COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 1

LIOUVILLE'S THEOREM FOR FIRST-ORDER PARTIAL DIFFERENTIAL OPERATORS

BY

L. R. HUNT*, J. J. MURRAY AND M. J. STRAUSS*
(LUBBOCK, TEXAS)

In the language of partial differential equations, Liouville's theorem states that all bounded differentiable functions $u: C \to C$ such that $L_1 u = 0$, where L_1 is the Cauchy-Riemann operator in the complex plane, are constants. The standard one-variable proof remains valid for bounded differentiable functions $u: C^n \to C$ solving the overdetermined system of differential equations $L_j u = 0$ $(1 \leqslant j \leqslant n)$, where each L_j is the Cauchy-Riemann operator in the j-th complex variable. Liouville's theorem can thus be considered as a statement about bounded classical solutions to systems of homogeneous first-order differential equations of a particular type.

Džuraev [4] examines single first-order linear differential equations $Lu = 0$, where $u: \boldsymbol{R}^3 \to \boldsymbol{C}$, and shows that under certain conditions any smooth u that is bounded on $\boldsymbol{R}^3$ and satisfies the homogeneous equation $Lu = 0$ is constant. He also shows that Hans Lewy's famous operator on $\boldsymbol{R}^3$ has an infinite number of linearly independent bounded solutions to the homogeneous equation.

Despite the fact that all of the operators in Džuraev's main theorem are solvable in the sense of Nirenberg and Treves [9] and despite his one counterexample (Lewy's), the dichotomy is not solvability versus non-solvability, as Džuraev also shows that adding any of an infinite number of operators of order zero (i.e. multiplications) of a certain sort to the non-solvable Lewy operator yields an operator satisfying the Liouville theorem; nor do all solvable operators satisfy Džuraev's hypotheses. There are many other non-solvable operators for which one could add appropriate zeroth-order terms and show Liouville's validity for the perturbed operator. Indeed, if one could find a zeroth-order term to add to the operator so that performing a coordinate change would convert the

* Supported in part by NSF Grant No. MCS76-05267.

equation into $\partial u/\partial \bar{z} - zu = 0$, where $\bar{z}$ denotes the complex conjugate of the complex variable z, in suitable new coordinates, or a system of such equations, then one finds that the only bounded solution $u(z, \bar{z}, t)$ of this equation in $\boldsymbol{R}^3$ is $u = 0$, verifying Liouville's theorem for the perturbed Lewy operator. In addition, in $\boldsymbol{R}^3 = \{(x_1, x_2, x_3)\}$ the solvable operator

$$M = \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3}$$

has many smooth bounded non-constant solutions to the homogeneous equation $Mu = 0$, particularly any bounded smooth function of x_1 alone, showing that Liouville's theorem does not hold for this solvable operator.

One must thus look for alternate means to investigate the general validity of Liouville's theorem for first-order linear differential operators. Since many [1] but by no means all [8] such operators can be expressed as the induced Cauchy-Riemann operators on some real k-dimensional manifold $X^k \subset \boldsymbol{C}^n$ for some $k, n \geqslant 2$, and since we have seen that the Liouville theorem is generally stated for holomorphic functions in $\boldsymbol{C}^n$, it is perhaps appropriate to examine the problem in this framework. Also, without loss of generality we will assume that $k = 2n-1$, so that we are dealing with a real hypersurface $X \subset \boldsymbol{C}^n$, since our proofs easily extend to lower dimensions.

Suppose that one considers a C^∞ real hypersurface X which separates $\boldsymbol{C}^n$ $(n > 1)$ into two parts X^+ and X^-. Then one can ask whether or not all bounded C^∞ CR-functions (functions satisfying the induced Cauchy-Riemann equations on X) must be constants. For X being a tube hypersurface in $\boldsymbol{C}^n$ over a two-sided hypersurface Γ in $\boldsymbol{R}^n$, Hill [6] and Carlson and Hill [3] have proved that X has the Liouville property for its CR-functions if and only if the convex hull of Γ in $\boldsymbol{R}^n$ is $\boldsymbol{R}^n$ itself (or, equivalently, the convex hull of X in $\boldsymbol{C}^n$ is $\boldsymbol{C}^n$ itself, or all CR-functions on X extend to holomorphic functions on $\boldsymbol{C}^n$).

We wish to study this problem for general real hypersurfaces X in $\boldsymbol{C}^n$. If all C^∞ CR-functions on X extend to holomorphic functions on $\boldsymbol{C}^n$, then we will show that X has the Liouville property for its CR-functions. However, if there exists a complex hyperplane in $\boldsymbol{C}^n$ which is bounded away from X, we will see that X does not have the Liouville property. From the last two statements one might conjecture that, in order for X to have the Liouville property, all CR-functions on X must extend to holomorphic functions on $\boldsymbol{C}^n$. However, using the Fatou [5] and Bieberbach [2] example, we will show that this is not the case. In fact, there exists a hypersurface Y in $\boldsymbol{C}^n$ such that all CR-functions on Y extend to holomorphic functions on an open subset V of $\boldsymbol{C}^n$ (and to no larger open set) with $C^n - \bar{V} \neq \emptyset$, and all bounded CR-functions on Y must be constants.

