COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 1

STRUCTURES FOR A LOGIC WITH ADDITIONAL GENERALIZED QUANTIFIER

BY

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1. Introduction. In 1957 Mostowski [2] introduced the notion of a generalized quantifier. Some generalized quantifiers became of special interest, e.g. the quantifiers Q_a, where $(Q_a x)\varphi(x)$ is interpreted as "there exist at least $\aleph_a$ many x such that $\varphi(x)$". Let L_{Q_a} be the language which is formed by adding, to the first-order predicate logic L with equality, the quantifier Q_a. The question arises whether there exists a set of axioms and deduction rules for L_{Q_a} such that every sentence deducible in L_{Q_a} is logically valid and conversely. For L_{Q_1} this question was answered positively by Keisler [1]. To prove this he introduced the following notion of a weak model for L_{Q_1}: $(\mathfrak{A}, p)$ is a *weak model* for L_{Q_1} if $\mathfrak{A}$ is a structure for L, and p is a set of subsets of A. In this paper we study these weak structures. When trying to consider the Model Theory for weak models in a more systematic way, one can observe that there are several reasonable ways to introduce the notion of a submodel and of an extension. The aim of this paper* is to suggest notions which seem to be the most natural (i.e., having the most natural properties) in the Model Theory.

In Section 3 we introduce four different notions of a submodel and of an extension.

In Section 5 we discuss the notion of a chain of models. In all considerations we distinguish four cases corresponding to the four definitions of a submodel and of an extension.

In Section 6 elementary extensions and elementary chains are studied. The first definition of an elementary extension turns out to be closely related to the definition of an elementary extension given by Keisler [1].

* This paper and its sequel (this volume, p. 161-173) form the master thesis of the author. He made it as a student of Prof. de Iongh, Nijmegen, under guidance of Dr. B. Z. Węglorz. The publication of this paper was possible when the author was in Wrocław, supported by grants of the Polish government and the Niels Stensen Stichting, Holland.

We prove some generalizations of theorems of [3]. The sequel of this paper will contain a study of the Compactness Theorem for weak models for L_{Q_1} and of both Löwenheim-Skolem Theorems. We shall prove there also a Łoś theorem for the models concerned.

From this paper and its sequel it follows that the first definition of a submodel and of an extension gives the best possibilities to generalize properties of structures for L to weak structures for L_{Q_1}.

2. Preliminaries. Throughout this paper the Greek letters ξ, ζ, δ, η denote ordinals, and $\alpha, \beta, \gamma, \varkappa, \lambda, \mu, \nu$ denote cardinals. $S(A)$ is the set of all subsets of the set A. Let L be a first-order predicate logic with equality. L is supposed to have no function symbols. The predicate letters of L are P_ξ for $\xi < \alpha$, where α is a cardinal fixed for the given language L. The set $\{c_\xi\colon \xi < \beta\}$, where β is also a cardinal fixed for the given language L, is the set of individual constants of L. The language L has countably many variables $v_0, v_1, \ldots$ The notions of a formula of L, of a sentence of L and of a structure for L, are as usual, as is the definition of $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$. Let L_Q be the language which we obtain from L by adding to L the quantifier Q. The set of formulas of L_Q is the smallest set X which has the following properties:

(i) all atomic formulas of L are in X;

(ii) if φ, ψ are in X, and y is a variable, then $\varphi\wedge\psi$, $\neg\varphi$, $(\exists y)\varphi$ and $(Qy)\varphi$ are in X.

A structure for L_Q is a pair $(\mathfrak{A}, p)$ such that $\mathfrak{A}$ is a structure for L and $p \subseteq S(A)$. The notion of an n-tuple $a_1, \ldots, a_n \in A$ satisfying a formula $\varphi(v_1, \ldots, v_n)$ of L_Q in $(\mathfrak{A}, p)$ is defined in the usual way, by induction on the construction of φ and is denoted by $(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n]$. The Q-clause in the definition is

$$(\mathfrak{A}, p) \models (Qv_m)\varphi[a_1, \ldots, a_n] \text{ iff } \{b \in A : (\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n]\} \in p,$$

where $\varphi(v_1, \ldots, v_n)$ is a formula of L_Q and $m \leqslant n$. The other clauses in the definition are the familiar ones for L.

Definition. (i) Let $(\mathfrak{A}, p)$ and $(\mathfrak{A}, q)$ be given. Then

$$(\mathfrak{A}, p) =_* (\mathfrak{A}, q)$$

if

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \text{ iff } (\mathfrak{A}, q) \models \varphi[a_1, \ldots, a_n]$$

for all formulas φ in L_Q, all n and all n-tuples $a_1, \ldots, a_n$ in A.

(ii) Let $(\mathfrak{A}, p)$ and $p^1 \subseteq S(A)$ be given. Then p^1 is called *minimal* with respect to the p-interpretation of Q in $\mathfrak{A}$ if $(\mathfrak{A}, p) =_* (\mathfrak{A}, p^1)$ and $p^1 \subseteq q$ for all $q \subseteq S(A)$ such that $(\mathfrak{A}, p^1) =_* (\mathfrak{A}, q)$.

