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ON THE ALGEBRAIC STRUCTURE OF THE PROCESS OF FORMING SUBSEQUENCES

BY

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TO MY DEAR TEACHER
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1. Introduction. The purpose of this paper is to present algebraic properties of the process of forming subsequences.

A subsequence, finite or infinite, of a given sequence is determined by the sequence of indices, i.e., an increasing function which maps the set of natural numbers N or its segment $N_n = \{1, 2, \ldots, n\}$ into N. We denote these functions by Greek letters $a, \beta, \ldots$ and call them *abstract sequences* or, simply, *sequences*.

The domain of the function a, which is always one of the sets N_n or N, will be denoted by Da and the image $a(Da)$ will be denoted by Ia. We denote by 0 the empty sequence with $Da = Ia = \emptyset$.

Obviously, the set Ia determines a entirely. If the cardinality of Ia is finite, $|Ia| = n$, then $Da = N_n$; if $|Ia| = \aleph_0$, then $Da = N$. In the first case we call a a *finite sequence*, in the second case — an *infinite sequence*.

The identity mapping $N \to N$ is denoted by ε, consequently, we have

$$D\varepsilon = I\varepsilon = N; \quad \varepsilon(n) = n \text{ for every } n \in N. \tag{1.1}$$

The inclusion mapping $N_n \to N$ is denoted by ι_n; hence

$$D\iota_n = I\iota_n = N_n; \quad \iota_n(k) = k \text{ for every } k \in N_n. \tag{1.2}$$

We denote the set of all sequences including 0 and ε by S.

The following proposition is an obvious consequence of the fact that Da contains together with every positive integer all smaller positive integers, and that a is an increasing function:

PROPOSITION 1.1. *Every element $\alpha \in S$ satisfies the following properties:*

(1.3) $\alpha(n) \geqslant n$ *for every* $n \in D\alpha$;

(1.4) *If* $D\alpha \neq I\alpha$, *then there exists a* $k \in D\alpha$ *such that* $\alpha(k) > k$;

(1.5) *If* $k+1 \in D\alpha$ *and* $\alpha(k) = m$, *then* $\alpha(k+1) \geqslant m+1$.

2. The semigroup structure of *S*. Since the elements of S are mappings of N or N_n into N, S has a natural semigroup structure with the composition of mappings taken as multiplication. More precisely, we define $\alpha\beta$ as follows:

$$D(\alpha\beta) = \beta^{-1}(D\alpha), \qquad \alpha\beta(k) = \alpha\big(\beta(k)\big) \text{ for } k \in D(\alpha\beta). \tag{2.1}$$

Note that $\beta^{-1}(D\alpha) = N$ if and only if both sequences are infinite. If one of the sequences is finite, we have $D(\alpha\beta) = N_n$, where n is the largest positive number $n \in D\beta$ such that $\beta(n) \in D\alpha$.

PROPOSITION 2.1. *The following inclusions hold:*

$$D\alpha\beta \subseteq D\alpha, \qquad D\alpha\beta \subseteq D\beta, \tag{2.2}$$

$$I\alpha\beta \subseteq I\alpha. \tag{2.3}$$

Proof. The second inclusion of (2.2) is obvious, since $\beta^{-1}(N) = D\beta$, and, therefore, for every subset $M \subseteq N$ we have $\beta^{-1}(M) \subseteq D\beta$. If $D\alpha = N$ or $D\alpha = \emptyset$, the first inclusion of (2.2) is also obvious. Suppose now that $D\alpha = N_n$. If $\beta(k) > n$ for every k, then $\beta^{-1}(D\alpha) = \emptyset \subseteq D\alpha$. Otherwise, let k denote the largest positive integer such that $\beta(k) \leqslant n$. Obviously, $k \leqslant n$ and $\beta^{-1}(D\alpha) = \beta^{-1}(N_n) = N_k \subseteq N_n = D\alpha$. Inclusion (2.3) is obvious.

PROPOSITION 2.2. *The empty sequence* 0 *is the zero of the semigroup* S.

