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UNIQUENESS OF REPRESENTATIONS OF A DISTRIBUTIVE LATTICE AS A FREE PRODUCT OF A BOOLEAN ALGEBRA AND A CHAIN

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1. Introduction. The main objective of this paper* is to answer a question which can be stated roughly as follows:

If a distributive lattice can be represented as a free product (= *coproduct in the category of distributive lattices with* $0, 1$) *of a Boolean algebra and a chain, then is this representation unique?* Before giving a precise formulation of this question we introduce some terminology and make some conventions.

Since we are concerned with free products only in the category $\mathscr{D}$ of distributive lattices with $0, 1$ and $(0, 1)$-lattice homomorphisms, by $L = L_1 * L_2$ we will always mean that L, L_1 and L_2 are in $\mathscr{D}$, and that L is the coproduct of L_1 and L_2 in $\mathscr{D}$. Furthermore, in order to avoid cumbersome formulations, we will always assume that L_1 and L_2 are $(0, 1)$-sublattices of L and that the corresponding injections are the inclusions $L_i \to L$, $i = 1, 2$. However for technical reasons, we prefer to carry out our arguments in the larger category $\mathscr{D}'$ of distributive lattices and lattice homomorphisms. So, with the exception already mentioned, all objects, subobjects, etc., are to be taken in $\mathscr{D}'$. Thus, for example if we say S is a subchain of $B * C$, where B is a Boolean algebra and C is a chain, then necessarily B and C have 0 and 1 but S is merely a linearly ordered subset of $B * C$. Finally, the symbols L, L_1, L_2, L', etc., C, C_1, C_2, C', etc., and B, B_1, B_2, B', etc., will always denote distributive lattices, chains, and Boolean algebras, respectively. The precise formulation of the question stated in the first paragraph is then:

(1) *Does* $B * C = B * C'$ *imply* $C = C'$?

Note that we require equality and not just isomorphism. This is of course due to the conventions made above.

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In [1] it was shown that if $L = B*C = B*C'$ and if C is finite, then $C = C'$. This generalized the uniqueness theorem ([2], Theorem 2.1) for the chain of constants in a Post algebra and gave rise to the question posed above. It is also an immediate consequence of a result in [1] that if $L = B*C = B'*C'$, then $B = B'$. On the other hand, there exist chains C and C' in $\mathscr{D}$ with the property that $B*C = B*C'$ but $C \neq C'$. We therefore consider the class $\mathscr{E}$ of all chains C in $\mathscr{D}$ such that $B*C = B*C'$ implies $C = C'$ for every B and every C' (the class $\mathscr{E}$ was first introduced in [1]). The main results of the present paper are:

(i) *If $L = B*C = B*C'$, then $B = B'$ and C and C' are isomorphic* (Theorem 1).

(ii) *A chain C belongs to $\mathscr{E}$ if and only if C has $0, 1$ and C is rigid* (Theorem 6). (A chain is *rigid* if it has no automorphisms but the identity map.) The complete answer to (1) is then given by

(iii) *If $L = B*C$, then this representation is unique if and only if C has $0, 1$ and either $B = 2$ or C is rigid.*

In order to obtain these results it appears necessary to give some characterization of rigid chains. In particular a sufficient condition for a chain to be rigid is that each convex subchain has a least or greatest element (see the remark after Theorem 3). This condition, however, is not necessary and a suitable counterexample is provided.

Many of the results obtained here carry over to the case where the Boolean algebra B in $L = B*C$ is replaced by an arbitrary distributive lattice in $\mathscr{D}$. We will consider generalizations of this type in the last section.

2. Preliminaries and a fundamental result. We will need the following facts; the first is well known and the second was proved in [1].

(2) $L = L_1 * L_2$ if and only if $L_1 \cup L_2$ generates L and for $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$, $a_1 a_2 \leqslant b_1 + b_2$ implies $a_1 \leqslant b_1$ or $a_2 \leqslant b_2$.

(3) $L = B*C$ if and only if each $x \in L$ can be uniquely represented in the form $x = \sum_{i=1}^{m} a_i c_i$, $m \geqslant 1$, where $\{c_1, \ldots, c_m\} \subseteq C$, $\{a_1, \ldots, a_m\} \subseteq B$ and $0 = c_1 < c_2 < \ldots < c_m$, $1 = a_1, a_2 > \ldots > a_m > 0$.

Our first theorem is crucial to our main result (Theorem 6) and is of interest in itself.

