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INDEPENDENT COMPLETE SUBALGEBRAS OF COLLAPSING ALGEBRAS

BY

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The present paper deals with complete Boolean product of complete Boolean algebras which is a natural extrapolation of the $(\mathfrak{m}, 0)$-product considered in [3]. We show that, for each cardinal a greater than the cardinality of the continuum, the collapse algebra $\mathrm{Col}\, a$ is a complete product of two Cohen algebras or two random algebras. Thus, for Cohen algebras or random algebras, respectively, complete Boolean products form a proper class and the greatest complete product does not exist.

Another result answers a question which was arisen by the previous problem. The theorem asserts that in the collapse algebra there is an infinite decomposition which is independent of a countable independent set of complete generators.

The complete Boolean product has been studied also in [1].

Before stating our results more precisely, let us introduce some notions and notation. In general, we shall follow the terminology used in [3] (however, the notation used here is not always strictly the same as in [3], e.g. we shall use $\wedge, \vee, 0, 1$ for Boolean operations and bound elements; for logical conjunction and disjunction we use $\wedge$ and $\vee$).

An ordinal ξ will be considered as the set of all ordinals less than ξ, a cardinal will be an initial ordinal. Thus, a natural number n is the set $\{0, 1, \ldots, n-1\}$. For $\varepsilon \in 2$ we get $\varepsilon = 0$ or $\varepsilon = 1$, and so for an element A in a Boolean algebra we have $(-1)^\varepsilon \cdot A = 1 \cdot A = A$ or $(-1)^\varepsilon \cdot A = (-1) \cdot A = -A$, respectively. If f is a function from A to B, we write $D(f) = A$ and $W(f) = B$; the set of all such functions is denoted by ${}^A B$. Following this definition, ${}^\omega 2$ is the set of all infinite sequences of $0, 1$. The set of all finite sequences of $0, 1$ will be denoted by ${}^{<\omega}2$.

The *collapse algebra* $\mathrm{Col}\, a$ is the complete Boolean algebra $\mathrm{RO}({}^\omega a)$ of all regular open subsets of the topological space ${}^\omega a$ which is the usual product of ω copies of a with the discrete topology. (Thus, the base of the topology in ${}^\omega a$ is formed by the sets $u_\varphi = \{f \in {}^\omega a; f \supseteq \varphi\}$ for all functions φ such that $D(\varphi)$ is a finite subset of ω and $W(\varphi) \subseteq a$.)

The *Cohen algebra* is the complete Boolean algebra of all Borel sets of reals modulo the ideal of meager sets. The *random algebra* is the complete Boolean algebra of all Borel sets of reals modulo the ideal of sets of measure zero.

Definition. A Boolean algebra $\mathscr{C}$ is a *complete Boolean product* of Boolean algebras $\mathscr{A}$, $\mathscr{B}$ if

(i) $\mathscr{C}$ is a complete Boolean algebra,

(ii) $\mathscr{A}$, $\mathscr{B}$ are regular subalgebras of $\mathscr{C}$,

(iii) $\mathscr{A}$, $\mathscr{B}$ are independent in $\mathscr{C}$,

(iv) $\mathscr{A} \cup \mathscr{B}$ completely generates $\mathscr{C}$.

The main results of the paper can now be formulated as follows:

THEOREM A (Boolean two-Cohen theorem). *Let a be a cardinal, $a \geqslant 2^{\omega}$. Then, in the collapse algebra $\mathscr{C} = \mathrm{Col}\, a$, there are regular subalgebras $\mathscr{A}$, $\mathscr{B}$ isomorphic to the Cohen algebra and such that $\mathscr{C}$ is a complete Boolean product of $\mathscr{A}$, $\mathscr{B}$.*

THEOREM B (Boolean two-random theorem). *Let a be a cardinal, $a \geqslant 2^{\omega}$. Then, in the collapse algebra $\mathscr{C} = \mathrm{Col}\, a$, there are regular subalgebras $\mathscr{A}$, $\mathscr{B}$ isomorphic to the random algebra and such that $\mathscr{C}$ is a complete Boolean product of $\mathscr{A}$, $\mathscr{B}$.*

The proofs of the theorems are given in Sections 2 and 3, they use the Boolean-valued models of set theory. In Section 1, a theorem on independent generators and decompositions (Theorem 1.4) and its application to the collapse algebra (Theorem 1.5) are proved.