Proof. $D(0\beta) = \beta^{-1}(\emptyset) = \emptyset$, $D(\beta 0) = \emptyset$. Hence, $0\beta = \beta 0 = 0$.

PROPOSITION 2.3. *The sequence* ε *is the unity of the semigroup* S.

Proof is obvious.

PROPOSITION 2.4. *Elements* ι_n, ε *and* 0 *are idempotents of the semigroup* S. *Conversely, every idempotent of* S *is either* 0 *or* ε *or one of the* ι_n*'s.*

Proof. The identities $\iota_n^2 = \iota_n$, $\varepsilon^2 = \varepsilon$ and $0^2 = 0$ are obvious. Suppose now that α is distinct from all ι_ns, from 0 and from ε. Then $D\alpha \neq \emptyset$ and $\alpha(k) > k$ for some $k \in D\alpha$. Let k be the smallest positive integer having this property. Now, if $\alpha(k) \in D\alpha$, then, by (1.5), $\alpha^2(k) = \alpha\big(\alpha(k)\big) > \alpha(k)$ and $\alpha^2 \neq \alpha$. On the other hand, if $\alpha(k) \notin D(\alpha)$, $D\alpha^2$ does not contain k while $D\alpha$ does, and again $\alpha^2 \neq \alpha$.

PROPOSITION 2.5. *If* $m \leqslant n$, *then* $\iota_m \iota_n = \iota_n \iota_m = \iota_m$. *Consequently, the set of elements* $0, \varepsilon, \iota_m$ *forms an abelian subsemigroup of* S.

Proof. Indeed, $D\iota_m\iota_n = \iota_m^{-1}(N_n) = N_m$, $D\iota_n\iota_m = \iota_n^{-1}(N_m) = N_m$ and for $k \leqslant m$ we have $\iota_m\iota_n(k) = \iota_n\iota_m(k) = k$.

PROPOSITION 2.6. *A sequence α commutes with an idempotent ι if and only if $I\alpha \subseteq D\iota$ or $D\iota \subseteq I\alpha$.*

Proof. The proposition is obviously true for $\iota = \varepsilon$ and $\iota = 0$. Let us assume that $\iota = \iota_n$ for some positive integer n. Consider the following cases:

1° If $I\alpha \subseteq N_n$, then, obviously, $\iota_n \alpha = \alpha$; also α is finite and $D\alpha = N_k$ for some $k \leqslant \alpha(k) \leqslant n$. Therefore, $D\alpha \subseteq N_n$ and $D\alpha\iota_n = D\alpha$ which implies $\alpha\iota_n = \alpha$. Consequently, in this case $\alpha\iota_n = \iota_n\alpha = \alpha$.

2° If $N_n \subseteq I\alpha$, then, obviously, $\alpha\iota_n = \alpha$. Further, $N_n \subseteq D\alpha$, because $|I\alpha| \geqslant n$ implies $|D\alpha| \geqslant n$, and $\iota_n\alpha = \alpha\iota_n = \iota_n$. Consequently, in this case too, ι_n and α commute.

Conversely, assume that neither $I\alpha \subseteq N_n$ nor $N_n \subseteq I\alpha$. Therefore, there exists a positive integer $k \in D\alpha$ such that $\alpha(k) > n$. Let k be the smallest of all such integers. Then $k \leqslant n$, since, otherwise, we would have $\alpha(N_n) \subseteq N_n$, which can happen only if $\alpha(N_n) = N_n$, i.e., if $N_n \subseteq I\alpha$. Consequently, $\alpha\iota_n(k)$ is well defined and so $k \in D\alpha\iota_n$ while $k \notin D\iota_n\alpha$. Hence $\alpha\iota_n \neq \iota_n\alpha$.

As a consequence one has the following

PROPOSITION 2.7. *The center of the semigroup S consists of 0 and ε only.*

Proof. If α is not an idempotent, then there exists the smallest positive integer k such that $k \in D\alpha$ and $\alpha(k) > k$. Since $\alpha(j) = j$ for every $j < k$, we have $k \notin I\alpha$. Consequently, neither $I\alpha \subseteq N_k$ nor $N_k \subseteq I\alpha$, and α does not commute with ι_k.

