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REPRESENTATIONS OF ORDERED SEMIGROUPS AND LATTICES BY BINARY RELATIONS

BY

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Semigroups of binary relations (i.e., sets of binary relations closed under the operation of multiplication of relations) form an important class of semigroups containing such subclasses as the class of semigroups of full or partial transformations. It is well known that every semigroup is isomorphic to a semigroup of binary relations. Every semigroup of binary relations is naturally ordered by the set-theoretical inclusion of relations and this ordering is *stable*, i.e., compatible with the operation of multiplication. In 1959, Zareckiĭ [13] proved that every ordered semigroup (by the definition, the order relation of an ordered semigroup is stable) is isomorphic to an inclusion-ordered semigroup of binary relations. Since then the following problem has remained open: does there exist an isomorphism of an ordered semigroup onto an inclusion-ordered semigroup of binary relations which preserves the greatest lower bounds existing in the given ordered semigroup. The main result of this paper gives an affirmative answer to this problem. Various corollaries follow from this result. Suppose that a set G is a semigroup with respect to a multiplication $\cdot$ and a semilattice (i.e., an idempotent and commutative semigroup) with respect to an operation $\wedge$, and the order relation of the semilattice $(G; \wedge)$ is stable with respect to the multiplication in $(G; \cdot)$ (in other words, the semigroup $(G; \cdot)$ is ordered in such a way that any pair of elements g, h of G has the g.l.b. $g \wedge h$. Then the algebra $(G; \cdot, \wedge)$ is isomorphic to an algebra of binary relations of the form $(\Phi; \circ, \cap)$, where Φ is a set of binary relations closed under the operations $\circ$ of multiplication and $\cap$ of intersection of relations. Moreover, all the relations in Φ can be chosen so that they satisfy certain special conditions.

Another important corollary is a representation theorem for lattices. If Φ is a set of quasi-order relations on a fixed set, then, for every $\varphi, \psi \in \Phi$, the intersection $\varphi \cap \psi$ is a quasi-order relation while the product $\psi \circ \varphi$ need not be a quasi-order relation. However, if we suppose that both $\varphi \cap \psi$ and $\psi \circ \varphi$ belong to Φ for any $\varphi, \psi \in \Phi$, then one can easily verify that the algebra $(\Phi; \circ, \cap)$ is a lattice in which the product $\psi \circ \varphi$ is the join $\varphi \vee \psi$. If Φ consists of equivalence relations, we obtain a subclass of the class of all modular lattices. This subclass was characterized by Jónsson [2]. We prove that every abstract lattice can isomorphically be represented as a lattice of the form $(\Phi; \circ, \cap)$, where all the elements of Φ may be chosen to be (partial) order relations (moreover, one can choose the elements of Φ to be strict, i.e., irreflexive, and dense order relations). Thus every lattice can be represented by simple set-theoretical objects. Till now there have been known "good" representation theorems for special classes of lattices only (e.g., the theorems on representation of distributive lattices as rings of sets and of Boolean algebras as fields of sets). The only general result on representation of arbitrary lattices is the theorem of Whitman [12] stating that every lattice is isomorphically embeddable in a partition lattice on an infinite set. However, the join operation in partition lattices is of a rather complicated nature and cannot be expressed by means of a first-order predicate calculus formula involving the joined partitions. Therefore, the Whitman representation is not a representation of lattices as relation algebras in the sense of [8]. The representation of arbitrary lattices given in this paper is a representation by relation algebras. This representation theorem has been published in a separate note [9] (the present paper was written in the summer of 1971 but could not be published for the reasons not under the authors' control).