Remark. The unpublished result called *two-Cohen theorem* was proved by R. Solovay:

Assume that ω_1^L is countable and x is a real. Then there are Cohen reals a, b such that $x = a+b$, i.e. $x \in L(a, b)$.

The analogous result is known for random reals.

The Boolean versions presented in this paper give information on subalgebras generated by reals a, b.

1. Independent generators. The main results of this section are Theorems 1.4 and 1.5. We start with reminding some definitions.

1.0. Definition. A subset D of a Boolean algebra is called a *decomposition* (of unit element) if

(i) $0 \neq A$ for any $A \in D$,

(ii) $A \wedge B = 0$ for any $A \neq B$, $A, B \in D$,

(iii) $\bigvee D = 1$.

1.1. Definition. A family $(D_l; l \in T)$ of decompositions in a Boolean algebra is called *independent* if $\bigwedge \{A_k; k \in n\} \neq 0$ holds true for any finite subset $\{l_0, l_1, \ldots, l_{n-1}\} \subseteq T$ and for any elements $A_k \in D_{l_k}$, $k \in n$.

1.2. Definition. We say that a subset S of a Boolean algebra is *independent* if the set of decompositions $\{\{A, -A\};\ A \in S\}$ is independent.

To get an equivalent formulation, let us introduce the notation

$$f_\varepsilon = \bigwedge \{(-1)^{\varepsilon(k)} f(k);\ k \in n\} \quad \text{for any } n \in \omega,\ \varepsilon \in {}^n2,\ f \in {}^nS.$$

Such an f_ε will be called a *constituent* over S if f is injective. Now, S is *independent* if any constituent over S is non-zero.

1.3. Definition. We say that a decomposition D is *independent* of a subset S of a Boolean algebra if, for any constituent f_ε over S and for any $A \in D$, $f_\varepsilon \neq 0$ implies $A \wedge f_\varepsilon \neq 0$.

Remark. The implication in the defining condition is necessary, as S itself need not be independent. For an independent set S it is equivalent to demand the set of decompositions $\{D, \{A, -A\};\ A \in S\}$ to be independent.

1.4. THEOREM. *Let $\mathscr{B}$ be a non-atomic Boolean algebra with countably many generators and let $\mathscr{Y}$ be a countable set generating $\mathscr{B}$. Then there is a countable independent set $\mathscr{X}$, generating $\mathscr{B}$, such that any constituent over $\mathscr{X}$ majorizes some non-zero constituent over $\mathscr{Y}$.*

COROLLARY. *Let $\mathfrak{m}$ be an infinite cardinal, $\mathscr{B}$ a non-atomic algebra, and $\mathscr{Y}$ a countable set $\mathfrak{m}$-generating (completely generating) $\mathscr{B}$. Then there is a countable independent set $\mathscr{X}$, $\mathfrak{m}$-generating (completely generating) $\mathscr{B}$, such that any decomposition independent of $\mathscr{Y}$ is also independent of $\mathscr{X}$.*

Proof. Let the assumptions of the corollary be fulfilled. We denote by $\mathscr{B}_0$ the subalgebra of $\mathscr{B}$ generated by $\mathscr{Y}$. The algebra $\mathscr{B}_0$ is non-atomic, for if $A \in \mathscr{B}_0$ were an atom in $\mathscr{B}_0$, then, for any $Y \in \mathscr{Y}$, $A \leqslant Y$ or $A \leqslant -Y$ would be true, which implies that A would be an atom in $\mathscr{B}$ as well. Thus, using the theorem we get an independent set $\mathscr{X}$, generating $\mathscr{B}_0$, which $\mathfrak{m}$-generates (completely generates) the algebra $\mathscr{B}$.

The condition concerning constituents enables us to prove independence of decompositions. Namely, let D be a decomposition independent of $\mathscr{Y}$, let $A \in D$ and let f_ε be a constituent over $\mathscr{X}$. Then there is a constituent g_φ over $\mathscr{Y}$ such that $0 \neq g_\varphi \leqslant f_\varepsilon$, and therefore also $0 \neq g_\varphi \wedge A \leqslant f_\varepsilon \wedge A$ holds true.