If α is an idempotent, say $\alpha = \iota_n$, define the sequence β as follows:

$$\beta(k) = \begin{cases} k & \text{for } k < n, \\ k+1 & \text{for } k \geqslant n. \end{cases}$$

Again, $I\beta$ does not contain n, and so neither $I\beta \subseteq N_n$ nor $N_n \subseteq I\beta$. Consequently, β and $\alpha = \iota_n$ do not commute.

Proposition 2.8 gives an algebraic characterization of finite elements of S. First we must prove two lemmas.

LEMMA 2.1. *The product of two infinite sequences is an infinite sequence.*

Indeed, $D\alpha\beta = \beta^{-1}(D\alpha) = \beta^{-1}(N) = N$.

LEMMA 2.2. *If β is not an idempotent element and if the smallest integer k such that $\beta(k) > k$ belongs to $D\alpha$, then $I\alpha\beta$ is a proper subset of $I\alpha$, $I\alpha\beta \subset I\alpha$.*

Proof. For $j < k$ we have $\beta(j) = j$ and, therefore, $\alpha\beta(j) = \alpha(j)$; on the other hand, $\alpha\beta(k) > \alpha(k)$, since $\beta(k) > k$. Consequently, $\alpha\beta(m) > \alpha(m)$ for all $m \geqslant k$ by (1.5). Consequently, $\alpha(k) \notin I\alpha\beta$ and $I\alpha\beta \neq I\alpha$.

PROPOSITION 2.8. *An element $a \neq \varepsilon$, 0 of S is finite if and only if the power a^n is an idempotent for some n.*

Proof. 1. Suppose a is finite, say $Da = N_n$, but $a \neq \iota_n$. Then there exists a smallest positive integer $k \in N_n$ such that $a(k) > k$. For all $j < k$ we have $a(j) = j$ and, consequently, $j \in Da^m$ for every m and $a^m(j) = j$. Now, if $a(k) \in Da^m$, then, by lemma 2.2, $|Da^{m+1}| < |Da^m|$. Since Da is finite, the sequence of numbers $|Da| > |Da^2| > \ldots$ cannot decrease indefinitely. Therefore, for some n we must have $a(k) \notin Da^{n-1}$, but then $Da^n = N_{k-1}$ and $a^n = \iota_{k-1}$ if $k > 1$, or $a^n = 0$ if $k = 1$.

2. To prove the converse, note that if $a \neq \varepsilon$ is infinite, then by lemmas 2.1 and 2.2 all powers of a^n are infinite distinct from ε and, therefore, not idempotent.

The relation of finite elements to finite idempotents is as follows:

PROPOSITION 2.9. *An element a of S is finite if and only if there exists a finite idempotent ι_n such that $a\iota_n = a$.*

Proof. If $a\iota_n = a$, then $Da = Da\iota_n \subseteq D\iota_n = N_n$. On the other hand, if a is finite, then for some n there is $Da = N_n$ and $a\iota_n = a$.

Note that, by proposition 2.5, if $a\iota_n = a$, then $a\iota_m = a$ for all $m > n$. Indeed, $m > n$ implies $\iota_n \iota_m = \iota_m \iota_n = \iota_n$; consequently, $a\iota_m = (a\iota_n)\iota_m = a(\iota_n \iota_m) = a\iota_n = a$. For $a \neq 0$ we call the greatest lower bound of positive integers having this property, the *numerical length of* a, in symbols $\mathrm{nl}\, a$. For $\beta = 0$ we put $\mathrm{nl}\,\beta = 0$ and, for an infinite sequence γ, $\mathrm{nl}\,\gamma = \infty$.

From this definition one immediately obtains

PROPOSITION 2.11. *The numerical length of a equals $\mathrm{nl}\, a = |Da| = |Ia|$. Consequently, a is of length n if and only if $Da = N_n$.*

PROPOSITION 2.12. *An element a of S is finite if and only if there exists a positive integer m such that $\iota_m a = a$. This integer is always not less than the numerical length of a.*

Proof. If a is finite of length n, then $Da = N_n$ and, consequently, $Ia \subseteq N_{a(n)}$, while $N_{a(n)-1}$ does not contain Ia. Therefore, $\iota_m a = a$ if and only if $m \geqslant a(n)$. Since $a(n) \geqslant n$, our second assertion is also proved.

