ is bounded from above for $(t, x)\in(0, T]\times(|x|\geqslant R_0)$, and $\sup\|C\|_{pq}<\infty$ where the norms are taken over cylinders in the family $F$, and $p$ and $q$ are such that

$$1\leqslant p, q\leqslant\infty \quad\text{and}\quad \frac{n}{2p}+\frac{1}{q}<1.$$

We say that $u(t, x)$ is a *weak solution* of $Lu = 0$ in $(0, T]\times E_n$ if $u\in L^\infty[\delta, T;\ L^2_{\mathrm{loc}}(E_n)]\cap L^2[\delta, T;\ H^{1,2}_{\mathrm{loc}}(E_n)]$ for all $\delta\in(0, T)$ and

$$\iint_{(0,T]\times E_n} (-u\varphi_t+A_{ij}u_{x_i}\varphi_{x_j}+A_ju\varphi_{x_j}-B_ju_{x_j}\varphi-Cu\varphi)\,dt\,dx = 0$$

for any $\varphi\in C_0^1\big((0, T)\times E_n\big)$.

It is known that, under conditions I and II, there exists a fundamental solution $\Gamma(t, x;\ \tau, y)$ of $Lu = 0$ defined for $(t, x), (\tau, y) \in (0, T] \times E_n$ $(\tau < t)$, which satisfies the estimates

$$(1)\qquad K^{-1}(t-\tau)^{-n/2}\exp\left(-a_1\frac{|x-y|^2}{t-\tau}\right) \leqslant \Gamma(t, x;\ \tau, y) \leqslant K(t-\tau)^{-n/2}\exp\left(-a_2\frac{|x-y|^2}{t-\tau}\right)$$

for all $(t, x), (\tau, y) \in (0, T) \times E_n$, where $K$, $a_1$ and $a_2$ are positive constants. The existence of $\Gamma$ and estimates (1) were proved by Aronson [1], [2], [3].

THEOREM 1. *If $u(t, x)$ is a non-negative weak solution of $Lu = 0$ in $(0, T] \times E_n$, then there exists a non-negative function $f(x)$ such that*

$$\lim_{t\to 0} u(t, x) = f(x) \qquad \textit{almost everywhere in } E_n$$

*and $e^{-\lambda|x|^2} f(x) \in L^1(E_n)$ for some $\lambda > 0$.*

Proof. It follows from the representation theorem (Theorem 12 in [3]) that there exists a unique non-negative Borel measure $\varrho$ such that, for all $(t, x) \in (0, T] \times E_n$,

$$u(t, x) = \int_{E_n} \Gamma(t, x;\ 0, y)\,\varrho(dy),$$

where

$$\varrho(E) = \int_E e^{2\gamma|y|^2}\hat{\varrho}(dy) \qquad (\gamma > 0),$$

$E$ runs over all Borel subsets of $E_n$, and the non-negative measure $\hat{\varrho}$ is finite. By the Lebesgue decomposition theorem ([13], Theorem 14.6, p. 33) there is a non-negative function $f(y) \in L^1_{\mathrm{loc}}(E_n)$ and a non-negative singular measure $\mu$ such that

$$\varrho(dy) = f(y)\,dy + \mu(dy).$$

It is known that almost every point of the integrable function $f$ is a Lebesgue point and symmetrical derivative of the singular measure $\mu$ is almost everywhere equal to zero. Therefore, for any $\varepsilon > 0$, there exist $a$, $b > 0$ such that

$$a^{-n}\int_{|y-x|\leqslant a} |\varphi(y) - \varphi(x)|\,dy + \mu(dy) < \varepsilon$$

for all $0 < a \leqslant 2b$. For every $0 < t \leqslant \min(2b, T)$ choose a positive integer $P(t)$ such that

$$2^{P-1}\sqrt{t} \leqslant b \leqslant 2^P\sqrt{t}.$$

Consider the inequality

$$(2)\quad |u(t,x)-\int_{E_n}\Gamma(t,x;\,0,y)f(x)dy|$$
$$\leqslant \int_{|y-x|<\sqrt{t}} \Gamma(t,x;\,0,y)[|f(y)-f(x)|dy+\mu(dy)]+$$
$$+\sum_{L=1}^{P} \int_{2^{l-1}\sqrt{t}<|y-x|<2^l\sqrt{t}} \Gamma(t,x;\,0,y)[|f(y)-f(x)]dy+\mu(dy)]+$$
$$+\int_{|y-x|\geqslant b}\Gamma(t,x;\,0,y)f(x)dy+\int_{|y-x|\geqslant b}\Gamma(t,x;0,y)\varrho(dy)=J_1+J_2+J_3+J_4.$$

