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REMARKS AND EXAMPLES CONCERNING UNORDERED BAIRE-LIKE AND ULTRABARRELLED SPACES

BY

P. DIEROLF, S. DIEROLF (MUNICH) AND L. DREWNOWSKI (POZNAŃ)

1. Following Saxon [12], p. 153, we call a locally convex space *Baire-like* (*unordered Baire-like*) if it cannot be covered by an increasing (arbitrary) sequence of nowhere dense absolutely convex subsets. Thus for a locally convex space we have the following implications:

$$\text{Baire} \Rightarrow \text{unordered Baire-like} \Rightarrow \text{Baire-like} \Rightarrow \text{barrelled}.$$

Recently, Saxon [12] (p. 158, Example 2.2, and p. 157, Example 1.4) showed that unordered Baire-like normed spaces need not be Baire and that Baire-like normed spaces need not be unordered Baire-like, whereas, on account of [1], p. 274, a metrizable locally convex space is barrelled iff it is Baire-like.

It is our first purpose here to give some more simple examples. In fact, Theorem 1 enables us to give examples of normed unordered Baire-like spaces which are not ultrabarrelled (and hence not Baire). Recall that, according to [10], p. 249, a linear topological space (X, β) is called *ultrabarrelled* if $\alpha \subset \beta$ for every linear topology α on X which is β-polar, i.e., has a base of neighbourhoods of zero consisting of β-closed sets. Further characterizations of ultrabarrelled spaces may be found in [5], p. 295 ff., and [14], p. 10 ff. Clearly, if X is Baire, then it is ultrabarrelled. On the other hand, ultrabarrelled linear topological spaces need not be Baire, as the strongest linear topology on an infinite-dimensional linear space shows. Furthermore, any strict inductive limit of an increasing sequence of Fréchet spaces is ultrabarrelled (see [5], p. 297, Corollary 2), hence barrelled, but clearly not Baire-like.

Our Theorem 2 provides examples of metrizable (and even normed) ultrabarrelled locally convex spaces which are not unordered Baire-like, and hence not Baire.

Summarizing, there seems to be no evident relation between "ultrabarrelled" and "unordered Baire-like", which is not surprising at all.

In view of this, and the above-mentioned result of [1], we note that the following characterization results from Corollary 3 of [4], p. 558:

A metrizable linear topological space X is ultrabarrelled iff it satisfies the following condition:

Let $(A_i^{(m)};\ i, m \in N)$ be a double sequence of closed balanced subsets of X such that

(a) $A_i^{(m)} \subset A_i^{(m+1)}$ and $A_{i+1}^{(m)} + A_{i+1}^{(m)} \subset A_i^{(m)}$ $(i, m \in N)$,

(b) $\bigcup \{A_i^{(m)};\ m \in N\}$ is absorbent $(i \in N)$.

Then for every $i \in N$ there exists $m \in N$ such that $A_i^{(m)}$ is a neighbourhood of zero in X.

In Section 3, we consider the space $m_0(\mathscr{A})$ of $\mathscr{A}$-simple scalar-valued functions defined on a set I, where $\mathscr{A}$ is a σ-algebra of subsets of I, equipped with the usual supremum-norm. In addition to some known curious properties of $m_0(\mathscr{A})$ like those of being barrelled, non-Baire, etc., we prove that it is not ultrabarrelled and contains no infinite-dimensional separable barrelled subspace. For a special case $I = N$ and $\mathscr{A} = \mathscr{P}(N)$, the σ-algebra of all subsets of N, these results are due to N. J. Kalton and A. Pełczyński, respectively (unpublished). The extension of Pełczyński's result to $m_0(\mathscr{A})$ requires no new techniques while our proof of the result of Kalton is entirely different from the original one (cf. Remark 2). In fact, we prove a stronger result: No infinite-dimensional subspace X of $m_0(\mathscr{A})$ admits a linear topology ξ stronger than the norm-topology such that (X, ξ) is either Baire or metrizable and ultrabarrelled. We also give an alternative proof of the main result of Batt et al. (see [2], Theorem 1) concerning summable sequences in $m_0(\mathscr{A})$.

We are grateful to Professor Kalton and to Professor Pełczyński for the permission to use their results in this paper.

2. Let us start by observing that a locally convex space (X, ξ) is unordered Baire-like iff it is barrelled and has the following property:

$(*)$ *Given a sequence $(A_n;\ n \in N)$ of absolutely convex closed sets covering X, some A_n is absorbent (i.e., is a barrel in (X, ξ)).*

Now, if $\xi \supset \eta$ are linear topologies on a linear space X such that (X, ξ) satisfies $(*)$, then so does (X, η). In particular, this is certainly the case where (X, ξ) is Baire.