The smallest positive integer m such that $\iota_m a = a$ is called the *numerical height of* a, in symbols $\mathrm{nh}\, a$. In the case of an infinite element β we set $\mathrm{nh}\,\beta = \infty$, also $\mathrm{nh}\, 0 = 0$. For finite elements a we have

$$\mathrm{nh}\, a = a(\mathrm{nl}\, a) = \max_{n \in Da}\{a(n)\}.$$

An obvious consequence of this definition is the following

PROPOSITION 2.13. *For every a we have $\mathrm{nl}\, a \leqslant \mathrm{nh}\, a$. The height of a finite a coincides with its length if and only if a is a finite idempotent.*

Using the notions of height and length of a sequence, one can reformulate proposition 2.1 without the explicit use of the domain and image. Namely,

PROPOSITION 2.14. *For any two sequences α and β we have*

$$\mathrm{nl}\,\alpha\beta \leqslant \mathrm{nl}\,\alpha, \quad \mathrm{nl}\,\alpha\beta \leqslant \mathrm{nl}\,\beta, \tag{2.4}$$

$$\mathrm{nh}\,\alpha\beta \leqslant \mathrm{nh}\,\alpha. \tag{2.5}$$

PROPOSITION 2.15. *The equation* $\mathrm{nl}\,\alpha\beta = \mathrm{nl}\,\beta$ *holds if and only if* $\mathrm{nh}\,\beta \leqslant \mathrm{nl}\,\alpha$.

Proof. The inequality $\mathrm{nh}\,\beta \leqslant \mathrm{nl}\,\alpha$ is equivalent to $I\beta \subseteq D\alpha$ which is equivalent to $D\alpha\beta = \beta^{-1}(D\alpha) = D\beta$. This, in turn, is equivalent to $\mathrm{nl}\,\alpha\beta = \mathrm{nl}\,\beta$.

3. The lattice structure of S. Let us define a partial order in S as follows:

$\alpha \leqslant \beta$ if there exists a sequence $\gamma \in S$ such that $\alpha = \beta\gamma$.

PROPOSITION 3.1. $\alpha \leqslant \beta$ *if and only if* $I\alpha \subseteq I\beta$.

Proof. Let $\alpha \leqslant \beta$. Then by (2.3) we have $I\alpha = I\beta\gamma \subseteq I\beta$. Conversely, let $I\alpha \subseteq I\beta$. Then we have also $D\alpha \subseteq D\beta$, and for every $k \in D\alpha$ there exists an n_k such that $\alpha(k) = \beta(n_k)$. Obviously, $n_k \geqslant k$ and $n_k > n_l$ for $k > l$. Let us now define a sequence γ with $D\gamma = D\alpha$ by the formula $\gamma(k) = n_k$. Then, of course, $\alpha = \beta\gamma$, and $\alpha \leqslant \beta$.

Since the relation $\leqslant$ is equivalent to the set-theoretic inclusion of the images, $(S, \leqslant)$ is, obviously, a partially ordered set.

PROPOSITION 3.2. *If $\alpha \leqslant \beta$, then there exists an element δ in S such that* $\alpha = \delta\beta$.

Proof. As in the preceding proof denote by n_k the positive integer for which $\alpha(k) = \beta(n_k)$. Define δ as follows:

$$D\delta = N_{\max\beta(D\alpha)}, \qquad D\delta = N \text{ if } D\alpha = N,$$

$$\delta(j) = \begin{cases} j & \text{for } j \leqslant \beta(1), \\ n_k + j - k & \text{for } \beta(k) \leqslant j < \beta(k+1) \text{ if } k+1 \in D\alpha, \\ & \quad \text{or for } j \geqslant \beta(k) \text{ if } k+1 \notin D\alpha. \end{cases}$$

Then $\delta\beta = \alpha$. Indeed, $D\delta\beta = \beta^{-1}(D\delta) = D\alpha$ and also, for $k \in D\delta\beta$, we have $\delta\beta(k) = \delta\big(\beta(k)\big) = n_k = \alpha(k)$.