Now we introduce several definitions. A *binary relation* on a set A is any subset ϱ of the Cartesian square $A \times A$. The *converse* (for ϱ) *binary relation* is denoted by $\overset{-1}{\varrho}$ (i.e., $(a_1, a_2) \in \overset{-1}{\varrho}$ iff $(a_2, a_1) \in \varrho$); $\sigma \circ \varrho$ denotes the *product* of the relations ϱ and σ (i.e., $(a_1, a_2) \in \sigma \circ \varrho$ iff $(a_1, a) \in \varrho$ and $(a, a_2) \in \sigma$ for some $a \in A$). A relation ϱ is called *asymmetric* if $\varrho \cap \overset{-1}{\varrho} = \emptyset$ and *antisymmetric* if $\varrho \cap \overset{-1}{\varrho} \subset \Delta_A$, where Δ_A denotes the *identity binary relation* on A, i.e., $\Delta_A = \{(a, a) : a \in A\}$. A relation ϱ is called *reflexive* if $\Delta_A \subset \varrho$, and *transitive* if $\varrho \circ \varrho \subset \varrho$. A reflexive and transitive binary relation is called a *quasi-order relation*, and an antisymmetric quasi-order relation is called an *order relation*. A relation ϱ is called *irreflexive* if $\varrho \cap \Delta_A = \emptyset$. An irreflexive and transitive binary relation is called a *strict order relation*. It is obvious that for transitive relations irreflexivity is equivalent to asymmetricity. If ϱ is a strict order relation, then $\varrho \cup \Delta_A$ is an order relation. Conversely, if σ is an order relation, then $\sigma \setminus \Delta_A$ is a strict order relation. A relation ϱ is called *full* if for every $a \in A$ there exists $a_1 \in A$ such that $(a, a_1) \in \varrho$. If both ϱ and $\overset{-1}{\varrho}$ are full, then ϱ is called *effective*. A strict order relation ϱ is called *dense* if, for every $a_1, a_2 \in A$ for which

$(a_1, a_2) \in \varrho$, there exists $a \in A$ such that $(a_1, a) \in \varrho$ and $(a, a_2) \in \varrho$ (i.e., if $\varrho \subset \varrho \circ \varrho$). An order relation ϱ is called *dense* if the corresponding strict order relation $\varrho \setminus \Delta_A$ is dense.

An isomorphism f of an ordered semigroup $(G; \cdot, \leqslant)$ into an ordered semigroup $(H; \cdot, \leqslant)$ (by the definition, an isomorphism of ordered semigroups preserves both the operation of multiplication and the order relation) is called *preserving intersections* if, for every non-empty subset $F \subset G$ for which the greatest lower bound $\bigvee F$ exists, $f(\bigvee F)$ is the greatest lower bound of the subset $f(F)$ in the ordered semigroup $(H; \cdot, \leqslant)$.

THEOREM 1. *For every ordered semigroup there exists a preserving intersections isomorphism into the inclusion ordered semigroup of all binary relations on a set. As images of the elements of the given ordered semigroup under the above isomorphism one can choose effective asymmetric binary relations.*

The proof of this theorem consists of a series of lemmas.

In this paper an ordinal number is a set of all smaller ordinals so that $0 = \emptyset$, $1 = \{\emptyset\}, \ldots, \omega = \{0, 1, \ldots\}$. An ordinal is called *initial* if it is infinite and if it is the smallest ordinal in the class of all ordinals having the same cardinality.

LEMMA 1. *For every initial ordinal a there exists a partition $(a_{i,j})_{i,j<a}$ of the set $a \setminus 2$ such that all subsets $a_{i,j}$ have the same cardinality as a and $i, j < \min a_{i,j}$ for every $a_{i,j}$.*

Proof. Since $a \setminus 2$ is infinite and infinite cardinals are idempotent under multiplication, there exists a partition $(\beta_{i,j})_{i,j<a}$ of the set $a \setminus 2$ into subsets $\beta_{i,j}$ having the same cardinality as a. Suppose that

$$a_{i,j} = \beta_{ij} \cap [\max\{i, j\} + 1, a),$$

$$\text{where } [\beta, a) = \{\gamma \colon \beta \leqslant \gamma < a\} \text{ for } (i, j) \neq (0, 0),$$

and suppose that

$$a_{0,0} = \beta_{0,0} \cup \bigcup_{(i,j)\neq(0,0)} \beta_{i,j} \setminus a_{i,j}.$$

The family $(a_{i,j})_{i,j<a}$ satisfies all conditions of the lemma. Thus Lemma 1 is proved.