It follows that any non-Baire barrelled space X which admits a stronger linear Baire topology will provide an example of a non-Baire unordered Baire-like locally convex space. Such spaces really do exist, as was already shown by Robertson [10], and from the results in Section 7 of [10], p. 255, we immediately have the following

THEOREM 1. *Let (X, ξ) be a non-locally convex F-space (i.e., a metrizable complete linear topological space) such that the strongest locally convex* *topology $c(\xi)$ weaker than ξ is Hausdorff. Then $(X, c(\xi))$ is a metrizable (hence Mackey) unordered Baire-like locally convex space which is not ultrabarrelled. Furthermore, if (X, ξ) is locally bounded, then $(X, c(\xi))$ is normed.*

The most simple spaces to which Theorem 1 applies are (as in [10], p. 256) the classical sequence spaces l_p, where $0 < p < 1$, equipped with the topology ξ defined by the F-norm

$$|(t_n;\, n \in N)|_p := \sum_{n\in N} |t_n|^p.$$

Then $(l_p, |\cdot|_p)$ is a locally bounded non-locally convex F-space which is continuously embedded as a dense subspace in $(l_1, |\cdot|_1)$, and its dual is identified in a standard way with l_∞, the dual of $(l_1, |\cdot|_1)$ (cf. [3], p. 822). Hence $c(\xi)$ is simply the topology induced on l_p by the norm $|\cdot|_1$, and thus $(l_p, |\cdot|_1)$ is a normed unordered Baire-like space which is not ultrabarrelled (and hence not Baire).

THEOREM 2. *Let (X, ξ) be an infinite-dimensional F-space which admits a biorthogonal sequence $((u_n, f_n);\, n \in N)$ such that*

$$Z := Y + \operatorname{lin}\{u_n;\, n \in N\}, \quad \textit{where } Y := \{x \in X;\, f_n(x) = 0 \textit{ for all } n \in N\},$$

is dense in X. Then (X, ξ) contains a dense subspace which is ultrabarrelled but not unordered Baire-like.

Proof. Let $\mathfrak{F}$ be an ultra-filter on N such that $\{n\} \notin \mathfrak{F}$ for all $n \in N$. For $A \in \mathfrak{F}$ let

$$E_A := \overline{Y + \operatorname{lin}\{u_n;\, n \notin A\}}^{\xi} \quad \text{and} \quad E := \bigcup \{E_A;\, A \in \mathfrak{F}\}.$$

Clearly, E is a linear subspace of X containing Z, hence E is dense in (X, ξ). Furthermore, E is the union of the sequence of closed hyperplanes $f_n^{-1}(0) \cap E \neq E$, whence E in its relative topology $\xi \cap E$ is not unordered Baire-like. Let η be the strongest linear topology on E such that $\eta \cap E_A = \xi \cap E_A$ for all $A \in \mathfrak{F}$, i.e., (E, η) is the linear inductive (or *-inductive as in [5], p. 286) limit of the F-spaces $(E_A, \xi \cap E_A)$ $(A \in \mathfrak{F})$. By Corollary 1 to Theorem 3.2 of [5], p. 297, (E, η) is ultrabarrelled. We are going to show that $\eta = \xi \cap E$. Clearly, it suffices to prove $\eta \subset \xi \cap E$.

Let U be an η-closed neighbourhood of zero in (E, η). We prove first the existence of an open neighbourhood V of zero in $(E, \xi \cap E)$ such that $V \cap Z \subset U$. Let $(V_n;\, n \in N)$ be a base of the neighbourhoods of zero in (X, ξ) satisfying $V_{n+1} \subset V_n$ $(n \in N)$, and assume that $V_n \cap Z \not\subset U$ for all $n \in N$. Since for every $r \in N$ the space $Y + \operatorname{lin}\{u_n;\, n > r\}$ is a closed subspace of finite codimension in $(Z, \eta \cap Z)$, we find inductively a partition $(I(k);\, k \in N)$ of N into disjoint consecutive finite sets $I(k)$ $(k \in N)$, a sequence $(y_k;\, k \in N)$ in Y, and a sequence $(\lambda_i;\, i \in N)$ of scalars such that

$$y_k + \sum_{i \in I(k)} \lambda_i u_i \in V_k \setminus U \quad \text{for all } k \in N.$$