Proof of Theorem 1.4. Let $\mathscr{Y} = \{Y_n;\ n \in \omega\}$ be a set of generators in a non-atomic Boolean algebra $\mathscr{B}$. For $n \in \omega$ we put

$$y_n = \{\bigwedge \{(-1)^{\varepsilon(k)} Y_k;\ k \in n\};\ \varepsilon \in {}^n2\} - \{0\},$$

$$y = \bigcup \{y_n;\ n \in \omega\}.$$

Any y_n is a decomposition in $\mathscr{B}$. As $\mathscr{B}$ is non-atomic, we get

$$(*) \qquad (\forall Y \in y_n)(\exists m \geqslant n)[(Y \wedge Y_m) \neq 0 \wedge (Y \wedge -Y_m) \neq 0].$$

(Otherwise, Y would be an atom in $\mathscr{B}$.)

The set of independent generators $\mathscr{X} = \{X_n;\ n \in \omega\}$ will be defined by induction on n.

Induction assumption. To define generators $\{X_k;\ k \in n\}$ we put

$$x_n = \big\{\bigwedge\{(-1)^{\varepsilon(k)}X_k;\ k \in n\};\ \varepsilon \in {}^n2\big\}.$$

Then we have

(i) x_n is a decomposition in $\mathscr{B}$, i.e. $X \neq 0$ for any $X \in x_n$,

(ii) $x_n \subseteq y$,

(iii) x_n is a refinement of y_n, i.e.

$$(\forall X \in x_n)(\forall k \in n)[(X \wedge Y_k) = 0 \vee (X \wedge -Y_k) = 0].$$

For any $X \in x_n$, let us denote by $m(X)$ the least natural number m the existence of which follows from (*). The generator X_n is defined as follows:

$$X_n = \bigvee\{X \wedge Y_{m(X)};\ X \in x_n\}.$$

It is easy to verify that

$$-X_n = \bigvee\{X \wedge -Y_{m(X)};\ X \in x_n\}.$$

Evidently, for any $X \in x_n$ we obtain

$$X \wedge X_n = X \wedge Y_{m(X)} \quad \text{and} \quad X \wedge -X_n = X \wedge -Y_{m(X)}.$$

Thus, we have

$$x_{n+1} = \{X \wedge Y_{m(X)}, X \wedge -Y_{m(X)};\ X \in x_n\},$$

which implies that the generators $X_0, X_1, \ldots, X_n$ fulfill (i)-(iii) of the induction assumption ((ii) and (iii) follow from the minimality of $m(X)$). The definition of $\mathscr{X}$ by induction is therefore complete.

From (i) it follows that $\mathscr{X}$ is an independent set in $\mathscr{B}$, (ii) gives the condition on majorizing constituents, and (iii) implies that $\mathscr{X}$ generates a subalgebra containing all Y_n $(n \in \omega)$, thus $\mathscr{X}$ generates $\mathscr{B}$.

Remark. Without the condition on majorizing constituents, the theorem can be proved in a much simpler way. We can use the fact that the only (up to isomorphism) non-atomic Boolean algebra with countably many generators contains an independent set of generators.

1.5. THEOREM. *Let $\alpha > \beta$ be cardinals, α being infinite. Then in the collapse algebra $\mathscr{C} = \operatorname{Col}\alpha$ there are a countable independent set $\mathscr{X}$ of complete generators and a decomposition $\mathscr{D}$, of cardinality β, independent of $\mathscr{X}$.*

Proof. It is proved in [4] that the algebra $\mathscr{C}$ is completely generated by a countable set of generators $\mathscr{Y} = \{Y_{nm};\ n, m \in \omega\}$, where $Y_{nm} = \{f \in {}^\omega\alpha;\ f(n) \leqslant f(m)\}$. It is easy to show that the set $\mathscr{D} = \{D_\xi;\ \xi \in \beta\}$, where $D_\xi = \{f \in {}^\omega\alpha;\ f(0) \equiv \xi \pmod{\beta}\}$, is a decomposition in $\mathscr{C}$ independent of $\mathscr{Y}$. The rest follows from Theorem 1.4 and its corollary.