Let $(G; \cdot, \leqslant)$ be an ordered semigroup. Adjoin a new element ∞ to G and suppose that $x\infty = \infty x = \infty$ and $x \leqslant \infty$ for all $x \in G^\infty = G \cup \{\infty\}$. Let a be an initial ordinal whose cardinality is not less than that of the set $G^\infty \times G^\infty$. Suppose that the set $G^\infty \times G^\infty$ is well ordered and its ordinal type is not larger than a. Let an element $g \in G$ be fixed as well as a family $(a_{i,j})_{i,j<a}$ satisfying the conditions of Lemma 1. Using transfinite induction we define a family $\mathfrak{A} = (g_{i,j})_{i,j<a}$ of elements of G^∞.

Base of induction. Suppose that $g_{0,0} = g_{0,1} = g_{1,1} = \infty$ and $g_{1,0} = g$.

Inductive step. Suppose that all $g_{i,j}$ for $i, j < \beta < \alpha$ have already been defined for some $\beta > 1$. Then $\beta \in a_{p,q}$ for some $p, q < \alpha$. Therefore, $p, q < \beta$, i.e., $g_{p,q}$ has already been defined. The set $a_{p,q}$ is a subset of α, hence it is well ordered by the order relation induced by that of α. Suppose that β is the γ-th element in the set $a_{p,q}$ (here γ is an ordinal). Since $a_{p,q} \subset \alpha$, the ordinal type of $a_{p,q}$ does not exceed that of α; since $a_{p,q}$ has the same cardinality as α and α is an initial ordinal, the ordinal type of $a_{p,q}$ is α. Therefore, $\gamma < \alpha$. Suppose that in the set of all pairs $(u, v) \in G^\infty \times G^\infty$ such that $g_{p,q} \leqslant uv$ there exists a pair (x, y) which is the γ-th element of the set. Then put $g_{p,\beta} = x$ and $g_{\beta,q} = y$. Otherwise, set $g_{p,\beta} = g_{\beta,q} = \infty$. If $p \neq i < \beta$ and $q \neq j < \beta$, set $g_{i,\beta} = g_{i,p}g_{p,\beta}$ and $g_{\beta,j} = g_{\beta,q}g_{q,j}$. Let $g_{\beta,\beta} = \infty$.

Thus the family $\mathfrak{A}$ has been defined. Now we are going to find certain properties of this family.

LEMMA 2. *For every* $i < \alpha$, $g_{i,i} = \infty$.

The proof is by an obvious transfinite induction over $i < \alpha$.

LEMMA 3. *If β is the γ-th element of the set $a_{p,q}$, then* $g_{p,q} \leqslant g_{p,\beta}g_{\beta,q}$.

Proof. Either $(g_{p,\beta}, g_{\beta,q}) = (x, y)$, where (x, y) is the γ-th element in the set of all the pairs $(u, v) \in G^\infty \times G^\infty$ such that $g_{p,q} \leqslant uv$, or $(g_{p,\beta}, g_{\beta,q}) = (\infty, \infty)$. In both cases $g_{p,q} \leqslant g_{p,\beta}g_{\beta,q}$.

LEMMA 4. *If $g_{i,k} \leqslant xy$ for some $i, k < \alpha$ and $x, y \in G^\infty$, then there exists an ordinal $j < \alpha$ such that $g_{i,j} = x$ and $g_{j,k} = y$.*

Proof. Suppose that $g_{i,k} \leqslant xy$ and the pair (x, y) is the β-th element of the set of all the pairs $(u, v) \in G^\infty \times G^\infty$ for which $g_{i,k} \leqslant uv$. Let j be the β-th element of the set $a_{i,k}$. By the definition of $\mathfrak{A}$, $g_{i,j} = x$ and $g_{j,k} = y$.