The converse of this statement does not hold as is shown by the following example where the sequences are given by their images:

Example 3.1. Let $I\beta = \{1, 2\}$, $I\delta = \{2, 3, 4\}$, and $\alpha = \delta\beta$. Then $I\alpha = \{2, 3\}$ and, consequently, $I\alpha \nsubseteq I\beta$ and $\alpha \nleqslant \beta$.

PROPOSITION 3.3. *The element 0 is the smallest and ε is the largest element of S.*

Proof is obvious.

Proposition 2.5 implies immediately the following

PROPOSITION 3.4. *The idempotents form a well ordered chain with respect to the order $\leqslant$. The order type is $\omega+1$. We have $\iota_m \leqslant \iota_n$ if and only if $m \leqslant n$.*

PROPOSITION 3.5. *Every set $M \subseteq S$ has the least upper bound and the greatest lower bound. Consequently, $(S, \leqslant)$ is an absolute lattice with largest and smallest element.*

Proof follows immediately from proposition 3.1.

We denote the least upper bound, called *join*, of the set M by $\bigvee M$ and the greatest lower bound, called *meet*, by $\bigwedge M$. If M is an indexed family $M = \{a_t | t \in T\}$, we also write $\bigvee\limits_{t\in T} a_t$ and $\bigwedge\limits_{t\in T} a_t$, respectively. In the finite case also $a_1 \vee \ldots \vee a_n$ and $a_1 \wedge \ldots \wedge a_n$.

Obviously,

$$I \bigvee_{t\in T} a_t = \bigcup_{t\in T} Ia_t, \qquad I \bigwedge_{t\in T} a_t = \bigcap_{t\in T} Ia_t. \tag{3.1}$$

PROPOSITION 3.6. *Operations of join and meet are left-distributive with respect to the semigroup multiplication:*

$$\beta\Big(\bigvee_{t\in T} a_t\Big) = \bigvee_{t\in T} (\beta a_t), \qquad \beta\Big(\bigwedge_{t\in T} a_t\Big) = \bigwedge_{t\in T} (\beta a_t). \tag{3.2}$$

Proof.

$$I\beta\Big(\bigvee_{t\in T} a_t\Big) = \beta\Big(I \bigvee_{t\in T} a_t\Big) = \beta\Big(\bigcup_{t\in T} Ia_t\Big) = \bigcup_{t\in T} \beta(Ia_t) = \bigcup_{t\in T} I(\beta a_t) = I \bigvee_{t\in T} (\beta a_t).$$

In the case of the meet the proof is similar, but we must use the fact that β is a one-to-one mapping in order to be able to conclude that

$$\beta\Big(\bigcap_{t\in T} Ia_t\Big) = \bigcap_{t\in T} \beta(Ia_t).$$

The following examples show that there is no right-distributivity even for joins and meets of two elements:

Example 3.2. Let $Ia_1 = \{1, 3, 5\}$, $Ia_2 = \{2, 4, 6\}$ and $I\beta = \{2, 3\}$. Then $I(a_1 \vee a_2) = \{1, 2, 3, 4, 5, 6\}$, $a_1 \vee a_2 = \iota_6$ and $(a_1 \vee a_2)\beta = \beta$. On the other hand, $Ia_1\beta = \{3, 5\}$, $Ia_2\beta = \{4, 6\}$ and, hence, $I(a_1\beta \vee a_2\beta) = \{3, 4, 5, 6\}$, $(a_1\beta) \wedge (a_2\beta) \neq (a_1 \vee a_2)\beta$.

Example 3.3. Let $a_1 = \iota_6$, $Ia_2 = \{4, 5, 6\}$ and $\beta = \iota_3$. Then $a_1 \wedge a_2 = a_2$ and $(a_1 \wedge a_2)\beta = a_2\beta = a_2$. On the other hand, $a_1\beta = \beta$, $a_2\beta = a_2$ and $(a_1\beta) \wedge (a_2\beta) = \beta \wedge a_2 = 0$.