THEOREM 1. *Suppose $L = B*C = B'*C'$. Then $B = B'$ and C is isomorphic with C'. Moreover, an isomorphism $f\colon C \to C'$ can be chosen such that $f(x) \leqslant x$ for each $x \in C$.*

Proof. The fact that $B = B'$ follows immediately from a result that was proved in [1] and which states that if $L = B*C$, then B is the center of L. Now each $x \in C$, $x \neq 0$, can be uniquely represented in the form

$$x = \sum_{i=1}^{n} b_i c_i', \quad n \geq 1, \tag{4}$$

where $\{c_1', \ldots, c_n'\} \subseteq C'$, $\{b_1, \ldots, b_n\} \subseteq B$, $0 < c_1' < \ldots < c_n'$ and $1 = b_1 > b_2 > \ldots > b_n > 0$.

Indeed, if in (3) $x \neq 0$, then $m \geq 2$ and we have $x \leq c_1 + a_2 + \ldots + a_m = 0 + a_2$, so, by (2), $a_2 = 1$ and therefore

$$x = \sum_{i=2}^{m} a_i c_i,$$

where $m \geq 2$, $1 = a_2 > \ldots > a_m > 0$ and $0 < c_2 < \ldots < c_m$. Now define $f\colon C \to C'$ as follows: $f(0) = 0$ and for $x \neq 0$ with unique representation (4), $f(x) = c_1'$. Obviously, $f(x) \leq x$ for all $x \in C$. We will show that f is an isomorphism. Let $x, y \in C \sim \{0\}$ where the unique representation of x is given by (4) and the unique representation of y by

$$y = \sum_{j=1}^{m} b_j' d_j', \quad m \geq 1,$$

where $\{d_1', \ldots, d_1'\} \subseteq C'$, $\{b_1', \ldots, b_m'\} \subseteq B$, $0 < d_1' < \ldots < d_m'$ and $1 = b_1' > b_2' > \ldots > b_m' > 0$.

Suppose first that $x \leq y$ and $m \geq 2$. Then

$$c_1' \leq b_1 c_1' \leq x \leq y \leq d_1' + (b_2' + \ldots + b_m') = d_1' + b_2'.$$

But $b_2' \neq 1$ so by (2) $c_1' \leq d_1'$. If $m = 1$, then we have immediately $c_1' \leq d_1'$ and thus in either case $f(x) \leq f(y)$. Next, suppose $f(x) \leq f(y)$ and $x \nleq y$. Then $y < x$ and hence $f(y) \leq f(x)$, so $c_1' = f(x) = f(y) = d_1'$. If $n \geq 2$, then $x \leq c_1' + b_2 = d_1' + b_2 = b_1' d_1' + b_2 \leq y + b_2$. But $x \nleq y$, so $b_2 = 1$, a contradiction. For $n = 1$, we have immediately the contradiction $x \leq y$. It remains to show that f is "onto". Let $y \in C'$, $y \neq 0$. Then y has the unique representation

$$y = \sum_{i=1}^{n} b_i c_i, \quad n \geq 1,$$

where $\{c_1, \ldots, c_n\} \subseteq C$, $\{b_1, \ldots, b_n\} \subseteq B$, $0 < c_1 < \ldots < c_n$ and $1 = b_1 > b_2 > \ldots > b_n > 0$.

Again c_n has the unique representation

$$c_n = \sum_{j=1}^{m} b_j' c_j', \quad m \geq 1,$$

where $\{c'_1, \ldots, c'_m\} \subseteq C'$, $\{b'_1, \ldots, b'_m\} \subseteq B$, $0 < c'_1 < c'_2 < \ldots < c'_m$ and $1 = b'_1 > b'_2 > \ldots > b'_m > 0$. Now if $m \geqslant 2$, then

$$y \leqslant c_1 + \ldots + c_n \leqslant c'_1 + (b'_2 + \ldots + b'_m) = c'_1 + b'_2.$$

But $b'_2 \neq 1$, so $y \leqslant c'_1$. If $m = 1$, then it is immediate that $c'_1 \leqslant y$. On the other hand, since $b'_1 = 1$, we have $b_n c'_1 = b_n b'_1 c'_1 \leqslant b_n c_n \leqslant y$. But $b_n \neq 0$, so $c'_1 \leqslant y$. Thus, $f(c_n) = c'_1 = y$, completing the proof of the theorem.