Let X be a C^∞ real hypersurface in C^n $(n > 1)$, and denote by $\bar{\partial}_X$ the Cauchy-Riemann operator induced on X by the complex structure of C^n. A C^∞-function for X is a *CR-function* on X if $\bar{\partial}_X f = 0$ on X.

Definition. The real hypersurface X in C^n $(n > 1)$ has the *Liouville property* if all bounded CR-functions on X are constants.

Our first result can be used to give many examples of real hypersurfaces in C^n which have the Liouville property.

PROPOSITION 1. *Suppose that X is a real hypersurface in C^n which has the property that all CR-functions on X extend to holomorphic functions on all of C^n. Then X has the Liouville property.*

Proof. Let f be a CR-function on X such that $|f(z)| \leqslant M < \infty$ for all points $z \in X$. Since f extends to a holomorphic function $\hat{f}$ on C^n with $\hat{f} | X = f$, we have $|\hat{f}(z)| \leqslant M$ for all $z \in C^n$ (the extension theory preserves bounds — see [3]). From the Liouville theorem on C^n we infer that $\hat{f}$, and hence f, is a constant.

Suppose that X is a real hypersurface in C^n, and that there exist a complex hyperplane (real codimension 2) A in C^n and an $\varepsilon > 0$ such that the Euclidean distance in C^n between X and A is greater than ε. If A is defined as the zero set of the linear holomorphic function $\varrho(z)$, then $1/\varrho(z)$ is a bounded CR-function on X which is not a constant, and X does not have the Liouville property. This leads us to consider necessary conditions for a real hypersurface X in C^n to have the Liouville property.

PROPOSITION 2. *If a real hypersurface X in C^n has the Liouville property, then there exists a dense subset of the set of complex hyperplanes in C^n which intersect X.*

Proof. Let A be a complex hyperplane in C^n. Given any $\varepsilon > 0$, there exists a point $p \in A$ such that the distance between p and the intersection of X with the 2-dimensional normal tangent space to A at p is less than $\varepsilon/2$. Otherwise, we would have a contradiction to the statement preceding this proposition. Now consider all translates of A in these normal directions and translated by a distance of less than ε. Certainly, at least one of these must intersect X.

An important case, in which X does not have the Liouville property, is when it can be foliated by a 1-parameter family of complex hyperplanes in C^n so that $X = \boldsymbol{R} \times C^{n-1}$. In this instance one has the Liouville property for each complex hyperplane, but not for all of X; for example, any bounded non-constant C^∞-function of the real parameters alone, constant in the complex variables, violates the Liouville property. Of course, such an X does not satisfy the necessary conditions of Proposition 2.

Now we come to the main result of this paper, a real hypersurface Y in C^n which has the Liouville property, but whose envelope of holomorphy as defined in [3] is not all of C^n. To construct such a Y we shall need the following result of Fatou [5] and Bieberbach [2] (stated for C^2, but easily extended to C^n).

THEOREM 1. *There exists a biholomorphic map*

$$\beta\colon C^n \to C^n$$

such that $C^n - \overline{\beta(C^n)} \neq \emptyset$.

Hence there exists an open subset of C^n *which is not in the image set* $\beta(C^n)$.

THEOREM 2. *There exists a hypersurface* Y *in* C^n *which has the Liouville property, but whose set of CR-functions extends to holomorphic functions on an open subset* V *of* C^n *(and to no larger open set) with* $C^n - \overline{V} \neq \emptyset$.

Proof. Let X be a real hypersurface in C^n such that all CR-functions on X extend to holomorphic functions on C^n (e.g., a tube hypersurface in C^n whose convex hull is all of C^n). Let $Y = \beta(X)$ and set $V = \beta(C^n)$. Now $Y \subset V$ and, since β is a biholomorphic map of C^n into C^n, V is a domain of holomorphy in C^n. Suppose that f is a CR-function on Y such that $|f(z)| \leqslant M < \infty$ for all $z \in Y$. Then $f \circ \beta$ is a CR-function on X such that $|f \circ \beta(z)| \leqslant M$ for all $z \in X$. By Proposition 1, $f \circ \beta$ is a constant, and hence f is a constant. Moreover, f extends to a holomorphic function on V. Since V is a domain of holomorphy, there exists some f which extends to no larger set than V (see [7]).

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TEXAS TECH UNIVERSITY
DEPARTMENT OF MATHEMATICS
LUBBOCK, TEXAS

Reçu par la Rédaction le 7. 5. 1977