Now let

$$M := \bigcup\{I(2k-1);\, k \in N\} \quad \text{and} \quad N := \bigcup\{I(2k);\, k \in N\}.$$

$\mathfrak{F}$ being an ultra-filter, either $M \in \mathfrak{F}$ or $N \in \mathfrak{F}$. Consequently, there is $A \in \mathfrak{F}$ such that $V_n \cap E_A \not\subset U$ for all $n \in N$ which contradicts $\eta \cap E_A = \xi \cap E_A$. Thus there is an open neighbourhood V of zero in $(E, \xi \cap E)$ such that $V \cap Z \subset U$. Since

$$\overline{V \cap (Z \cap E_A)}^{\eta} \supset V \cap E_A \quad \text{for all } A \in \mathfrak{F},$$

we have $V \subset \bar{U}^{\eta} = U$, which proves that U is a neighbourhood of zero in $(E, \xi \cap E)$.

Theorem 2 applies especially when (X, ξ) is a separable Fréchet space, since then in X there exists a biorthogonal sequence $\big((u_n, f_n);\, n \in N\big)$ such that $\operatorname{lin}\{u_n;\, n \in N\}$ is dense in (X, ξ) (a result of Klee, cf. [9], p. 118). Thus, in particular, we may construct dense ultrabarrelled not unordered Baire-like subspaces in every Banach space with basis. We do not know whether or not the assertion of Theorem 2 holds for all infinite-dimensional Fréchet spaces. (**P 1031**)

3. If $\mathscr{A}$ is an algebra of subsets of a set I, we denote by $m_0(\mathscr{A})$ the linear space of all $\mathscr{A}$-simple scalar-valued functions defined on I; τ denotes the topology induced on $m_0(\mathscr{A})$ by the usual supremum-norm $\|\cdot\|_\infty$. If $\mathscr{A}$ is infinite, then $\big(m_0(\mathscr{A}), \tau\big)$ is easily seen to be non-Baire, and thus not complete. If $\mathscr{A}$ is a σ-algebra, then, as in [12], p. 157, Example 1.4, one proves easily that $\big(m_0(\mathscr{A}), \tau\big)$ is barrelled.

For the proof of Theorem 3 we use the following two lemmas, whereby we write $\operatorname{card}(A)$ for the cardinality of the set A.

LEMMA 1. *If X is a linear subspace contained in*

$$X_n := \big\{x \in m_0(\mathscr{A});\, \operatorname{card}\big(x(I)\big) \leqslant n\big\},$$

then $\dim X \leqslant n$.

Proof. Suppose that this is not true and take any $y \in X$ with

$$\operatorname{card}\big(y(I)\big) = m := \max\big\{\operatorname{card}\big(z(I)\big);\, z \in X\big\}.$$

Since $\dim X > n \geqslant m$, there exists $z \in X$ such that, for some $t \in y(I)$,

$$\operatorname{card}\Big(z\big(y^{-1}(t)\big)\Big) \geqslant 2.$$

Then taking $\varepsilon > 0$ sufficiently small and setting $x := y + \varepsilon z$, we have $\operatorname{card}\big(x(I)\big) > m$ and $x \in X$, which is a contradiction.

LEMMA 2. *If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, then $m_0(\mathscr{B})$ is a closed subspace of* $\big(m_0(\mathscr{A}), \tau\big)$.

Proof. Take any non-zero $x \in \overline{m_0(\mathscr{B})}$ and write it in the form

$$x = \sum_{i=0}^{m} t_i \chi_{A_i}$$

with pairwise disjoint $A_i \in \mathscr{A}$, $t_i \neq t_j$ for $i \neq j$, and $t_0 = 0$. Let

$$a := \min\{|t_i - t_j|;\, i \neq j\},$$

and then choose any $y \in m_0(\mathscr{B})$ such that $\|x-y\|_\infty < a/2$; say

$$y = \sum_{k=0}^{n} s_k \chi_{B_k},$$

where B_k are pairwise disjoint members of $\mathscr{B}$, $s_k \neq s_l$ for $k \neq l$, and $s_0 = 0$.

If $B_k \cap A_i \neq \emptyset$ for some i and k, then $B_k \subset A_i$. Otherwise, for some $j \neq i$ we would have $B_k \cap A_j \neq \emptyset$, and hence

$$|s_k - t_i| < a/2 \quad \text{and} \quad |s_k - t_j| < a/2,$$

so that $|t_i - t_j| < a$, which is impossible. It follows that each A_i is the union of a subsequence of $B_0, B_1, \ldots, B_n$ so that $A_i \in \mathscr{B}$, and thus x is $\mathscr{B}$-simple.