For the idempotents the meet coincides with the product. Namely, we have even a stronger result.

PROPOSITION 3.7. *If a is an idempotent, then $Ia\beta = Ia \cap I\beta$, and, consequently, $a\beta = a \wedge \beta$.*

Proof. Since a is an idempotent, thus an inclusion map, we have $Ia\beta = a(I\beta \cap Da) = Ia \cap I\beta = I(a \wedge \beta)$.

The converse is not true as the following example shows:

Example 3.4. $Ia = \{1, 2, 5\}$ and $I\beta = \{3, 7, 5\}$ so that neither a nor β are idempotent, but $Ia\beta = \{5\} = Ia \cap I\beta = I(a \wedge \beta)$.

Also the proposition will not hold if we assume that β is an idempotent, not a.

Example 3.5. $Ia = \{4, 5, 6\}$ and $\beta = \iota_3$. Here we have $a \wedge \beta = 0$, but $a\beta = a$.

However, we have the following

PROPOSITION 3.8. *If β is an idempotent, then $D(a\beta) = Da \cap D\beta$.*

Proof. $Da\beta = \beta^{-1}(Da) = D\beta \cap Da$, since β is the inclusion map of $D(\beta)$ into $\boldsymbol{N}$.

It follows from proposition 3.7 that if a is idempotent, then $a\beta \leqslant \beta$ and, therefore, there exists an element $\gamma \in \boldsymbol{S}$ such that $a\beta = \beta\gamma$. That sequence γ can be specified to be an idempotent, and we have the following

PROPOSITION 3.9. *For every idempotent ι and every $a \in \boldsymbol{S}$ there exists an idempotent $\varkappa$ such that $\iota a = a\varkappa$. Namely, one can take $\varkappa = \iota_{\mathrm{nl}(\iota a)}$.*

Proof. By proposition 3.8, we have $D(a\varkappa) = Da \cap D\varkappa$. By definition of $\varkappa$, $D\varkappa = D\iota a \subseteq Da$, because of (2.2). Thus $D(a\varkappa) = D\iota a$. Since both ι and $\varkappa$ are inclusion mappings on their corresponding domains, $a\varkappa = \iota a$.

In particular, if both factors a and β are idempotents and $a \leqslant \beta$, we have $a \wedge \beta = a\beta = a$. Consequently, the ordering $\leqslant$ in the set $\boldsymbol{S}_0$ of idempotents coincides with the natural order of the band $\boldsymbol{S}_0$.

Because of (3.1), there is a simple relationship between the numerical heights of sequences and their join and meet.

PROPOSITION 3.10. *It is*

$$\mathrm{nh} \bigvee_{t \in T} a_t = \sup_{t \in T} \mathrm{nh}\, a_t, \qquad \mathrm{nh} \bigwedge_{t \in T} a_t = \inf_{t \in T} \mathrm{nh}\, a_t.$$

The situation with domains and thus with lengths is less simple.

PROPOSITION 3.11. *It is*

$$\mathrm{nl} \bigvee_{t \in T} a_t = \Big|\bigcup_{t \in T} Ia_t\Big| = \Big|I\big(\bigvee_{t \in T} a_t\big)\Big|,$$

$$\mathrm{nl} \bigwedge_{t \in T} a_t = \Big|\bigcap_{t \in T} Ia_t\Big| = \Big|I\big(\bigwedge_{t \in T} a_t\big)\Big|.$$

Proof. $\mathrm{nl} \bigvee_{t \in T} a_t = \big|D \bigvee_{t \in T} a_t\big| = \big|I \bigvee_{t \in T} a_t\big|$, and similarly for the meet.

We need two more results.

PROPOSITION 3.12. *If* $\mathrm{nl}\,\alpha\iota = \mathrm{nl}\,\beta\iota$ *and* $\mathrm{nh}\,\alpha\iota = \mathrm{nh}\,\beta\iota$ *for every idempotent* ι, *then* $\alpha = \beta$.