3. Rigid chains. In this section we will prove some results concerning chains to be used later. Recall that a chain C is *rigid* if C has no proper automorphisms. A subchain S of a chain C is *convex* if $x, y \in S$, $z \in C$ and $x \leqslant z \leqslant y$ imply $z \in S$. We call an automorphism f of a chain C *strictly increasing* (*strictly decreasing*) if $f(x) > x$ $(f(x) < x)$ for all $x \in C$.

We also recall the definition of ordinal sum of chains. Let I be a chain and $\{C_i\colon i \in I\}$ a set of disjoint chains. Then the *ordinal sum* $C = \bigoplus_{i \in I} C_i$ of the set $\{C_i\colon i \in I\}$ is defined as follows: C is the set-theoretic union of the C_i, $i \in I$; and C is made into a chain by $a < b$ in C provided that $\{a, b\} \subseteq C_i$ for some i and $a < b$ in C_i, or $a \in C_i$, $b \in C_j$ and $i < j$. Finally, ω and ω^* will, as usual, denote the first infinite ordinal and its dual, respectively.

LEMMA 2 (cf. [3]). *An automorphism of a convex subchain of a chain C can be extended to an automorphism on C.*

Proof. Let f be an automorphism on a convex subchain S of C. Define $g\colon C \to C$ by $g(x) = f(x)$ for $x \in S$ and $g(x) = x$ for $x \notin S$. Using the convexity of S it is routine to verify that g is an automorphism.

The basic result on rigid chains that we will prove can best be stated in the following form (cf. [3]):

THEOREM 3. *For a chain C, the following conditions are equivalent:*

(i) *C is not rigid.*

(ii) *C has a non-void convex subchain S which has a strictly increasing (decreasing) automorphism.*

(iii) *C has non-void convex subchain T such that $T = \bigoplus_{n \in \omega^* + \omega} T_n$, where $\{T_n\}_{n \in \omega^* + \omega}$ is a set of isomorphic subchains of C.*

Proof. (i) $\to$ (ii). Let $f\colon C \to C$ be a non-trivial automorphism. Without loss of generality we may assume that there exists a $c_0 \in C$ such that $f(c_0) > c_0$ (otherwise take f^{-1}). Setting $f^0(c_0) = c_0$, observe that for integers m, n with $m < n$, $f^m(c_0) < f^n(c_0)$. Let

$$S = \{x \in C\colon f^n(c_0) \leqslant x < f^{n+1}(c_0) \quad \text{for some integer } n\}.$$

Now $f^0(c_0) \leqslant c_0 < f^1(c_0)$, so $S \neq \emptyset$. To see that S is convex, suppose $x, y \in S$, $z \in C$ and $x \leqslant z \leqslant y$. From the definition of S, there exist integers m and n such that

$$f^n(c_0) \leqslant x < f^{n+1}(c_0) \quad \text{and} \quad f^m(c_0) \leqslant y < f^{m+1}(c_0),$$

so

$$f^n(c_0) \leqslant x \leqslant z \leqslant y < f^{m+1}(c_0).$$

Thus there is a greatest integer p with the property that $f^p(c_0) \leqslant z$ and a least integer q such that $z < f^q(c_0)$. By the definition of p, $z < f^{p+1}(c_0)$, so by the definition of q, $q \leqslant p+1$. But $q < p+1$ implies $q \leqslant p$, so

$$z < f^q(c_0) \leqslant f^p(c_0) \leqslant z.$$

Hence $q = p+1$ and so $z \in S$. If $f^n(c_0) \leqslant x < f^{n+1}(c_0)$ for some n, then $f^{n+1}(c_0) \leqslant f(x) < f^{n+2}(c_0)$ which shows that $f|S$ maps S into itself and also that $f|S$ is strictly increasing. Finally, if $f^n(c_0) \leqslant x < f^{n+1}(c_0)$, we have

$$f^{n-1}(c_0) \leqslant f^{-1}(x) < f^n(c_0) \text{ so } f^{-1}(x) \in S$$

and hence $f|S$ is the desired automorphism on S. (Note that $f^{-1}|S$ is a strictly decreasing automorphism on S.)