2. Reducibility to Cohen algebras. In this section we give a proof of Theorem A. First we present some notation and facts we shall need.

2.0. The functions from ${}^{\omega}2$ will be considered as real numbers. The base for a topology in ${}^{\omega}2$ will be formed by the sets $u_\varphi = \{f \in {}^{\omega}2; f \supseteq \varphi\}$ for all functions φ such that $D(\varphi)$ is a finite subset of ω and $W(\varphi) \subseteq 2$. The topological space defined in this way will be denoted by R.

We define a binary operation $+$ on R as follows: for $a, b, c \in {}^{\omega}2$ we set $a+b = c$ if and only if $a(n)+b(n) = c(n)(\mathrm{mod}\,2)$ for any $n \in \omega$. The operation $+$ is continuous in both variables.

Further, we define a σ-additive measure μ on R by setting $\mu(u_\varphi) = 2^{-n}$ if $\mathrm{card}\,D(\varphi) = n$. Thus we get $\mu({}^{\omega}2) = \mu(u_0) = 1$. Every Borel set in R is μ-measurable.

2.1. Steinhaus proved in [6] the following theorem:

Let A and B be subsets of R, $\mu(A) > 0$ and $\mu(B) > 0$. Then the set $H = \{a+b;\ a \in A \wedge b \in B\}$ contains an interval u_ϑ, $\vartheta \in {}^{<\omega}2$.

We shall use the following modification of the Steinhaus theorem:

2.2. THEOREM. *Let A and B be G_δ-sets in R, dense in intervals u_φ and u_ψ, $\varphi, \psi \in {}^{<\omega}2$. Then the set $H = \{a+b;\ a \in A \wedge b \in B\}$ contains an interval u_ϑ, $\vartheta \in {}^{<\omega}2$.*

Proof. Let C be a dense subset of R. Then there is a $c \in C$ such that u_φ and $u_\psi + c = \{f+c;\ f \in u_\psi\}$ have a non-empty intersection. That intersection is an interval u_χ, $\chi \in {}^{<\omega}2$. The sets $A \cap u_\chi$ and $(B+c) \cap u_\chi$ are G_δ and dense in u_χ. The Baire theorem implies that their intersection is G_δ and dense in u_χ as well. Therefore, there exists an $a \in (A \cap u_\chi) \cap \big((B+c) \cap u_\chi\big)$. Here we have $a = b+c$ and $a+b = c$ for some $b \in B$. It means that $H \cap C \neq \emptyset$ for any set C, dense in R. Therefore, the complement of H cannot be dense in R, and so H contains an interval u_ϑ, $\vartheta \in {}^{<\omega}2$.

2.3. In the proofs of Theorems A and B we shall use the method of Boolean-valued models, which is presented in details, e.g., in [2]. The terminology and notation introduced there will be used in Sections 2 and 3.

Thus, V denotes the universal class of all sets and for any complete Boolean algebra $\mathscr{C}$ we have a Boolean-valued model $V^{\mathscr{C}}$. If $a \in V^{\mathscr{C}}$ is a real number in the sense of the model, then a determines in $\mathscr{C}$ the elements $A_n = \|a(n) = 0\|$, $n \in \omega$, and a complete subalgebra $\mathscr{A}$, completely generated by $\{A_n;\ n \in \omega\}$.

Solovay proved in [5] that $\mathscr{A}$ is isomorphic to the Cohen algebra if and only if, in the sense of the model $V^{\mathscr{C}}$, a belongs to any dense G_δ-subset of R which belongs to V (then a is called the *Cohen number* over V). An analogous result is valid for the random algebra, subsets of measure 1 and random number over V.

In case of the collapse algebra $\mathscr{C} = \mathrm{Col}\, a$, there is a function $f \in V^{\mathscr{C}}$, collapsing a to ω, i.e. such that, in the sense of $V^{\mathscr{C}}$, f is a surjection of ω onto a.