The following Lemmas 1 and 2 show that, for every $(\mathfrak{A}, p)$, there is exactly one $p^1 \subseteq S(A)$ which is minimal with respect to the p-interpretation of Q in $\mathfrak{A}$.

LEMMA 1. *Let* $(\mathfrak{A}, p)$ *be given and let* $p_0 = \{x \in p$: *there are a* $\varphi(v_1, \ldots, v_n)$ *in* L_Q, *an* n, *an* $m \leqslant n$, *and an* n*-tuple* $a_1, \ldots, a_n \in A$ *such that* $x = \{b \in A: (\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n]\}\}$. *Then* $(\mathfrak{A}, p_0) =_* (\mathfrak{A}, p)$.

The proof follows by induction on the construction of φ.

LEMMA 2. *Let* $(\mathfrak{A}, p)$ *and* $(\mathfrak{A}, q)$ *be given and let* p_0 *be as in Lemma* 1. *If* $(\mathfrak{A}, p) =_* (\mathfrak{A}, q)$, *then* $p_0 \subseteq q$.

Proof. Let $x \in p_0$. Then $x \in p$, and so we have

$$(\mathfrak{A}, p) \models (Qv_m)\varphi[a_1, \ldots, a_n]$$

and, by $(\mathfrak{A}, p) =_* (\mathfrak{A}, q)$,

$$x = \{b \in A: (\mathfrak{A}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\}$$

and

$$(\mathfrak{A}, q) \models (Qv_m)\varphi[a_1, \ldots, a_n].$$

Hence $x \in q$.

Let $(\mathfrak{A}, p)$ be given. If p is minimal with respect to the p-interpretation of Q in $\mathfrak{A}$, then we will call p, simply, *minimal*.

3. Substructures and extensions. We introduce now four different notions of a substructure and of an extension. Let $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ be given structures for L_Q.

(a) $(\mathfrak{A}, p) \subseteq_1 (\mathfrak{B}, q)$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and, for every $x \in p$, there exists a $y \in q$ such that $x = y \cap A$.

(b) $(\mathfrak{A}, p) \subseteq_2 (\mathfrak{B}, q)$ if $(\mathfrak{A}, p) \subseteq_1 (\mathfrak{B}, q)$ and, for every $y \in q$, $y \cap A \in p$.

(c) $(\mathfrak{A}, p) \subseteq_3 (\mathfrak{B}, q)$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and $p \subseteq q$.

(d) $(\mathfrak{A}, p) \subseteq_4 (\mathfrak{B}, q)$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and $p = q \cap S(A)$.

Note that if $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ are such that, for a certain ordinal α,

$$p = \{X \subseteq A: |X| > \aleph_\alpha\} \quad \text{and} \quad q = \{X \subseteq B: |X| > \aleph_\alpha\},$$

then, considering $\mathfrak{A}$ and $\mathfrak{B}$ as structures for L_{Q_α}, we have $(\mathfrak{A}, p) \subseteq_4 (\mathfrak{B}, q)$ iff $\mathfrak{A} \subseteq \mathfrak{B}$.

If $(\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$, where $i \in \{1, 2, 3, 4\}$, then $(\mathfrak{A}, p)$ is called an *i-submodel* of $(\mathfrak{B}, q)$ and $(\mathfrak{B}, q)$ is called an *i-extension* of $(\mathfrak{A}, p)$.

The following shows the relation between structures for L_Q and certain second-order structures.

Let $\boldsymbol{K}$ be the class of structures $\langle C, A, P, E, P_\xi^C, c_\zeta^C \rangle_{\xi<\alpha, \zeta<\beta}$ such that the following conditions are satisfied:

(i) C is a set.

(ii) A is a unary relation over C, which is not empty.

(iii) P is a unary relation over C.

(iv) $P \cap A = \emptyset$ and $P \cup A = C$.

(v) E is a binary relation over C such that $A(x)$ and $P(y)$ whenever $E(x, y)$.

(vi) For every $\xi < \alpha$, P_ξ^C is a relation over C such that $A(x_i)$ for all $i \in \{1, \ldots, n\}$ whenever $P_\xi^C(x_1, \ldots, x_n)$.

(vii) For every $\zeta < \beta$, c_ζ^C is an element of C such that $A(c_\zeta^C)$.

(viii) The following sentence holds:

$$\forall v_0 \forall v_1 \Big(\big(P(v_0) \wedge P(v_1)\big) \to \Big((v_0 = v_1) \rightleftarrows (\forall v_2)\big(E(v_2, v_0) \rightleftarrows E(v_2, v_1)\big)\Big)\Big).$$

We can translate the formulas of L_Q into formulas of $L_{\boldsymbol{K}}$, i.e., the language of $\boldsymbol{K}$, as follows:

$$\varphi^* = \varphi \quad \text{if } \varphi \text{ is an atomic formula of } L_Q,$$
$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*, \qquad (\varphi \vee \psi)^* = \varphi^* \vee \psi^*,$$
$$(\neg\varphi)^* = \neg(\varphi^*), \qquad (\exists x \varphi)^* = (\exists x)\big(A(x) \wedge \varphi^*\big),$$
$$(Qx\varphi)^* = (\exists x)\Big(P(x) \wedge (\forall y)\big(E(y, x) \rightleftarrows \varphi^*(y)\big)\Big).$$