Using the upper bound (1) of $\Gamma$, it is not difficult to verify that (for details see [8] or [11])

$$(3)\qquad \lim J_i = 0 \quad \text{for } i=1,2,3,4.$$

Since (Theorem 10 [3])

$$\lim_{t\to 0}\int_{E_n}\Gamma(t,x;\,0,y)dy=1 \quad \text{for all } x\in E_n,$$

we infer, by (2) and (3), that

$$\lim_{t\to 0}\int_{E}\Gamma(t,x;\,0,y)\varrho(dy)=f(x).$$

Using the lower bound (1), we conclude that

$$e^{-\lambda|x|^2}f(x)\in L^1(E_n)$$

for some $\lambda>0$.

THEOREM 2. *Let $u(t,x)$ be a non-negative weak solution of $Lu=0$ in $(0,T]\times E_n$. If $\limsup\limits_{t\to 0} u(t,x)<\infty$ for all $x\in E_n$, then there exists a non-negative function $f(x)$ such that*

$$e^{-\lambda|x|^2}f(x)\in L^1(E_n) \quad \textit{for some} \quad \lambda>0,$$
$$u(t,x)=\int_{E_n}\Gamma(t,x;\,0,y)f(y)dy$$

*for all $x\in E_n$, and $\lim\limits_{t\to 0} u(t,x)=f(x)$ almost everywhere in $E_n$.*

Proof. From inequality (1) it follows that

$$\operatorname{const} t^{-n/2}\varrho(|y-x|<\sqrt{t})\leqslant t^{-n/2}\int_{|y-x|<\sqrt{t}}\varrho(dy)$$
$$\leqslant \int_{|y-x|<\sqrt{t}}\Gamma(t,x;\,0,y)\varrho(dy)\leqslant u(t,x).$$

Hence symmetrical upper derivative of measure $\varrho$ is finite for all $x \in E_n$. In view of Kato's result [11], the measure $\varrho$ is absolutely continuous with respect to Lebesgue measure, therefore $\mu = 0$.

**3. Fatou property for solutions in a bounded cylinder.** Let $\Omega$ be a fixed bounded open domain in $E_n$ and $Q = (0, T] \times \Omega$. Throughout this section it will be assumed that coefficients satisfy the following conditions:

I′. $A_j, B_j \in L^{pq}(Q)$, where $p$ and $q$ satisfy

$$2 < p, q \leqslant \infty \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < \frac{1}{2}.$$

II′. $C \in L^{pq}(Q)$, where $p$ and $q$ satisfy

$$1 < p, q \leqslant \infty \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < 1.$$

A function $u(t, x)$ is said to be a *weak solution* of $Lu = 0$ in $Q$ if $u \in L^\infty[\delta, T;\ L^2(\Omega)] \cap L^2[\delta, T;\ H_0^{1,2}(\Omega)]$ for all $\delta \in (0, T)$ and if $u(t, x)$ satisfies

$$\iint_Q (-u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + A_j u \varphi_{x_j} - B_j u_{x_j} \varphi - Cu\varphi)\, dt\, dx = 0$$

for any $\varphi \in C_0^1(Q)$.