LEMMA 5. *For every $i, j, k < \alpha$, at least one of the elements $g_{i,j}$ and $g_{j,i}$ is ∞, and* $g_{i,k} \leqslant g_{i,j}g_{j,k}$.

The proof will be given by simultaneous transfinite induction over $n = \max\{i, j, k\}$. If $n \leqslant 1$, the statement of the lemma can easily be verified by a straightforward computation. Suppose now that the lemma is true if $i, j, k < \beta$, where $\beta > 1$. Then $\beta \in a_{p,q}$. By Lemma 3, $g_{p,q} \leqslant g_{p,\beta}g_{\beta,q}$. Multiplying this inequality by $g_{i,p}$ on the left and/or by $g_{q,k}$ on the right and using the induction hypothesis, which implies that

$$g_{i,k} \leqslant g_{i,p}g_{p,k} \leqslant g_{i,p}g_{p,q}g_{q,k},$$

and the equalities

$$g_{i,\beta} = g_{i,p}g_{p,\beta} \quad \text{and} \quad g_{\beta,k} = g_{\beta,q}g_{q,k},$$

we obtain

$$g_{i,k} \leqslant g_{i,\beta}g_{\beta,k} \quad \text{for every } i, k < \beta.$$

In particular, using Lemma 2, we obtain $\infty = g_{i,i} \leqslant g_{i,\beta} g_{\beta,i}$ which shows that at least one of the elements $g_{i,\beta}$ and $g_{\beta,i}$ is ∞. If $i = j$ or $j = k$, then

$$g_{i,k} \leqslant \infty = g_{i,j} g_{j,k}.$$

If $i = k = \beta$, then

$$g_{i,k} = \infty = g_{\beta,j} g_{j,\beta} = g_{i,j} g_{j,k}.$$

Therefore, it remains to consider the case where $i \neq j \neq k$ and one of the elements i, k is β while the two remaining elements are less than β.

Let $k = \beta$. If $i = p$, then

$$g_{p,\beta} \leqslant \infty = g_{p,j} g_{j,p} g_{p,\beta} = g_{p,j} g_{j,\beta}.$$

If $i \neq p$, then, using the induction hypothesis, we obtain

$$g_{i,\beta} = g_{i,p} g_{p,\beta} \leqslant g_{i,j} g_{j,p} g_{p,\beta} = g_{i,j} g_{j,\beta}.$$

Now let $i = \beta$. If $k = q$, then

$$g_{\beta,q} \leqslant \infty = g_{\beta,q} g_{q,j} g_{j,q} = g_{\beta,j} g_{j,k}.$$

If $k \neq q$, then, by the induction hypothesis, we obtain

$$g_{\beta,k} = g_{\beta,q} g_{q,k} \leqslant g_{\beta,q} g_{q,j} g_{j,k} = g_{\beta,j} g_{j,k}.$$

Here we have supposed that $j \neq p$ in the case where $k = \beta$, and that $j \neq q$ in the case where $i = \beta$. Otherwise,

$$g_{i,\beta} = g_{i,p} g_{p,\beta} \quad \text{and} \quad g_{\beta,k} = g_{\beta,q} g_{q,k}.$$

Thus we have proved that Lemma 5 is true for $i, j, k \leqslant \beta$ which completes the proof.