THEOREM 3. *Let $\mathscr{A}$ be an infinite algebra of subsets of a set I. Then*:

(a) *No infinite-dimensional subspace X of $m_0(\mathscr{A})$ admits a linear Baire topology ξ which is stronger than $\tau \cap X$. Thus, in particular, no infinite-dimensional subspace X of $m_0(\mathscr{A})$ admits an F-space topology ξ stronger than $\tau \cap X$.*

(b) *No infinite-dimensional subspace X of $m_0(\mathscr{A})$ admits a metrizable ultrabarrelled linear topology ξ which is stronger than $\tau \cap X$. Thus, in particular, no infinite-dimensional subspace of $m_0(\mathscr{A})$ is ultrabarrelled.*

(c) *Every separable subspace of $(m_0(\mathscr{A}), \tau)$ is of at most countable dimension. Thus no infinite-dimensional separable subspace of $(m_0(\mathscr{A}), \tau)$ is barrelled.*

Proof. Our proof of (a) and (b) uses some ideas found in [13], p. 981, and in [8], Section 4. For each $n \in N$ let

$$X_n := \{x \in m_0(\mathscr{A});\ \operatorname{card}(x(I)) \leqslant n\}.$$

Suppose that X is an infinite-dimensional subspace of $m_0(\mathscr{A})$ and ξ is a linear topology on X satisfying $\xi \supset \tau \cap X$. Since each X_n is τ-closed, $X_n \cap X$ is ξ-closed and, clearly, balanced for all $n \in N$. Furthermore, we have

$$X_n \cap X + X_n \cap X \subset X_{n^2} \cap X \qquad (n \in N)$$

and

$$X = \bigcup\{X_n \cap X;\, n \in N\}.$$

If ξ is metrizable and ultrabarrelled, then by Corollary 2 of [6], p. 683, there exists $k \in N$ such that $X_k \cap X$ is a ξ-neighbourhood. This implies

$$X = \bigcup \{m \cdot (X_k \cap X);\ m \in N\} \subset X_k,$$

hence $\dim X \leqslant k$ by Lemma 1.

If ξ is a Baire topology, we also infer that some $X_k \cap X$ is a ξ-neighbourhood, which leads to the same contradiction. Thus we have proved (a) and (b).

To show (c) it is enough to prove the first assertion, since it is well known that a metrizable locally convex space of countably infinite dimension cannot be barrelled.

Let X be a separable subspace of $m_0(\mathscr{A})$, and let D be a countable dense subset of $(X, \tau \cap X)$. Then there exists a countable subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $D \subset m_0(\mathscr{B})$. Since $m_0(\mathscr{B})$ is closed by Lemma 2, we have $X \subset m_0(\mathscr{B})$, and hence

$$\dim X \leqslant \dim m_0(\mathscr{B}) \leqslant \aleph_0.$$

Remarks. (a) Using similar methods, H. Pfister has independently proved that $\big(m_0(\mathscr{P}(I)), \tau\big)$ is not ultrabarrelled and that every Banach disk in $m_0(\mathscr{P}(I))$, provided with the relative product topology from K^I, must be finite dimensional (unpublished).

(b) Kalton's original proof of the fact that $\big(m_0(\mathscr{P}(N)), \tau\big)$ is not ultrabarrelled is very ingenious so that we would like to present it here. We shorten $m_0(\mathscr{P}(N))$ to m_0. Suppose that (m_0, τ) is ultrabarrelled and let $(F, |\cdot|)$ be the F-space, constructed by Rolewicz and Ryll-Nardzewski (see [11], p. 329 ff.), containing a sequence $(x_n;\ n \in N)$ which is subseries summable but not bounded multiplier summable. For each $n \in N$ define $T_n\colon l_\infty \to F$ by

$$T_n(t) := \sum_{i=1}^{n} t_i x_i \qquad (t = (t_i;\ i \in N) \in l_\infty).$$

Then each T_n is a continuous linear map of $(l_\infty, \|\cdot\|_\infty)$ into $(F, |\cdot|)$, and $(T_n;\ n \in N)$ converges pointwise on m_0. Hence, by the Banach-Steinhaus theorem for ultrabarrelled spaces (cf. [10], p. 250), $(T_n | m_0;\ n \in N)$ is equicontinuous. Since m_0 is dense in $(l_\infty, \|\cdot\|_\infty)$, we infer that $(T_n;\ n \in N)$ is also equicontinuous and converges pointwise on l_∞. This implies that $(x_n;\ n \in N)$ is bounded multiplier summable, which is a contradiction.