Assumptions I′ and II″ imply the existence of a weak Green function $G(t, x;\ \tau, y)$ of $Lu = 0$ defined for $(t, x)$, $(\tau, y) \in Q$ $(\tau < t)$. Let $\Omega'$ be a convex subset of $\Omega$ such that the distance from $\Omega'$ to the boundary of $\Omega$ is positive. Then there exist positive constants $\eta$, $C_1$, $C_2$ $a_1$ and $a_2$ such that

$$G(t, x;\ \tau, y) \geqslant C_1 (t-\tau)^{-n/2} \exp\left(-a_1 \frac{|x-y|^2}{t-\tau}\right) \tag{4}$$

for all $x, \xi \in \Omega'$ and $\tau \leqslant t \leqslant \tau + \eta$, and

$$G(t, x;\ \tau, y) \leqslant C_2 (t-\tau)^{-n/2} \exp\left(-a_2 \frac{|x-y|^2}{t-\tau}\right) \tag{5}$$

for all $(t, x) \in (0, T) \times \Omega$.

The existence of the Green function and estimates (4) and (5) are due to Aronson ([3], Theorem 9). We shall use these results to derive a representation formula and the Fatou property for any non-negative weak bounded solution of $Lu = 0$ in $Q$.

THEOREM 3. *If $u(t, x)$ is a non-negative weak bounded solution of $Lu = 0$ in $Q$, then there exists a non-negative Borel measure $\varrho$ such that,* *for all* $(t, x) \in Q$,

$$u(t, x) = \int_{\Omega} G(t, x;\ 0, y)\varrho(dy).$$

Proof. Our proof is similar to that of Krzyżański [12]. In [4] it is shown that every weak solution $u$ of $Lu = 0$ in $Q$ has a representative which is continuous in $Q$. Therefore we shall assume that $u$ denotes the continuous representative of a given weak solution. Hence there is no difficulty in talking about the value of $u$ at any point of its domain. For arbitrary $s \in (0, T)$, $u(t, x)$ is a solution of the boundary value problem

$$Lv = 0 \text{ for } (t, x) \in Q, \qquad v(s, x) = u(s, x) \quad \text{ for } x \in \Omega$$

and

$$v(t, x) = 0 \quad \text{ for } (t, x) \in (s, T] \times \partial\Omega,$$

where $\partial\Omega$ is the boundary of $\Omega$. It follows from Theorem 9 in [3] that

$$u(t, x) = \int_{\Omega} G(t, x;\ s, y) u(s, y) dy \tag{6}$$

for $(t, x) \in (s, T] \times \Omega$. For an arbitrary fixed $x_0 \in \Omega$ and for each $s \in (0, T)$ define the Borel measure

$$\varrho_s(E) = \int_{E} G(T, x_0;\ s, y) u(s, y) dy. \tag{7}$$

In view of (6),

$$\varrho_s(E) \leqslant \varrho_s(\Omega) = u(T, x_0)$$

for all Borel subsets $E$ of $\Omega$. Since the measures $\varrho_s$ are uniformly bounded there exists a sequence $s_j \to 0$ such that corresponding measures $\varrho_{s_j}$ converge to a Borel measure $\hat{\varrho}$. In particular,

$$\lim_{j\to\infty} \int_{\Omega} f(y)\varrho_{s_j}(dy) = \int_{\Omega} f(y)\hat{\varrho}(dy)$$

for any $f \in C_0^0(\Omega)$. Set $\Omega = \bigcup_{r\geqslant 1} K_r$, where each $K_r$ is a compact set and $K_r \subset K_{r+1}$ for all $r$. For each $K_r$ we define a function $h_r \in C_0^0(\Omega)$ such that $h_r(x) = 1$ for $x \in K_r$. Using (7) we get from (6) the equality

$$u(t, x) = \int_{\Omega} G(t, x;\ s_j, y)\big(1 - h_r(y)\big)u(s_j, y)dy +$$
$$+ \int_{\Omega} G(t, x;\ s_j, y) G(T, x_0;\ s_j, y)^{-1} h_r(y)\varrho_{sj}(dy) = J_1 + J_2.$$

for all $(t, x) \in (s_j, T] \times \Omega$. Since $u$ is bounded we find that $\lim_{r\to\infty} J_1 = 0$ uniformly with respect to $s_j \in (0, t/2]$. In view of the choice of $h_r$

$$G(T, x_0;\ s_j, y)^{-1} h_r(y) \in C_0^0(\Omega)$$

for all $s_j < T$ and $r$. Hence

$$\lim_{j\to\infty} J_2 = \int_\Omega G(t, x;\ 0, y)G(T, x_0;\ 0, y)^{-1}h_r(y)\hat{\varrho}(dy).$$

Thus it follows from the monotone convergence theorem that $u$ is given by the formula

$$u(t, x) = \int_\Omega G(t, x;\ 0, y)\varrho(dy),$$

where $\varrho(dy) = G(T, x_0;\ 0, y)^{-1}\hat{\varrho}(dy)$.