LEMMA 6. *For every $x \in G^\infty$ and every $i < a$ there exist $j, k < a$ such that $g_{i,j} = g_{k,i} = x$.*

Proof. Let $x \in G^\infty$ and $i < a$. Suppose that the pair (x, ∞) is the p-th element and the pair (∞, x) is the q-th element of the set of all pairs $(u, v) \in G^\infty \times G^\infty$ such that $\infty = g_{i,i} \leqslant uv$. If j is the p-th element and k is the q-th element of the set $a_{i,i}$, then, by the definition of $\mathfrak{A}$, $g_{i,j} = x$ and $g_{k,i} = x$. Thus Lemma 6 is proved.

Now we define a mapping P of G into the set of all binary relations on the set a (i.e., on the set of all ordinals less than a). By the definition,

$$(i, j) \in \mathsf{P}(x) \Leftrightarrow g_{i,j} \leqslant x.$$

LEMMA 7. *P is a preserving intersections isomorphism of $(G; \cdot, \leqslant)$ into the inclusion ordered semigroup of all binary relations on a. For every $x \in G$, $\mathsf{P}(x)$ is an effective asymmetric binary relation.*

Proof. If $(i, k) \in \mathsf{P}(xy)$, i.e., if $g_{i,k} \leqslant xy$, then, by Lemma 4, there exists $j < a$ such that $g_{i,j} = x$ and $g_{j,k} = y$. Therefore, $(i, j) \in \mathsf{P}(x)$ and $(j, k) \in \mathsf{P}(y)$, whence $(i, k) \in \mathsf{P}(y) \circ \mathsf{P}(x)$. Conversely, if $(i, k) \in \mathsf{P}(y) \circ \mathsf{P}(x)$, then $(i, j) \in \mathsf{P}(x)$ and $(j, k) \in \mathsf{P}(y)$ for some $j < a$, i.e., $g_{i,j} \leqslant x$ and $g_{j,k} \leqslant y$. Multiplying these inequalities and using the second statement of Lemma 5 we obtain $g_{i,k} \leqslant g_{i,j} g_{j,k} \leqslant xy$, whence $(i, k) \in \mathsf{P}(xy)$. Thus $\mathsf{P}(xy) = \mathsf{P}(y) \circ \mathsf{P}(x)$ for all $x, y \in G$.

We have proved that P is a homomorphic representation of $(G; \cdot)$ by binary relations on a. The form of the last equality is due to the fact that factors in a product of two elements of an abstract semigroup $(G; \cdot)$ are written from the left to the right, while the factors in a product of two binary relations are written from the right to the left, so that we prefer the above form to $\mathsf{P}(xy) = \mathsf{P}(x) \circ \mathsf{P}(y)$.

Suppose that $\mathsf{P}(x) \subset \mathsf{P}(y)$. By Lemma 6, there exist $i, j < a$ such that $g_{i,j} = x$. Therefore, $(i, j) \in \mathsf{P}(x)$, whence $(i, j) \in \mathsf{P}(y)$, i.e., $x = g_{i,j} \leqslant y$. If $\mathsf{P}(x) = \mathsf{P}(y)$, then $\mathsf{P}(x) \subset \mathsf{P}(y)$ and $\mathsf{P}(y) \subset \mathsf{P}(x)$, so that $x \leqslant y$ and $y \leqslant x$, i.e., $x = y$. Thus P is a one-to-one mapping.

Let a subset $H \subset G$ have the greatest lower bound $z = \bigwedge H$. If

$$(i, j) \in \bigcap_{h \in H} \mathsf{P}(h),$$

i.e., if $(i, j) \in \mathsf{P}(h)$ for each $h \in H$ or, which is the same, if $g_{i,j} \leqslant h$ for each $h \in H$, then (and only then) $g_{i,j} \leqslant z$, i.e., $(i, j) \in \mathsf{P}(z)$. Therefore

$$\mathsf{P}(\bigwedge H) = \bigcap_{h \in H} \mathsf{P}(h).$$

If $x \leqslant y$, then $x = x \wedge y$, whence

$$\mathsf{P}(x) = \mathsf{P}(x) \cap \mathsf{P}(y), \quad \text{i.e.,} \quad \mathsf{P}(x) \subset \mathsf{P}(y).$$

Therefore, $x \leqslant y \Leftrightarrow \mathsf{P}(x) \subset \mathsf{P}(y)$ and P is a preserving intersections isomorphism of $(G; \cdot, \leqslant)$ into the inclusion ordered semigroup of all binary relations on a.