If $(\mathfrak{A}, p)$ is a structure for L_Q, then $(\mathfrak{A}, p)_{\mathrm{II}}$ is the structure

$$\langle C, A^C, P^C, E^C, P_\xi^C, c_\zeta^C\rangle,$$

where $C = A \cup p$, $A^C = A$, $P^C = p$, $E^C(x, y)$ iff $x \in y$, $\langle x_1, \ldots, x_n\rangle \in P_\xi^C$ iff $\langle x_1, \ldots, x_n\rangle \in P_\xi^{(\mathfrak{A},p)}$, and $c_\zeta^C = c_\zeta^{(\mathfrak{A},p)}$. Let $\langle C, A^C, P^C, E^C, P_\xi^C, c_\zeta^C\rangle$ be a structure in $\boldsymbol{K}$, denoted by $\bar{A}$. Then $\bar{A}_{\mathrm{I}}$ is the structure $(\mathfrak{A}, p)$, where

$$\mathfrak{A} = \langle \{x \in C\colon \bar{A} \models A[x]\}, P_\xi^C, c_\zeta^C\rangle$$

and $y \in p$ iff there exists a $Y \in C$ such that

$$\bar{A} \models P[Y] \quad \text{and} \quad y = \{x \in C\colon \bar{A} \models E[x, Y]\}.$$

Then we have immediately the following: for every formula φ in L_Q, for every structure $(\mathfrak{A}, p)$ for L_Q, and for all $a_1, \ldots, a_n \in A$,

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad (\mathfrak{A}, p)_{\mathrm{II}} \models \varphi^*[a_1, \ldots, a_n];$$

for every formula φ in L_Q, for every structure $\bar{A}$ in $\boldsymbol{K}$, and for all $a_1, \ldots, a_n$ in C such that $\bar{A} \models A[a_i]$ for all $i \in \{1, \ldots, n\}$,

$$\bar{A} \models \varphi^*[a_1, \ldots, a_n] \quad \text{iff} \quad \bar{A}_{\mathrm{I}} \models \varphi[a_1, \ldots, a_n].$$

The notion $\subseteq_1$ corresponds to the notion $\subseteq$ for structures of $\boldsymbol{K}$. Let

$$\bar{A} = \langle C_A, A^{CA}, P^{CA}, E^{CA}, P_\xi^{CA}, c_\zeta^{CA}\rangle \in \boldsymbol{K}$$

and let $\bar{B} = \langle C_B, \ldots, c_\zeta^{CB}\rangle \in \boldsymbol{K}$ be such that $\bar{A} \subseteq \bar{B}$. We shall prove that $\bar{A}_{\mathrm{I}} \subseteq_1 \bar{B}_{\mathrm{I}}$. For let $\bar{A}_{\mathrm{I}} = (\mathfrak{A}, p)$ and $\bar{B}_{\mathrm{I}} = (\mathfrak{B}, q)$. Obviously, $\mathfrak{A} \subseteq \mathfrak{B}$, so take $x \in p$. Hence there exists a $Y \in C_A$ such that

$$\bar{A} \models P[Y] \quad \text{and} \quad x = \{z \in C_A : \bar{A} \models E[z, Y]\}.$$

Since $\bar{A} \subseteq \bar{B}$, we have $Y \in C_B$ and $\bar{B} \models P[Y]$. Let

$$y = \{z \in C_B : \bar{B} \models E[z, Y]\}.$$

Then $y \in q$ and $y \cap A = x$, which proves $(\mathfrak{A}, p) \subseteq_1 (\mathfrak{B}, q)$.

If $(\mathfrak{A}, p) \subseteq_1 (\mathfrak{B}, q)$, then there are $\bar{A}$ and $\bar{B}$ in $\boldsymbol{K}$ such that $\bar{A}_{\mathrm{I}} = (\mathfrak{A}, p)$, $\bar{B}_{\mathrm{I}} = (\mathfrak{B}, q)$, and $\bar{A} \subseteq \bar{B}$. The only difficulty is to construct the universes and the relations P^{C_A} and P^{C_B}.

Let $p = \{d_\xi : \xi < \varkappa\}$. Then it is possible to write

$$q = \{e_\xi : \xi < \varkappa\} \cup \{e_\xi : \varkappa \leqslant \xi < \lambda\},$$

where $e_\xi \cap A = d_\xi$ for all $\xi < \varkappa$. Put $C_A = A \cup \{e_\xi : \xi < \varkappa\}$, $P^{C_A}(x)$ iff $x = e_\xi$ for some $\xi < \varkappa$, $E^{C_A}(x, y)$ iff $y = e_\xi$ for some $\xi < \varkappa$ and $x \in d_\xi$. Let $C_B = B \cup \{e_\xi : \xi < \lambda\}$, $P^{C_B}(x)$ iff $x = e_\xi$ for some $\xi < \lambda$, $E^{C_B}(x, y)$ iff $x \in y$. All other relations and the realizations of the individual constants are the same as in $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$. Now we have $\bar{A} \subseteq \bar{B}$, $\bar{A}_{\mathrm{I}} = (\mathfrak{A}, p)$ and $\bar{B}_{\mathrm{I}} = (\mathfrak{B}, q)$. Roughly speaking, the structures $\bar{A}$ and $\bar{B}$ are constructed by choosing right names for the elements of p and q.