(ii) $\rightarrow$ (iii). Suppose g is a strictly increasing automorphism on S. Pick an element $c_0 \in S$ such that $g(c_0) > c_0$. Using the same argument as in the proof of (i) $\rightarrow$ (ii) it follows that

$$T = \{x \in S\colon\ g^n(c_0) \leqslant x < g^{n+1}(c_0) \text{ for some integer } n\}$$

is a convex subchain of S and hence also convex in C. Clearly $T = \bigcup_{n \in \omega^* + \omega} T_n$, where

$$T_n = \{x \in S\colon\ g^n(c_0) \leqslant x < g^{n+1}(c_0)\}$$

for each n. It is easily verified that if $n, m \in \omega^* + \omega$, $n \neq m$, then $T_n \cap T_m = \emptyset$ and also that if $x \in T_n$, $y \in T_m$, $m \neq n$, then $x < y$ if and only if $n < m$. It follows that $T = \bigoplus_{n \in \omega^* + \omega} T_n$.

Finally, it is obvious that $g|T_n$ is an isomorphism of T_n onto T_{n+1}.

(iii) $\rightarrow$ (i). By hypothesis there exists an isomorphism $f_n\colon T_n \rightarrow T_{n+1}$ for each $n \in \omega^* + \omega$. It is immediate from the definition of ordinal sum that $f\colon T \rightarrow T$ defined by $f(x) = f_n(x)$ if $x \in T_n$ is a strictly increasing automorphism. Finally, by Lemma 2, f can be extended to a proper automorphism on C, proving that C is not rigid.

Theorem 3 is very often useful in the investigation of whether or not a chain is rigid. Observe that it follows from part (ii) of this theorem that in order for a chain C to be rigid it is sufficient that every non-void convex subchain of C have a least or greatest element. Thus, ordinals and dual ordinals are rigid. However, this condition is not necessary. In fact, the next theorem exhibits a rigid chain which itself has no least or greatest element.

THEOREM 4. *Let $\{C_n\}_{n \in \omega}$ be a sequence of dual ordinals with $|C_n| < |C_{n+1}|$ for $n = 0, 1, 2, \dots$ and let $|C_0| \geqslant \aleph_0$. Then $C = \bigoplus_{n \in \omega} C_n$ is rigid.*

Proof. Suppose C is not rigid; then it follows from Theorem 3 that C has a non-void convex subchain T such that $T \to \bigoplus_{n\in\omega^*+\omega} T_n$, where $\{T_n\}_{n\in\omega^*+\omega}$ is a set of isomorphic subchains of C. If $T \subseteq \bigoplus_{n=0}^{k} C_n$ for some integer $k \geqslant 0$, then, by Theorem 3, $\bigoplus_{n=0}^{k} C_n$ would not be rigid, but this is impossible since $\bigoplus_{n=0}^{k} C_n$ is a dual ordinal. It follows that for each $n\in\omega$, there exists $n'\in\omega$, $n' > n$, such that $T \cap C_{n'} \neq \emptyset$. Let n_0 be the smallest $i\in\omega$ such that $T \cap C_i \neq \emptyset$ and let x be the largest element of C_{n_0}. Since $T \cap C_{n_0'} \neq \emptyset$ for some $n_0' > n_0$, we have by the convexity of T that $x\in T$. Hence there exists $n_1\in\omega^*+\omega$ such that $x\in T_{n_1}$. It follows that $T_{n_1-1} \subseteq \bigcup_{k=0}^{n_0} C_k$. But $T_{n_1-1} \cap C_k = \emptyset$ for $0 \leqslant k \leqslant n_0-1$, so $T_{n_1-1} \subseteq C_{n_0}$ and thus

$$|T_n| \leqslant |C_{n_0}| \quad \text{for each } n\in\omega^*+\omega \tag{5}$$

since all the T_n are isomorphic. Now we claim that $\bigcup_{n>n_0} C_n \subseteq T$. Indeed, let $z\in C_n$ for some $n > n_0$. Then there exists $n' > n$ such that $T \cap C_{n'} \neq \emptyset$, say $y\in T \cap C_{n'}$. But $x \leqslant z \leqslant y$ since $n_0 < n < n'$ and $x, y\in T$ so, by convexity, $z\in T$. Thus

$$|C_{n_0+1}| < |\bigcup_{n>n_0} C_n| \leqslant |T| = |\bigcup_{n\in\omega^*+\omega} T_n|,$$

But $|C_{n_0}| \geqslant \aleph_0$, so by (5)

$$|C_{n_0+1}| < |\bigcup_{n\in\omega^*+\omega} T_n| \leqslant \aleph_0 \cdot |C_{n_0}| \leqslant |C_{n_0}|, \text{ a contradiction.}$$