To complete the proof it remains to notice that, by the first statement of Lemma 5, if $(i, j) \in \mathsf{P}(x)$ for some $i, j < a$ and $x \in G$, i.e., if $g_{i,j} \leqslant x$, then $g_{i,j} \neq \infty$ so that $g_{j,i} = \infty$. By the definition of P, $(j, i) \notin \mathsf{P}(x)$ (moreover, $(j, i) \notin \mathsf{P}(y)$ is true for each $y \in G$). Therefore, $\mathsf{P}(x)$ is asymmetric. By Lemma 6, $\mathsf{P}(x)$ is effective for each $x \in G$.

Evidently, Lemma 7 completes the proof of Theorem 1. In fact, we have proved a somewhat stronger assertion (see Corollary 1).

Let $(A; <)$ be a *strictly linearly ordered set* (i.e., A is a set, $<$ is a strict order relation on A, and $<$ is linear, that is, for every $a_1, a_2 \in A$, $a_1 < a_2$, or $a_1 = a_2$, or $a_2 < a_1$). Binary relations on A can be considered as *multi-valued transformations* of A: $(a_1, a_2) \in \varrho \subset A \times A$ means that a_2 is an image of a_1 under the multi-valued transformation ϱ. If $(a_1, a_2) \in \varrho$ implies $a_1 < a_2$ (i.e., if ϱ is included into the strict order relation $<$ or, which is the same, if any image of any element is greater than the element itself), ϱ is called *strictly extensive.*

PROPOSITION 1. *Let Φ be a set of binary relations on a set A. Then on A there exists a strict linear order relation such that all elements of Φ are strictly extensive if and only if the semigroup Ψ of binary relations on A generated by the set Φ consists only of asymmetric binary relations.*

Proof. Necessity. Let Φ be a set of strictly extensive binary relations on a strictly linearly ordered set $(A; <)$. It is easy to see that the product of two strictly extensive binary relations is strictly extensive (in effect, if $\varphi \subset <$ and $\psi \subset <$, then $\psi \circ \varphi \subset < \circ < \subset <$). Therefore, all the elements of the semigroup Ψ are strictly extensive. Now all strictly extensive binary relations are asymmetric, since $<$ is asymmetric.

Sufficiency. Let all the binary relations from the semigroup Ψ generated by Φ be asymmetric. Let τ denote the transitivity relation [5] of Ψ, i.e., τ is the set-theoretical union of all the relations from Ψ. Clearly, τ is transitive (if $(a_1, a_2) \in \tau$ and $(a_2, a_3) \in \tau$, then $(a_1, a_2) \in \varphi$ and $(a_2, a_3) \in \psi$ for some $\varphi, \psi \in \Psi$ and, therefore, $(a_1, a_3) \in \psi \circ \varphi \in \Psi$ and $(a_1, a_3) \in \tau$). If $(a, a) \in \tau$, then $(a, a) \in \varphi$ for some $\varphi \in \Psi$. Therefore, τ is irreflexive, i.e., τ is a strict order relation on A. By the Zorn Lemma, every strict order relation on A is included into a maximal strict order relation. It is easy to see that maximal strict order relations are linear. Let $<$ be a linear strict order relation on A in which τ is included. For every $\varphi \in \Phi$, it follows from $(a_1, a_2) \in \varphi$ that $(a_1, a_2) \in \tau$, thus $a_1 < a_2$. Therefore, Φ is a set of strictly extensive binary relations on a linearly strictly ordered set $(A; <)$, which completes the proof.