4. Mappings. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L and let $f: \mathfrak{A} \to \mathfrak{B}$ be a one-to-one homomorphism. Then $f(\mathfrak{A})$ denotes the structure $\langle C, P_\xi^C, c_\zeta^C \rangle$ such that

(i) $x \in C$ iff $x = f(y)$ for some $y \in A$;
(ii) $\langle f(a_1), \ldots, f(a_n) \rangle \in P_\xi^C$ iff $\langle a_1, \ldots, a_n \rangle \in P_\xi^A$;
(iii) $c_\zeta^C = f(c_\zeta^A)$.

Let $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ be given. We shall call f a *pseudo-mapping* between $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ if $\operatorname{dom} f = A \cup p$, $\operatorname{rng} f \subseteq B \cup q$, each element of A is mapped on an element of B, and each element of p is mapped on an element of q.

If f is a pseudo-mapping of $(\mathfrak{A}, p)$ into $(\mathfrak{B}, q)$, then for each element x in p we have to distinguish between its value by f and its image. If $X \subseteq A$, then the image of X, $f * X$, is the set $\{f(x) : x \in X\}$. So the value by f of an element of p does not need to be equal to its image.

Definition. $f: (\mathfrak{A}, p) \to (\mathfrak{B}, q)$ is a *mapping* if f is a pseudo-mapping between $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ such that $f * x = f(x)$ for all $x \in p$.

From this it follows immediately that if the restriction of f to A is one-to-one, then the restriction of f to p is one-to-one.

If $f: (\mathfrak{A}, p) \to (\mathfrak{B}, q)$ is a mapping, then $f(\mathfrak{A}, p)$ denotes the structure $(\mathfrak{D}, r)$, where $\mathfrak{D} = f(\mathfrak{A})$ and $r = \{f(x) : x \in p\}$.

Definition. Let $f: (\mathfrak{A}, p) \to (\mathfrak{B}, S(B))$ be a mapping, q a subset of $S(B)$, and $i \in \{1, 2, 3, 4\}$. Let, moreover, the restriction of f to A be a one-to-one homomorphism. Then

(a) f is an *i-embedding* of $(\mathfrak{A}, p)$ into $(\mathfrak{B}, q)$ if $f(\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$;

(b) f is an *isomorphism* between $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$ if $f(\mathfrak{A}, p) = (\mathfrak{B}, q)$.

THEOREM 4.1. *Let f be an isomorphism between $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$. Then, for all formulas φ in L_Q and all $a_1, \ldots, a_n \in A$,*

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \quad \textit{iff} \quad (\mathfrak{B}, q) \models \varphi[f(a_1), \ldots, f(a_n)].$$

The proof is trivial.

If f is a mapping between $\mathfrak{A}$ and $\mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are structures for L, and if $p \subseteq S(A)$, then f induces a mapping between $(\mathfrak{A}, p)$ and $\big(\mathfrak{B}, S(B)\big)$ by $f(x) = f * x$ for all $x \in p$. This induced mapping will be also denoted by f. Conversely, if f is a mapping between $(\mathfrak{A}, p)$ and $(\mathfrak{B}, q)$, then the restriction of f to A is a mapping between $\mathfrak{A}$ and $\mathfrak{B}$. Also this mapping will be denoted by f.

5. Chains of models. In this section the notions of a chain of models and the union of a chain are defined in the four cases corresponding to the four definitions of an extension and of a submodel. The notion of a chain of models has sense only if we can define a structure which can be considered as the limit of that chain. Especially, in the cases of $\subseteq_1$ and $\subseteq_2$ we must be careful with the relations between p's.

Definition. Let $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ be a collection of structures such that $A_\xi \subseteq A_\eta$ for all $\xi < \eta$. A collection $(y_\xi)_{\xi_0\leqslant\xi<\delta}$ is a *tower* in $(p_\xi)_{\xi<\delta}$ if $y_\xi \in p_\xi$ for all ξ, $\xi_0 \leqslant \xi < \delta$, and $y_\xi = y_\eta \cap A_\xi$ for all ξ, η with $\xi_0 \leqslant \xi \leqslant \eta < \delta$. Such a tower is said to *begin* with y_{ξ_0}.

A collection $(p_\xi)_{\xi<\delta}$ is called *closed* with respect to towers if, for every $\xi_0 < \delta$ and $y_{\xi_0} \in p_{\xi_0}$, there exists a tower in $(p_\xi)_{\xi<\delta}$ which begins with y_{ξ_0}.

Now we can define the notions of a chain.

Definition. Let $i \in \{1, 2\}$. A collection $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ is an *i-chain* if

(i) $(\mathfrak{A}_\xi, p_\xi) \subseteq_i (\mathfrak{A}_\eta, p_\eta)$ for all $\xi \leqslant \eta < \delta$;

(ii) a collection $(p_\xi)_{\xi<\delta}$ is closed with respect to towers.