As for examples of non-rigid chains we have the integers, the rationals and the reals. For a less trivial example consider the Cantor discontinuum C. Let $S = C \sim \{0,1\}$ and suppose that each member of S is represented by a ternary expansion. For $n = 0, -1, -2, \ldots$ let $S_n = \{x\in S\colon\ 2\cdot 3^{n-2} \leqslant x < 2\cdot 3^{n-1}\}$ and for $n = 1, 2, 3, \ldots$ let $S_n = \{x\in S\colon\ 2\cdot 3^{-1} + \ldots + 2\cdot 3^{-n} \leqslant x < 2\cdot 3^{-1} + \ldots + 2\cdot 3^{-(n+1)}\}$. Then $S = \bigoplus_{n\in\omega^*+\omega} S_n$ and each S_n is isomorphic with $C \sim \{1\}$.

4. Characterization of the class $\mathscr{E}$. We will start this section by stating a slightly modified form of a theorem proved in [1].

LEMMA 5. *Let C be a chain with $0, 1$. Suppose $C \sim \{0,1\}$ contains a non-void convex subchain S such that S has a strictly increasing automorphism. Then for each Boolean algebra $B \neq 2$ there exists a chain C' such that $B * C = B * C'$ and $C \neq C'$.*

Recall the definition of the class $\mathscr{E}$. This is the class of all chains C with $0, 1$ such that $B * C = B * C'$ implies $C = C'$ for every Boolean algebra B and every chain C'. Our main result can be formulated as follows.

THEOREM 6. *A chain C with $0, 1$ belongs to $\mathcal{E}$ if and only if C is rigid.*

Proof. First suppose C is rigid, but that $C \notin \mathcal{E}$. Then there exist B and C' such that $B*C = B*C'$, $C \neq C'$. By Theorem 1, there exist isomorphisms $f\colon C \to C'$ and $g\colon C' \to C$ such that $f(x) \leqslant x$ for each $x \in C$ and $g(x') \leqslant x'$ for each $x' \in C'$. Since $C \neq C'$, there is an element $x_0 \in C$ such that $f(x_0) < x_0$. Let $h = g \circ f\colon C \to C$. Then $h(x_0) = g(f(x_0)) \leqslant f(x_0) < x_0$. Hence h is a proper automorphism of C and so C is not rigid, a contradiction.

Conversely, suppose $C \in \mathcal{E}$, but C is not rigid. Then by Theorem 3, C contains a non-void convex subchain S which has an increasing automorphism. It follows from Lemma 5 that $C \notin \mathcal{E}$, a contradiction.

To return to the question of uniqueness, recall that the representation $L = B*C$ is *unique*, if $L = B*C = B'*C'$ implies $B = B'$ and $C = C'$. The following corollary is now immediate from Theorems 1 and 6.

COROLLARY 7. *Let $L = B*C$. Then this representation is unique if and only if either $B = 2$ or C is rigid.*

Remark. It is clear that this corollary contains, as a special case, the uniqueness theorem for Post algebras mentioned in the introduction.

5. Generalizations. In this section we will give a sketch of a generalization of the results we have obtained, by replacing the Boolean algebra B in $B*C$ with an arbitrary object of $\mathcal{D}$. Since most of the proofs are analogous to those in the previous sections, we will omit details.

First an inspection of the proof of (3), as it was given in [1], shows that

(6) $L = L_1 * C$ if and only if each $x \in L$ can be uniquely represented in the form

$$x = \sum_{i=1}^{m} a_i c_i, \qquad m \geqslant 1,$$

where $\{c_1, \ldots, c_m\} \subseteq C$, $\{a_1, \ldots, a_m\} \subseteq L_1$, $0 = c_1 < c_2 < \ldots < c_m$ and $1 = a_1$, $a_2 > \ldots > a_m > 0$.

It is then possible to prove, using the same argument as in the proof of Theorem 1:

THEOREM 8. *Suppose $L = L_1 * C = L_1 * C'$. Then C and C' are isomorphic. Moreover, the isomorphism $f\colon C \to C'$ can be chosen so that $f(x) \geqslant x$ for all $x \in C$.*

Remark. It is not true that if $L_1 * C = L_1' * C'$ then necessarily $L_1 = L_1'$ (indeed $L' * L \doteq L * L'$) but in the case of chains we have the next best possible result.