(c) Replacing the phrase "no infinite-dimensional subspace" in Theorem 3 (a) and (b) by "no subspace X satisfying $\sup\{\mathrm{card}(x(I));\ x \in X\} = \infty$", we may also prove the assertions of Theorem 3 (a) and (b) for the space $m_0(\mathscr{A}, E)$ of all $\mathscr{A}$-simple functions with values in a normed space $(E, \|\cdot\|)$, provided with the topology induced by the supremum-norm.

(d) It would be interesting to know the coarsest ultrabarrelled topology on $m_0(\mathscr{A})$, stronger than τ. (**P 1032**)

Finally, we present an alternative and somewhat shorter proof for the main result of Batt et al. [2], Theorem 1, which — up to some variants — was also obtained independently by H. Pfister (unpublished).

THEOREM 4. *Let $\mathscr{A}$ be a σ-algebra of subsets of a set I, and let τ and π be the topologies on $m_0(\mathscr{A})$ of uniform and pointwise convergence on I, respectively. Let $(x_n;\, n \in N)$ be a sequence in $m_0(\mathscr{A})$. Then:*

(a) *If $(x_n;\, n \in N)$ is bounded multiplier (BM) summable in $(m_0(\mathscr{A}), \pi)$, then* $\dim \operatorname{lin}\{x_n;\, n \in N\} < \infty$.

(b) *If $(x_n;\, n \in N)$ is subfamily (SF) summable in $(m_0(\mathscr{A}), \tau)$, then* $\dim \operatorname{lin}\{x_n;\, n \in N\} < \infty$.

Proof. We shorten $m_0(\mathscr{A})$ to m_0. We first show that (b) $\Rightarrow$ (a). Suppose that $(x_n;\, n \in N)$ is BM-summable for π and define a linear map $T\colon l_\infty \to m_0$ by

$$T(t) := \sum_{i=1}^{\infty} t_i x_i \qquad (t = (t_i;\, i \in N) \in l_\infty),$$

where the sum is taken with respect to π. By the Banach-Steinhaus the, orem, $T\colon (l_\infty, \|\cdot\|_\infty) \to (m_0, \pi)$ is continuous. Since τ is π-polar, $T\colon (l_\infty, \|\cdot\|_\infty$- $\to (m_0, \tau)$ is continuous as well. It follows that $(x_n;\, n \in N)$ is τ-bounded) and thus, again by the π-polarity of τ, $(s_n x_n;\, n \in N)$ is SF-summable with respect to τ for each $(s_n;\, n \in N) \in l_1$. Choosing $(s_n;\, n \in N) \in l_1$ such that $s_n \neq 0$ for all $n \in N$, from (b) we obtain

$$\dim \operatorname{lin}\{s_n x_n;\, n \in N\} = \dim \operatorname{lin}\{x_n;\, n \in N\} < \infty.$$

Now suppose that (b) is false; then without loss of generality we may assume the sequence $(x_n;\, n \in N)$ to be linearly independent. Since $(x_n;\, n \in N)$ is SF-summable, the formula

$$m(A) := \sum_{n \in A} x_n \qquad (A \subset N)$$

defines a countably additive vector measure $m\colon \mathscr{P}(N) \to (m_0, \tau)$. Then, with X_n as in the proof of Theorem 3, a result of [7], p. 46, Lemma 2, implies $m(A) \in X_r$ for some $r \in N$ and all finite subsets A of N, and we may suppose that r is the smallest integer for which this holds. Choose a finite subset A of N such that

$$x := \sum_{n \in A} x_n$$

assumes precisely r distinct values. Then, as $\|x_m\|_\infty \to 0$ $(m \to \infty)$ and $(x_n;\, n \in N)$ is linearly independent, there exists $m \in N \setminus A$ such that x_m assumes at least two different values on the set $x^{-1}(t)$ for some $t \in x(I)$

and, at the same time, $\|x_m\|_\infty$ is small enough to assure that

$$x+x_m = \sum_{n\in A\cup\{m\}} x_n$$

assumes at least $r+1$ distinct values. This, however, contradicts the choice of r.

We refer to [2] for various consequences of Theorem 4.

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MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
MÜNCHEN

INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY
POZNAŃ

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