2.4. Proof of Theorem A. Let us fix a sequence $(h_\xi;\ \xi \in a)$ containing all open dense subsets of R. The existence of such a sequence follows from the assumption $a \geqslant 2^\omega$ (the sequence need not be injective). The collapsing function in $V^{\mathscr{C}}$ will be denoted by f. Let $\mathscr{X} = \{X_n;\ n \in \omega\}$ be an independent set of complete generators in $\mathscr{C}$, let $\mathscr{D} = \{D_n; n \in \omega\}$ be a decomposition in $\mathscr{C}$ independent of $\mathscr{X}$. We denote by x a real number in $V^{\mathscr{C}}$ such that $\|x(n) = i\| = (-1)^i X_n = X_n^i$ for $i \in 2,\ n \in \omega$.

CLAIM. *For any $\varphi \in {}^{<\omega}2$ there exists a real number $a_\varphi \in V^{\mathscr{C}}$, Cohen over V, such that $a_\varphi \supseteq \varphi$ and the number $b_\varphi = a_\varphi + x$ is Cohen over V.*

We prove the Claim in the model $V^{\mathscr{C}}$. The intersection of open dense sets $h_{f(n)}$, $n \in \omega$, is a dense G_δ-subset in R. Denoting by $A = B$ its intersection with $u_\varphi = u_\varphi$ and using Theorem 2.2, we get an interval u_ϑ, $\vartheta \in {}^{<\omega}2$, such that $u_\vartheta \subseteq \{a+b;\ a \in A \wedge b \in B\}$. There exists a rational number r such that $x+r \in u_\vartheta$, so we have $a \in A$ and $b \in B$ such that $x+r = a+b$, $x = a+(b+r)$. Evidently, $a = a_\varphi$ and $b+r = b_\varphi$ are Cohen over V. The condition $a_\varphi \supseteq \varphi$ is also fulfilled. The Claim is proved.

Now, let $\sigma = (\sigma(n);\ n \in \omega)$ be a sequence of all finite sequences belonging to ${}^{<\omega}2$. Using the notation introduced in the Claim, we define the real numbers $a, b \in V^{\mathscr{C}}$ as follows:

$$\|a = a_{\sigma(n)}\| = D_n, \qquad \|b = b_{\sigma(n)}\| = D_n.$$

For $i \in 2$ and $n \in \omega$ we set

$$A_n^i = \|a(n) = i\|, \qquad B_n^i = \|b(n) = i\|.$$

The subalgebras completely generated by $\{A_n^0;\ n \in \omega\}$ and $\{B_n^0;\ n \in \omega\}$ or, which is the same, by $\{A_n^1;\ n \in \omega\}$ and $\{B_n^1;\ n \in \omega\}$ will be denoted by $\mathscr{A}$ and $\mathscr{B}$, respectively.

We complete the proof of the theorem in the following steps:

(i) For any $i \in 2$ and $n \in \omega$,

$$A_n^i = \bigvee \{D_p \wedge \|a_{\sigma(p)}(n) = i\|;\ p \in \omega\},$$

$$B_n^i = \bigvee \{D_p \wedge \|b_{\sigma(p)}(n) = i\|;\ p \in \omega\}.$$

(ii) The equalities

$$A_n^i = -A_n^{i+1}, \qquad B_n^i = -B_n^{i+1}, \qquad X_n^i = -X_n^{i+1},$$

$$X_n^{i+j} = (A_n^i \wedge B_n^j) \vee (A_n^{i+1} \wedge B_n^{j+1}), \qquad X_n^i = (A_n^j \wedge B_n^{i+j}) \vee (A_n^{j+1} \wedge B_n^{i+j+1})$$

hold for any $i, j \in 2$, $n \in \omega$ and for any permutation of A, B and X (the addition of i and j is made modulo 2).

(iii) $\{A_n^i; n \in \omega\} \cup \{X_n^i; n \in \omega\}$ for $i \in 2$ is an independent subset of $\mathscr{C}$.

(iv) $\{A_n^i; n \in \omega\} \cup \{B_n^i; n \in \omega\}$ for $i \in 2$ is an independent subset of $\mathscr{C}$.

(v) $\mathscr{A}$ and $\mathscr{B}$ are isomorphic to the Cohen algebra.

(vi) $\mathscr{A} \cup \mathscr{B}$ completely generates $\mathscr{C}$.

(vii) $\mathscr{A}$ and $\mathscr{B}$ are independent subalgebras of $\mathscr{C}$.