Proof. Suppose $\alpha \neq \beta$. Then either 1° $D\alpha \neq D\beta$, say $D\alpha \subset D\beta$, or 2° the domains coincide, but there exists a number $k \in D\alpha = D\beta$ such that $\alpha(k) \neq \beta(k)$, say $\alpha(k) < \beta(k)$. In the first case $\mathrm{nl}\,\alpha < \mathrm{nl}\,\beta$, so $\mathrm{nl}\,\alpha\varepsilon \neq \mathrm{nl}\,\beta\varepsilon$; in the second case $\mathrm{nh}\,\alpha\iota_k = \alpha(k) \neq \mathrm{nh}\,\beta\iota_k = \beta(k)$.

PROPOSITION 3.13. *For every* α *and* $n \leqslant \mathrm{nl}\,\alpha$ *we have* $\alpha(n) = \mathrm{nh}\,\alpha\iota_n$.

Proof. If $n \leqslant \mathrm{nl}\,\alpha$, then $n \in D\alpha$ and $D\alpha\iota_n = D\alpha \cap N_n$, where n is the largest integer in $D\alpha\iota_n$. Consequently, $\alpha(n)$ is the largest integer in $I\alpha\iota_n$ and $\alpha(n) = \mathrm{nh}\,\alpha\iota_n$.

4. Replacing natural numbers by idempotents. The order isomorphism between the natural numbers and the finite idempotents (see proposition 3.4) allows us to remove from the theory the extrinsic elements — the natural numbers — and operate strictly within the set S. We define the *abstract length* $\mathrm{al}\,\alpha$ and *abstract height* $\mathrm{ah}\,\alpha$ of a sequence α as

$$\mathrm{al}\,\alpha = \bigwedge \{\iota \in S_0 \mid \alpha\iota = \alpha\}, \tag{4.1}$$

$$\mathrm{ah}\,\alpha = \bigwedge \{\iota \in S_0 \mid \iota\alpha = \alpha\}, \tag{4.2}$$

where S_0 is the set of all idempotent elements of the semigroup S.

The relation between the abstract length and height and the numerical length and height is obvious.

PROPOSITION 4.1. *The equations* $\mathrm{al}\,\alpha = 0$, $\mathrm{nl}\,\alpha = 0$, $\mathrm{ah}\,\alpha = 0$, $\mathrm{nh}\,\alpha = 0$, $\alpha = 0$ *are equivalent*; $\mathrm{al}\,\alpha = \varepsilon$ *if and only if* $\mathrm{nl}\,\alpha = \infty$; $\mathrm{al}\,\alpha = \iota_n$ *if and only if* $\mathrm{nl}\,\alpha = n$.

Also immediately from the definition one has

PROPOSITION 4.2. *For every* α, $D\alpha = D(\mathrm{al}\,\alpha) = I(\mathrm{al}\,\alpha)$, $I\alpha \subseteq D(\mathrm{ah}\,\alpha) = I(\mathrm{ah}\,\alpha)$.

We may replace numbers by finite idempotents of S and propositions 3.13 and 4.1 will allow to reconstruct the representation of elements of S as increasing mappings of segments of the chain $S_0 - \{0, \varepsilon\}$ of the form $\{\iota \in S_0 \mid \iota \leqslant \iota^*\}$ into S_0. Namely, for every $\alpha \in S$, α can be considered as a mapping of $D^*\alpha = \{\iota \in S_0 \mid \iota \leqslant \mathrm{al}\,\alpha,\ \iota < \varepsilon\}$ into S_0 such that, for every $\iota \in S_0$, $\iota \neq 0, \varepsilon$,

$$\alpha(\iota) = \mathrm{ah}\,\alpha\iota. \tag{4.3}$$

5. The Boolean structure. Since a sequence α is entirely determined by its image $I\alpha$, S inherits also the Boolean structure of the power set $P(N)$. Thus a complement α' of a sequence α will be defined by the formulae $I\alpha' = N - I\alpha$.

Together with the join and meet operations of section 3 this converts the set S into an absolutely additive Boolean algebra with the smallest element 0 and the largest element ε.