Let $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ be an i-chain, where $i \in \{1, 2\}$. The *union* of such a chain is the structure $(\mathfrak{A}, p)$, where

$$\mathfrak{A} = \bigcup_{\xi<\delta} \mathfrak{A}_\xi \quad \text{and} \quad x \in p,$$

iff there exists a $\xi_0 < \delta$ such that $x \cap A_\xi \in p_\xi$ for all $\xi \geqslant \xi_0$. Roughly speaking, p consists of the unions of towers in $(p_\xi)_{\xi<\delta}$.

In the case $i = 3$ the definition is much simpler, because so is the definition of $\subseteq_3$. A collection $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ is called a *3-chain* if $(\mathfrak{A}_\xi, p_\xi) \subseteq_3 (\mathfrak{A}_\eta, p_\eta)$ for all $\xi \leqslant \eta < \delta$. Its union is the structure $(\mathfrak{A}, p)$, where

$$\mathfrak{A} = \bigcup_{\xi<\delta} \mathfrak{A}_\xi \quad \text{and} \quad p = \bigcup_{\xi<\delta} p_\xi.$$

In the case $i = 4$ we do not define the notion of a chain because of the difficulty in defining (in a reasonable way) the union.

In analogy with the case $i = 3$, $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ is a *4-chain* if $(\mathfrak{A}_\xi, p_\xi) \subseteq_4 (\mathfrak{A}_\eta, p_\eta)$ for all $\xi \leqslant \eta < \delta$. One might expect that the union of a 4-chain should have, as in the case $i = 3$, the property that if a structure is a 4-extension of each element of the chain, then it is a 4-extension of the union. However, in general this is not true.

Example. Let $X_i = \{n \in \boldsymbol{Z}\colon n \leqslant i\}$ for all $i \in \boldsymbol{N}$. Let $(\mathfrak{A}_i, p_i) = \langle X_i, p_i \rangle$, where $x \in p_i$ iff $x = X_j$ for $j \leqslant i$. Then $(\mathfrak{A}_i, p_i) \subseteq_4 (\mathfrak{A}_j, p_j)$ for all $i \leqslant j$. The structures $(\mathfrak{A}, p)$ and $(\mathfrak{A}, q)$, where $\mathfrak{A} = \langle \boldsymbol{Z} \rangle$, $p = \{X_j\colon j \in \boldsymbol{N}\}$ and $q = p \cup \{\boldsymbol{Z}\}$, are both 4-extensions of each element of the chain and so cannot be both 4-extensions of the union, because the union has the universe $\boldsymbol{Z}$.

Remarks. 1. We can generalize the above-given definitions and define the notion of an i-directed system. Then we have to introduce the notion of a special tower in a directed set S. A set $Y \subseteq S$ is a *special tower* in S if Y is a maximal chain in $\{b \in S\colon b \geqslant a\}$ for some $a \in S$, i.e., Y is a tower in S, and for every $u \in S$ there is a $t \in Y$ such that $u \leqslant t$. Then Y is said to *begin* with a. The set S is *closed* with respect to special towers if for every $a \in S$ there is a special tower in S that begins with a. Not every directed set has this property, for example the set of all finite subsets of ω_1 has not. We can only speak about a 2-directed system if the corresponding directed set is closed with respect to special towers.

2. If a 1-chain $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ is given, then there is a chain $(\overline{\mathfrak{A}}_\xi)_{\xi<\delta}$ of second-order structures such that $\overline{A}_{\xi\mathrm{I}} = (\mathfrak{A}_\xi, p_\xi)$ for all $\xi < \delta$. Conversely, if $(\overline{A}_\xi)_{\xi<\delta}$ is a chain of second-order structures, then $(\overline{A}_{\xi\mathrm{I}})_{\xi<\delta}$ is a 1-chain.

6. Elementary extensions and elementary chains. In this section we define the notion of an elementary extension and of an elementary chain. We state some generalizations of theorems in [3].

Definition. Let $i \in \{1, 2, 3, 4\}$. Then

$$(\mathfrak{A}, p) \prec_i (\mathfrak{B}, q) \quad \text{if} \quad (\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$$

and, for all $\varphi \in L_Q$ and $a_1, \ldots, a_n \in A$,

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \quad \text{if} \quad (\mathfrak{B}, q) \models \varphi[a_1, \ldots, a_n].$$

If $(\mathfrak{A}, p) \prec_i (\mathfrak{B}, q)$, then $(\mathfrak{A}, p)$ is called an *elementary i-submodel* of $(\mathfrak{B}, q)$, and $(\mathfrak{B}, q)$ is an *elementary i-extension* of $(\mathfrak{A}, p)$.

Keisler [2] gave the following definition of an elementary substructure $(\mathfrak{A}, p)$ of $(\mathfrak{B}, q)$: $(\mathfrak{A}, p) \prec (\mathfrak{B}, q)$ if, for all $\varphi \in L_Q$ and $a_1, \ldots, a_n \in A$,

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad (\mathfrak{B}, q) \models \varphi[a_1, \ldots, a_n].$$

There is a relation between this definition and the definition of $\prec_1$ above. Namely, if p is minimal (see Section 2), then

$$(\mathfrak{A}, p) \prec (\mathfrak{B}, q) \quad \text{iff} \quad (\mathfrak{A}, p) \prec_1 (\mathfrak{B}, q).$$

The following theorem is a generalization of Vaught's test for elementary extensions:

THEOREM 6.1. *Let us assume that* $(\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$, *where* $i \in \{1, 2, 3, 4\}$. *Then* $(\mathfrak{A}, p) \prec_i (\mathfrak{B}, q)$ *iff the following conditions are satisfied*:

(i) *For all* $\varphi \in L_Q$ *and* $a_1, \ldots, a_n \in A$, *if* $(\mathfrak{B}, q) \models (\exists v_m)\varphi[a_1, \ldots, a_n]$, *then there exists an* $a_m \in A$ *such that*

$$(\mathfrak{B}, q) \models \varphi[a_1, \ldots, a_m, \ldots, a_n].$$

(ii) *For all* $\varphi \in L_Q$ *and* $a_1, \ldots, a_n \in A$,

$$\{b \in A\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in p \text{ iff } \{b \in B\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in q.$$

Proof. Suppose $(\mathfrak{A}, p) \prec_i (\mathfrak{B}, q)$. Let

$$\{b \in A\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in p.$$

Then also

$$\{b \in A\colon (\mathfrak{A}, p) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in p,$$

and so

$$(\mathfrak{A}, p) \models (Qv_m)\varphi[a_1, \ldots, a_n],$$

whence

$$(\mathfrak{B}, q) \models (Qv_m)\varphi[a_1, \ldots, a_n].$$

This implies

$$\{b \in B\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in q.$$

Let

$$\{b \in B\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in q.$$

Then

$$(\mathfrak{B}, q) \models (Qv_m)\varphi[a_1, \ldots, a_n],$$

and so

$$(\mathfrak{A}, p) \models (Qv_m)\varphi[a_1, \ldots, a_n].$$

According to the definition,

$$\{b \in A\colon (\mathfrak{A}, p) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in p,$$

and this implies

$$\{b \in A\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in p.$$

Suppose conditions (i) and (ii) are satisfied. By induction on the construction of φ, we are going to prove that, for all $a_1, \ldots, a_n \in A$ and all

φ in L_Q,

$$(\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad (\mathfrak{B}, q) \models \varphi[a_1, \ldots, a_n].$$

Only the case $\varphi = (Qv_m)\psi$ is not trivial.

Suppose $\varphi = (Qv_m)\psi$ and let $(\mathfrak{A}, p) \models (Qv_m)\psi[a_1, \ldots, a_n]$. Then, by the definition,

$$\{b \in A\colon (\mathfrak{A}, p) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in p$$

and, by the induction hypothesis on ψ,

$$\{b \in A : (\mathfrak{B}, q) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in p.$$

By condition (ii) it follows that

$$\{b \in B\colon (\mathfrak{B}, q) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in q,$$

and so

$$(\mathfrak{B}, q) \models (Qv_m)\psi[a_1, \ldots, a_n].$$

Let $(\mathfrak{B}, q) \models (Qv_m)\psi[a_1, \ldots, a_n]$. Then, by the definition,

$$\{b \in B\colon (\mathfrak{B}, q) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in q,$$

whence, by condition (ii),

$$\{b \in A\colon (\mathfrak{B}, q) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in p.$$

With the induction hypothesis on ψ we have

$$\{b \in A\colon (\mathfrak{A}, p) \models \psi[a_1, \ldots, b, \ldots, a_n]\} \in p,$$

and so $(\mathfrak{A}, p) \models (Qv_m)\psi[a_1, \ldots, a_n]$.

If we take $p = S(A) \setminus \{\emptyset\}$ in $(\mathfrak{A}, p)$, then the quantifier $\exists$ can be treated in the same way as the quantifier Q. For example, we can write down Theorem 6.1 uniformly if we rewrite condition (ii) in the form:

For all $\varphi \in L_Q$ and $a_1, \ldots, a_n \in A$,

$$\{b \in A\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in S(A) \setminus \{\emptyset\} \text{ iff}$$
$$\{b \in B\colon (\mathfrak{B}, q) \models \varphi[a_1, \ldots, b, \ldots, a_n]\} \in S(B) \setminus \{\emptyset\}.$$

Let $(\mathfrak{A}, p)$ be a structure for L_Q and C a subset of A. Then $L_{Q,C}$ is the language which we obtain from L_Q by adding an individual constant c_a for each $a \in C$. So $(\mathfrak{A}, a, p)_{a \in C}$ is a structure for $L_{Q,C}$, where the realization of c_a in $(\mathfrak{A}, a, p)_{a \in C}$ is equal to a for all $a \in C$.

LEMMA 6.2. *Let* $(\mathfrak{A}, p) \subseteq_i (\mathfrak{B}, q)$, *where* $i \in \{1, 2, 3, 4\}$. *Then*

$$\mathrm{Th}(\mathfrak{A}, a, p)_{a \in A} = \mathrm{Th}(\mathfrak{B}, a, q)_{a \in A} \quad \textit{iff} \quad (\mathfrak{A}, p) \prec_i (\mathfrak{B}, q).$$

Let $\bar{L}_Q$ be the language obtained from L_Q by adding a predicate letter P_φ for each formula φ in L_Q, i.e., P_φ is an n-ary predicate letter iff φ has n free variables. If $(\mathfrak{A}, p)$ is a structure for L_Q, then $(\mathfrak{A}, \varphi^{\mathfrak{A}}, p)_{\varphi \in L_Q}$ is the structure for L_Q in which each P_φ is realized in A as $\varphi^{\mathfrak{A}}$, i.e.,

$$\langle a_1, \ldots, a_n\rangle \in P_\varphi \quad \text{iff} \quad (\mathfrak{A}, p) \models \varphi[a_1, \ldots, a_n].$$