Assertion (i) follows directly from the definitions of a, b, A_n^i, B_n^i.

Equalities (ii) follow from (i) and from the fact that $a_\varphi(n) + b_\varphi(n) = x(n)$ (mod 2), and thus

$$\big(\|a_\varphi(n) = i\| \wedge \|b_\varphi(n) = j\|\big) \vee \big(\|a_\varphi(n) = i+1\| \wedge \|b_\varphi(n) = j+1\|\big) = \|x(n) = i+j\|$$

for any $i, j \in 2$, $n \in \omega$, $\varphi \in {}^{<\omega}2$ and for any permutation of a_φ, b_φ, x.

To prove (iii), it suffices to show that, for any $n \in \omega$, $\varphi, \varepsilon \in {}^n 2$,

$$\bigwedge \{A_k^{\varphi(k)} \wedge X_k^{\varepsilon(k)}; k \in n\} \neq 0.$$

We have

$$\bigwedge\big\{\bigvee \{D_p \wedge \|a_{\sigma(p)}(k) = \varphi(k)\|; p \in \omega\} \wedge X_k^{\varepsilon(k)}; k \in n\big\} \geqslant \bigwedge \{D_{\bar p} \wedge \|a_\varphi(k) = \varphi(k)\| \wedge X_k^{\varepsilon(k)}; k \in n\},$$

where $\bar p \in \omega$ is such that $\sigma(\bar p) = \varphi$. Then $\|a_\varphi(k) = \varphi(k)\| = 1$ for any $k \in n$ and, since $\mathscr{D}$ is independent of $\mathscr{X}$, we get

$$\bigwedge \{D_{\bar p} \wedge X_k^{\varepsilon(k)}; k \in n\} = D_{\bar p} \wedge \bigwedge \{X_k^{\varepsilon(k)}; k \in n\} \neq 0.$$

For the proof of (iv), let us choose $n \in \omega$, $\varphi, \psi \in {}^n 2$. Using (ii) and (iii) by direct computation we get

$$\bigwedge \{A_k^{\varphi(k)} \wedge B_k^{\psi(k)}; k \in n\} = \bigwedge \{A_k^{\varphi(k)} \wedge X_k^{\varepsilon(k)}; k \in n\} \neq 0$$

if we set $\varepsilon = \varphi + \psi$, $\varepsilon \in {}^n 2$.

Assertion (v) follows from Solovay's result mentioned in 2.3 and from the fact that, in $V^{\mathscr{C}}$, a and b are Cohen over V.

(vi) is an immediate consequence of (ii) and of the fact that $\mathscr{X}$ completely generates $\mathscr{C}$.

Finally, to prove (vii) we use (iv) and observe that, the algebras $\mathscr{A}$ and $\mathscr{B}$ being isomorphic to the Cohen algebra, they contain dense subsets of elements of the form $\bigwedge \{A_k^{\varphi(k)}; k \in n\}$ and $\bigwedge \{B_k^{\psi(k)}; k \in n\}$, respectively, for $\varphi, \psi \in {}^n 2$, $n \in \omega$. Thus Theorem A is proved.

3. Reducibility to random algebras. This section is devoted to the proof of Theorem B. As a consequence of the duality between measure and category, the proofs of Theorems A and B are rather analogous. However, a new approach is used for the independence of the subalgebras.

3.0. Let $\mathscr{X} = (X_n; n \in \omega)$ be a family of complete generators in a complete Boolean algebra $\mathscr{C}$ and let $x \in V^{\mathscr{C}}$ be a real number in the sense of the model $V^{\mathscr{C}}$, such that $\|x(n) = i\| = (-1)^i X_n$ for any $i \in 2$, $n \in \omega$.

Then, for any rational $r \in V^{\mathscr{C}}$, the number $x' = x+r$ defines a family $\mathscr{X}' = (X'_n; n \in \omega)$ (by setting $\|x'(n) = 0\| = X'_n$) completely generating $\mathscr{C}$. It is a consequence of the equality $V(x) = V(x')$.