PROPOSITION 5.1. *The complement has the following properties:*

$$(\alpha')' = \alpha, \tag{5.1}$$

$$\alpha \vee \alpha' = \varepsilon, \quad \alpha \wedge \alpha' = 0, \tag{5.2}$$

$$(\alpha\beta)' = \alpha' \vee (\alpha\beta'), \tag{5.3}$$

$$\alpha\beta' = \alpha \wedge (\alpha\beta)'. \tag{5.4}$$

Proof. (5.1) and (5.2) are simply the axioms of a Boolean algebra. To prove (5.3) note that $(\alpha\beta) \vee (\alpha\beta') \vee \alpha' = \alpha(\beta \vee \beta') \vee \alpha' = (\alpha\varepsilon) \vee \alpha' = \alpha \vee \alpha' = \varepsilon$, and, further, $(\alpha\beta) \wedge \big((\alpha\beta') \vee \alpha'\big) = [(\alpha\beta) \wedge (\alpha\beta')] \vee [(\alpha\beta) \wedge \alpha'] = [\alpha(\beta \wedge \beta')] \vee [(\alpha\beta) \wedge \alpha'] = 0$, since $\alpha\beta \leqslant \alpha$, and, therefore, $(\alpha\beta) \wedge \alpha' = 0$. This implies that $(\alpha\beta') \vee \alpha' = (\alpha\beta)'$.

To prove (5.4) it suffices to take the meets of both sides of (5.3) with α.

It might be of some interest to notice that $\iota_{n+m} = (\iota_n' \iota_m')'$. Thus addition of natural numbers can be reconstructed in S_0 by the definition

$$\iota + \varkappa = (\iota' \varkappa')' \tag{5.5}$$

expressed in terms of the semigroup operation and complementation only.

With the Boolean structure a Boolean ring structure is associated in a known manner with the meet as multiplication and the symmetric difference $\alpha \Delta \beta = (\alpha \wedge \beta') \vee (\alpha' \wedge \beta)$, as addition.

PROPOSITION 5.1. *The semigroup multiplication is left-distributive with respect to the symmetric difference* $\alpha(\beta \Delta \gamma) = (\alpha\beta) \Delta (\alpha\gamma)$.

Proof. By the definition,

$$\begin{aligned}
(\beta \Delta \gamma) &= \alpha[(\beta \wedge \gamma') \vee (\beta' \wedge \gamma)] \\
&= [\alpha(\beta \wedge \gamma')] \vee [\alpha(\beta' \wedge \gamma)] && \text{by (3.2)} \\
&= [(\alpha\beta) \wedge (\alpha\gamma')] \vee [(\alpha\beta') \wedge (\alpha\gamma)] && \text{by (3.2)} \\
&= [(\alpha\beta) \wedge \alpha \wedge (\alpha\gamma)'] \vee [\alpha \wedge (\alpha\beta)' \wedge \alpha\gamma] && \text{by (5.4)} \\
&= [(\alpha\beta) \wedge (\alpha\gamma)'] \vee [(\alpha\beta)' \wedge \alpha\gamma] = (\alpha\beta) \Delta (\alpha\gamma).
\end{aligned}$$

There is, of course, no right distributivity.

Example 5.1. $I\alpha = \{1, 2, 3\}$, $I\beta = \{1, 5, 6\}$, and $I\gamma = \{3, 4\}$. Then $I\alpha\gamma = \{3\}$, $I\beta\gamma = \{6\}$, $I(\alpha\gamma \Delta \beta\gamma) = \{3, 6\}$, $I(\alpha \Delta \beta) = \{2, 3, 5, 6\}$, and $I(\alpha \Delta \beta)\gamma = \{5, 6\}$. Thus $(\alpha \Delta \beta)\gamma \neq (\alpha\gamma) \Delta (\beta\gamma)$.

Properties expressed by propositions 5.1 and 3.6 can be summarized as follows:

PROPOSITION 5.2. *The set S is endowed with three binary operations, the symmetric difference $a \Delta \beta$, a group operation, and two multiplications $a\beta$ and $a \wedge \beta$. $(S, \Delta, \cdot)$ forms a nearring, and $(S, \Delta, \wedge)$ forms a Boolean ring. The nearring multiplication is left-distributive with respect to the Boolean multiplication.*

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