LEMMA 6.3. *Let* $i \in \{1, 2, 3, 4\}$. *Then*

$$(\mathfrak{A}, p) \prec_i (\mathfrak{B}, q) \quad \textit{iff} \quad (\mathfrak{A}, \varphi^{\mathfrak{A}}, p)_{\varphi\in L_Q} \subseteq_i (\mathfrak{B}, \varphi^{\mathfrak{B}}, q)_{\varphi\in L_Q}.$$

Elementary chains of models. Let $i \in \{1, 2, 3, 4\}$. An *elementary i-chain* is an i-chain $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ such that

$$(\mathfrak{A}_\xi, p_\xi) \prec_i (\mathfrak{A}_\eta, p_\eta) \quad \text{for all } \xi \leqslant \eta.$$

A notion which turns out to be useful is that of a simple elementary chain. This is defined by an elementary submodel and of an extension given by Keisler [1]:

A sequence of structures $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ is a *simple elementary chain* if

$$(\mathfrak{A}_\xi, p_\xi) \prec (\mathfrak{A}_\eta, p_\eta) \quad \text{for all } \xi \leqslant \eta.$$

THEOREM 6.4 (Keisler [1], Lemma 2.5). *Let* $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ *be a simple elementary chain. Let* $(\mathfrak{A}, p)$ *be such that*

$$\mathfrak{A} = \bigcup_{\xi<\delta} \mathfrak{A}_\xi \quad \textit{and} \quad x \in p$$

if there is an $\eta < \delta$ *such that* $x \cap A_\xi \in p_\xi$ *for all* $\xi \geqslant \eta$. *Then* $(\mathfrak{A}_\xi, p_\xi) \prec (\mathfrak{A}, p)$ *for all* $\xi < \delta$.

Immediately from this we obtain

THEOREM 6.5. *Let* $i \in \{1, 2, 3, 4\}$ *and let* $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$ *be an elementary i-chain. Let* $(\mathfrak{A}, p)$ *be the structure as defined in Theorem* 6.4. *Then*

(a) $(\mathfrak{A}_\xi, p_\xi) \prec_i (\mathfrak{A}, p)$ *for all* $\xi < \delta$;

(b) *if* $i \in \{1, 2\}$, *then* $(\mathfrak{A}, p)$ *is equal to the union of the chain* $(\mathfrak{A}_\xi, p_\xi)_{\xi<\delta}$, *considered as an i-chain.*

In the case $i = 3$ the union of an elementary 3-chain does not need to be an elementary 3-extension of each element of the chain.

Example. Let L have no predicate letters and no individual constants. For $i \in \omega$, let $X_i = \{n \in \boldsymbol{Z}\colon n \leqslant i\}$ and $(\mathfrak{A}_i, p_i) = \langle X_i, S(X_i)\rangle$.

CLAIM 0. $(\mathfrak{A}_i, p_i) \prec_3 (\mathfrak{A}_{i+1}, p_{i+1})$ *for all* $i \in \omega$.

Proof. Obviously, $(\mathfrak{A}_i, p_i) \subseteq_3 (\mathfrak{A}_{i+1}, p_{i+1})$ for all $i \in \omega$. Now we use Theorem 6.1. Condition (ii) is satisfied, because $p_i = S(X_i)$. So let

$$(\mathfrak{A}_{i+1}, p_{i+1}) \models (\exists v_m)\varphi[a_1, \ldots, a_n] \quad \text{for } a_j \leqslant i,\ j \in \{1, \ldots, n\}.$$

Let

$$(\mathfrak{A}_{i+1}, p_{i+1}) \models \varphi[a_1, \ldots, i+1, \ldots, a_n].$$

Take $b \in X_i$, $b \neq i+1$, $b \neq a_1, \ldots, b \neq a_n$, and let $f\colon X_{i+1} \to X_{i+1}$ be a mapping such that $f(i+1) = b$, $f(b) = i+1$, and $f(x) = x$ hold for $x \neq i+1$ and $x \neq b$. Then f is an isomorphism of $(\mathfrak{A}_{i+1}, p_{i+1})$ onto itself. From Theorem 4.1 it follows that

$$(\mathfrak{A}_{i+1}, p_{i+1}) \models \varphi[f(a_1), \ldots, f(i+1), \ldots, f(a_n)],$$

and so

$$(\mathfrak{A}_{i+1}, p_{i+1}) \models \varphi[a_1, \ldots, b, \ldots, a_n].$$

This proves condition (i) of Theorem 6.1.

The union of the chain $(\mathfrak{A}_i, p_i)_{i \in \omega}$ is $\langle \boldsymbol{Z}, p \rangle$, where $x \in p$ iff $x \subseteq X_i$ for some $i \in \omega$. So we have $\boldsymbol{Z} \notin p$, from which we may conclude $\langle \boldsymbol{Z}, p \rangle \models \neg(Qx)(x = x)$. But we do have $(\mathfrak{A}_i, p_i) \models (Qx)(x = x)$ for all $i \in \omega$. From this it follows that, for no $i \in \omega$, $(\mathfrak{A}_i, p_i) \prec_3 \langle \boldsymbol{Z}, p \rangle$ holds.