THEOREM 9. *If $C_1 * C_2 = C_1' * C_2'$, then $C_1 = C_i'$ and $C_2 = C_j'$, where $\{i, j\} = \{1, 2\}$.*

Proof. We first show that if $x \in C_1 \cup C_2$, then x is join-irreducible (i.e. $x \leqslant a+b$, $a, b \in C_1 * C_2 \Rightarrow x \leqslant a$ or $x \leqslant b$; or equivalently $x = a+b \Rightarrow x = a$ or $x = b$). It suffices to show that if $x, x_i \in C_1$, $y_i \in C_2$, $1 \leqslant i \leqslant n$ and $x \leqslant \sum_{i=1}^{n} x_i y_i$, then $x \leqslant x_{i_0} y_{i_0}$ for some i_0. Indeed, since C_1 is a chain, this is true if $x = 0$ or $y_i = 1$ for all i. Thus suppose $x \neq 0$ and $S = \{i\colon y_i \neq 1\} \neq \emptyset$. But $x \leqslant \sum_{i=1}^{n} y_i$ and $x \neq 0$ implies $\sum_{i=1}^{n} y_i = 1$, so $y_i = 1$ for some i. Hence $\{1, 2, \ldots, n\} \sim S \neq \emptyset$ and we have $x \leqslant (\sum_{i \in S} y_i) + (\sum_{i \notin S} x_i)$. Now $x \neq 0$ and $\sum_{i \in S} y_i \neq 1$, so $x \leqslant \sum_{i \notin S} x_i$ and we have that for some $i_0 \in S$ $x \leqslant x_{i_0} = x_{i_0} y_{i_0}$ since $y_{i_0} = 1$.

Thus each member of $C_1 \cup C_2$ is join-irreducible; dually, these elements are meet-irreducible ($x \in C_1 * C_2$ is meet-irreducible if $x \geqslant ab \Rightarrow x \geqslant a$ or $x \geqslant b$ for $a, b \in C_1 * C_2$; or equivalently, $x = ab \Rightarrow x = a$ or $x = b$). But clearly any element of $C_1 * C_2$ which is both meet and join-irreducible is necessarily in $C_1 \cup C_2$, so that $C_1 \cup C_2$ is exactly the set of elements of $C_1 * C_2$ which are both meet and join-irreducible. In particular, $C_1 * C_2 = C_1' * C_2' \Rightarrow C_1 \cup C_2 = C_1' \cup C_2'$. Suppose $C_1 \nsubseteq C_1'$ and $C_1 \nsubseteq C_2'$, then there exists $x \in C_1 \sim C_1'$ and $y \in C_1 \sim C_2'$. Since $x, y \in C_1$, we may assume that $x \leqslant y$. Now $x \in C_2'$ since $x \notin C_1'$ and $y \in C_1'$. Hence $x = 0$ or $y = 1$, a contradiction. Thus $C_1 \subseteq C_1'$ or $C_1 \subseteq C_2'$. The result now follows easily.

We now introduce the class $\mathscr{E}^*$ of chains which are defined as follows. $\mathscr{E}^*$ is the class of all chains C such that $L * C = L * C'$ implies $C = C'$ for every L and C'. It is obvious that $\mathscr{E}^* \subseteq \mathscr{E}$. Conversely, suppose $C \in \mathscr{E}$ and $C \notin \mathscr{E}^*$. Then again applying Theorem 8 and using a similar argument as in the "if" part of the proof of Theorem 6 (replacing B with a suitable L) one can show that C is not rigid, whence by Theorem 6, $C \notin \mathscr{E}$, a contradiction. Therefore we have the following theorem:

THEOREM 10. *$\mathscr{E}^* = \mathscr{E}$ and thus $C \in \mathscr{E}^*$ if and only if C is rigid.*

Finally, we claim that Lemma 5 will still hold true if one replaces B by any object L of $\mathscr{D}$ whose center is not the two-element Boolean algebra (the reader may check this by a careful inspection of the proof in Theorem 4.2 in [1]).

The following corollary is then a generalization of Corollary 7.

COROLLARY 11. (i) *Suppose $L = L_1 * C = L_1 * C'$ and C is rigid. Then $C = C'$.*

(ii) *Suppose $L = L_1 * C$ and suppose the center of L_1 is not the two-element Boolean algebra. Then there exists a chain C' such that $L = L_1 * C = L_1 * C'$ and $C \neq C'$.*

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