3.1. Proof of Theorem B. Similarly as in 2.4, let us fix a sequence $(h_\xi; \xi \in a)$ containing all subsets of R which are of measure 1. Further we take a sequence $\mathscr{X} = (X_n; n \in \omega)$ independently generating $\mathscr{C}$ and a decomposition $\mathscr{D} = (D_\xi; \xi \in a)$ in $\mathscr{C}$ independent of $\mathscr{X}$. Assume that elements $f, x \in V^{\mathscr{C}}$ have the same meaning as in 2.4.

CLAIM. *For any* $A, B \subseteq R$, $\mu(A), \mu(B) > 0$ *there are real numbers* $a_{AB}, b_{AB} \in V^{\mathscr{C}}$, *random over* V, *and a rational number* $r_{AB} \in V^{\mathscr{C}}$ *such that* $a_{AB} \in A$, $b_{AB} \in B$ *and* $a_{AB}+b_{AB} = x+r_{AB}$.

We prove the Claim in $V^{\mathscr{C}}$. The intersection of all subsets $h_{f(n)}$, $n \in \omega$, of measure 1 in R is a subset of measure 1. Its intersection with sets A and B will be denoted by $\bar{A}$ and $\bar{B}$, respectively. Using the Steinhaus theorem, we get an interval u_ϑ, $\vartheta \in {}^{<\omega}2$, such that $u_\vartheta \subseteq \{a+b; a \in \bar{A} \wedge b \in \bar{B}\}$. There exists a rational number r_{AB} such that $x+r_{AB} \in u_\vartheta$, so we have $a_{AB} \in \bar{A} \subseteq A$, $b_{AB} \in \bar{B} \subseteq B$, $a_{AB}+b_{AB} = x+r_{AB}$. It follows from 2.3 that a_{AB} and b_{AB} are random over V, which completes the proof of the Claim.

Now, let $\tau = (\tau(\xi); \xi \in a)$ be a sequence containing all pairs $\tau(\xi) = (A, B)$ such that $A, B \subseteq R$, $\mu(A) > 0$, $\mu(B) > 0$. The existence of such a sequence follows again from the assumption $a \geqslant 2^\omega$. The real numbers $a, b \in V^{\mathscr{C}}$ are defined as follows: $\|a = a_{\tau(\xi)}\| = D_\xi$, $\|b = b_{\tau(\xi)}\| = D_\xi$ for any $\xi \in a$. For $i \in 2$, $n \in \omega$ we set

$$A_n^i = \|a(n) = i\|, \quad B_n^i = \|b(n) = i\|.$$

Since the reals a and b are, in $V^{\mathscr{C}}$, random over V, we infer, using again Solovay's result mentioned in 2.3, that the subalgebras $\mathscr{A}$ and $\mathscr{B}$, completely generated in $\mathscr{C}$ by $\{A_n^0; n \in \omega\}$ and $\{B_n^0; n \in \omega\}$, respectively, are isomorphic to the random algebra.

If we define $r \in V^{\mathscr{C}}$ by setting $\|r = r_{\tau(\xi)}\| = D_\xi$ for any $\xi \in a$, we get in $V^{\mathscr{C}}$ the equality $a+b = x+r$. Here r is rational in $V^{\mathscr{C}}$, and by 3.0 we infer that the subalgebras $\mathscr{A}$ and $\mathscr{B}$ completely generate $\mathscr{C}$.

We complete the proof of Theorem B by showing the independence of $\mathscr{A}$ and $\mathscr{C}$. Let $0 \neq \bar{A} \in \mathscr{A}$ and $0 \neq \bar{B} \in \mathscr{B}$. Since $\mathscr{A}$ and $\mathscr{B}$ are isomorphic to the random algebra, i.e. to the algebra of all Borel subsets in R modulo the sets of measure 0, there exist subsets A and B of R such that $\mu(A) > 0$, $\mu(B) > 0$ and such that $\bar{A} = \|a \in A\|$, $\bar{B} = \|b \in B\|$. We have

$$\|a \in A\| \geqslant \|a = a_{AB}\| = D_\xi \quad \text{and} \quad \|b \in B\| \geqslant \|b = b_{AB}\| = D_\xi$$

for ξ such that $\tau(\xi) = (A, B)$. Consequently, we have $\bar{A} \wedge \bar{B} \geqslant D_\xi \neq 0$. Thus Theorem B is proved.

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