THEOREM 4. *If $u(t, x)$ is a non-negative bounded weak solution of $Lu = 0$ in $Q$, then there exists a non-negative function $f \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$u(t, x) = \int_\Omega G(t, x;\ 0, y)f(y)dy$$

*or all $(t, x) \in Q$ and $\lim_{t\to 0} u(t, x) = f(x)$ almost everywhere in $\Omega$.*

Proof. By the Lebesgue decomposition theorem there exist a non-negative function $f \in L^1_{\mathrm{loc}}(\Omega)$ and a singular measure $\mu$ such that

$$\varrho(dy) = f(y)dy + \mu(dy).$$

Let $x$ be a fixed Lebesgue point of $f$ at which symmetrical derivative of the measure $\mu$ is equal to zero. Set

$$\begin{aligned} u(t, x) &= \int_{B(x)} G(t, x;\ 0, y)\varrho(dy) + \int_{\Omega - B(x)} G(t, x;\ 0, y)\varrho(dy) \\ &= J_1 + J_2, \end{aligned}$$

where $B(x)$ is a closed ball with the center $x$ such that $\mathrm{dist}(\partial\Omega, B(x)) > 0$. Using the metod from the proof of Theorem 1 we obtain $\lim_{t\to 0} J_1 = f(x)$. Applying estimate (5) we find that $\lim_{t\to 0} J_2 = 0$. Since $u$ is bounded in $Q$, using estimate (4) we conclude that symmetrical upper derivative of $\varrho$ is finite for all $x \in \Omega$, hence $\mu = 0$.

## REFERENCES


[1] D. G. Aronson, *On the Green's function for second order parabolic equations with discontinuous coefficients*, Bulletin of the American Mathematical Society 69 (1963), p. 841-847.

[2] — *Bounds for the fundamental solution of parabolic equation*, ibidem 73 (1967), p. 890-896.

[3] — *Non-negative solutions of linear parabolic equations*, Annali della Scuola Normale Superiore di Pisa 22 (1968), p. 607-694.

[4] — and J. Serrin, *Local behaviour of solutions of quasilinear parabolic equations*, Archive for Rational Mechanics and Analysis 25 (1967), p. 81-122.

[5] M. Brelot and J. L. Doob, *Limites angulaires et limites fines*, Annales de l'Institut Fourier 13 (1963), p. 395-413.

[6] A. P. Calderón, *On the behaviour of harmonic functions at the boundary*, Transactions of the Amercian Mathematical Society 68 (1950), p. 45-54.

[7] L. Carleson, *On the existence of boundary values for harmonic functions of several variables*, Arkiv för Mathematik 4 (1962), p. 393-399.

[8] J. Chabrowski, *Certaines propriétés des solutions non négatives d'un système parabolique d'équations*, Annales Polonici Mathematici 24 (1971), p. 145-151.

[9] — *Les propriétés des solutions faibles non négatives de l'équation parabolique*, Nagoya Mathematical Journal 39 (1970), p. 119-125.

[10] R. A. Hunt and R. L. Wheeden, *On the boundary values of harmonic functions*, Transactions of the American Mathematical Society 132 (1968), p. 307-322.

[11] M. Kato, *On positive solutions of the heat equation*, Nagoya Mathematical Journal 30 (1967), p. 203-207.

[12] M. Krzyżański, *Sur les solutions non négatives de l'équation linéaire normale parabolique*, Revue Roumaine du Mathématiques Pures et Appliquées 9 (1964), p. 393-408.

[13] S. Saks, *Theory of the integral*, Monografie Matematyczne, Warszawa–Lwów 1937.

[14] K. O. Widman, *On the boundary values of harmonic functions in $R^3$*, Arkiv för Mathematik 5 (1964), p. 221-230.

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