In the case $i = 4$ quite often the following situation occurs:

$(\mathfrak{A}_\xi, p_\xi)_{\xi < \delta}$ is an elementary 4-chain and there is not a structure $(\mathfrak{A}, p)$ with the following two properties:

(i) $(\mathfrak{A}, p)$ is an elementary 4-extension of each $(\mathfrak{A}_\xi, p_\xi)$;

(ii) if $(\mathfrak{B}, q)$ is an elementary 4-extension of each $(\mathfrak{A}_\xi, p_\xi)$, then $(\mathfrak{A}, p) \prec_4 (\mathfrak{B}, q)$.

Example. For $i \in \omega$, let $\{n \in \boldsymbol{Z}\colon n \leqslant i\} = X_i$. Let $(\mathfrak{A}_i, p_i) = \langle X_i, \emptyset \rangle$, $(\mathfrak{A}, p_1) = \langle \boldsymbol{Z}, \emptyset \rangle$ and $(\mathfrak{A}, p_2) = \langle \boldsymbol{Z}, \{\{n \in \boldsymbol{Z}\colon 2 \mid n\}\} \rangle$.

CLAIM 1. *$(\mathfrak{A}_i, p_i) \prec_4 (\mathfrak{A}_{i+1}, p_{i+1})$ for all $i \in \omega$.*

This can be proved in the same way as Claim 0.

CLAIM 2. *$(\mathfrak{A}_i, p_i) \prec_4 (\mathfrak{A}, p_1)$ for all $i \in \omega$.*

This $(\mathfrak{A}, p_1)$ is exactly the structure $(\mathfrak{A}, p)$ defined in Theorem 6.4. So this claim follows immediately from Theorem 6.5.

CLAIM 3. *For all $\varphi \in L_Q$ and $a_1, \ldots, a_n \in \boldsymbol{Z}$,*

$$(\mathfrak{A}, p_1) \models \varphi[a_1, \ldots, a_n] \quad \textit{iff} \quad (\mathfrak{A}, p_2) \models \varphi[a_1, \ldots, a_n].$$

The proof follows by induction on the construction of φ. Only the case $\varphi = (Qv_m)\psi$ is not trivial. We shall prove that

$$\text{not}(\mathfrak{A}, p_1) \models (Qv_m)\psi[a_1, \ldots, a_n] \quad \text{and} \quad \text{not}(\mathfrak{A}, p_2) \models (Qv_m)\psi[a_1, \ldots, a_n].$$

Obviously, not$(\mathfrak{A}, p_1) \models (Qv_m)\psi[a_1, \ldots, a_n]$, because $p_1 = \emptyset$. Let

$$(\mathfrak{A}, p_2) \models (Qv_m)\psi[a_1, \ldots, a_n].$$

Then

$$\{b \in \boldsymbol{Z}\colon (\mathfrak{A}, p_2) \models \psi[a_1, \ldots, b, \ldots, a_n]\} = \{n \in \boldsymbol{Z}\colon 2 \mid n\}.$$

Take an even number b such that $b \neq a_1, \ldots, b \neq a_n$. Then

$$(\mathfrak{A}, p_2) \models \psi[a_1, \ldots, b, \ldots, a_n].$$

The induction hypothesis on ψ gives

$$(\mathfrak{A}, p_1) \models \psi[a_1, \ldots, b, \ldots, a_n].$$

Take an odd number c unequal to a_i for all $i \in \{1, \ldots, n\}$. Let $f \colon \boldsymbol{Z} \to \boldsymbol{Z}$ be a mapping which interchanges b and c and maps each other element onto itself. Then f is an isomorphism of $(\mathfrak{A}, p_1)$ onto itself. So we have

$$(\mathfrak{A}, p_1) \models \psi[a_1, \ldots, c, \ldots, a_n].$$

Applying the induction hypothesis on ψ we get

$$(\mathfrak{A}, p_2) \models \psi\, [a_1, \ldots, c, \ldots, a_n],$$

but this is in contradiction with

$$(\mathfrak{A}, p_2) \models (Qv_m)\psi[a_1, \ldots, a_n].$$

Hence

$$\text{not } (\mathfrak{A}, p_2) \models (Qv_m)\psi[a_1, \ldots, a_n].$$

CLAIM 4. *$(\mathfrak{A}_i, p_i) \prec_4 (\mathfrak{A}, p_2)$ for all $i \in \omega$.*

This follows immediately from Claims 2 and 3. Now suppose there is $(\mathfrak{A}', q)$ which has properties (i) and (ii). Then $(\mathfrak{A}', q) \prec_4 (\mathfrak{A}, p_1)$ and $(\mathfrak{A}', q) \prec_4 (\mathfrak{A}, p_2)$. Obviously, $\boldsymbol{Z} \subseteq A'$ and so we have $\mathfrak{A}' = \mathfrak{A}$, from which it follows that $q = p_1$, and $q = p_2$. However, this is impossible.

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Reçu par la Rédaction le 26. 10. 1973