COROLLARY 1. *For every ordered semigroup $(G; \cdot, \leqslant)$ there exists a preserving intersections isomorphism onto an inclusion ordered semigroup of strictly extensive effective binary relations on a densely linearly strictly ordered set $(A; <)$ having no endpoints. The cardinality of A can be chosen to be countably infinite for finite G and the same as the cardinality of G for infinite G. If G is finite or countably infinite, then $(A; <)$ can be chosen to be the chain of all rational numbers.*

Proof. Let P be the isomorphism of $(G; \cdot, \leqslant)$ onto a semigroup of binary relations on a set B which has been constructed in the proof of Theorem 1. Let $\tau = \bigcup_{g \in G} \mathsf{P}(g)$. It follows from the last lines of the proof that τ is a strict order relation on B. Let $<$ be a linear strict order relation on B, $\tau \subset <$. If Q^+ is the set of non-negative rational numbers, then one can order the set $A = B \times Q^+$ lexicographically:

$$(b_1, q_1) < (b_2, q_2) \quad \text{iff} \quad b_1 < b_2 \text{ or } b_1 = b_2 \text{ and } q_1 < q_2.$$

Clearly, $(A;<)$ is a densely linearly strictly ordered set. Now A has the smallest element iff B has such an element. However, B cannot have the smallest element (otherwise, the binary relations $\mathsf{P}(g)$ for $g \in G$ would not have been effective, since the smallest element of B cannot have an inverse image under $\mathsf{P}(g)$). Now, since A has no largest element, the chain $(A;<)$ has no endpoints. For every $g \in G$ define the following binary relation $\mathsf{P}'(g)$ on A:

$$\big((b_1, q_1), (b_2, q_2)\big) \in \mathsf{P}'(g) \quad \text{iff} \quad (b_1, b_2) \in \mathsf{P}(g) \text{ and } q_1 = q_2.$$

Then one can readily verify that P' is the needed isomorphism. The last sentence of the corollary follows from the Cantor Theorem: any densely strictly ordered countable chain without endpoints is isomorphic to the chain of rational numbers.

COROLLARY 2. *An ordered semigroup $(G; \cdot, \leqslant)$ has an isomorphism satisfying the properties listed in Corollary 1 and such that to every element of G there corresponds a strict order relation under this isomorphism if and only if $g^2 \leqslant g$ for every $g \in G$.*

Proof. If φ is such an isomorphism, then

$$\varphi(g^2) = \varphi(g) \circ \varphi(g) \subset \varphi(g),$$

since $\varphi(g)$ is a strict order relation. Therefore $g^2 \leqslant g$. Conversely, if P' is the representation constructed in the proof of Corollary 1 and $g^2 \leqslant g$, then

$$\mathsf{P}'(g) \circ \mathsf{P}'(g) \subset \mathsf{P}'(g),$$

i.e., $\mathsf{P}'(g)$ is transitive. Thus $\mathsf{P}'(g)$ is a strict order relation.

COROLLARY 3. *An ordered semigroup $(G; \cdot, \leqslant)$ has an isomorphism satisfying the properties listed in Corollary 1 and mapping each element of G onto a strict dense order relation if and only if $(G; \cdot)$ is an idempotent semigroup, i.e., satisfies the identity $g^2 = g$.*

Proof. In fact, an irreflexive binary relation is a dense strict order relation iff it is idempotent. This fact together with Corollary 1 imply Corollary 3.

A binary relation ϱ on an ordered set $(A; \leqslant)$ is called *extensive* if $\varrho \subset \leqslant$, i.e., if $(a_1, a_2) \in \varrho$ implies $a_1 \leqslant a_2$. An ordered semigroup $(G; \cdot, \leqslant)$ is called *positively ordered* if $g_1 \leqslant g_1 g_2$ and $g_2 \leqslant g_1 g_2$ for all $g_1, g_2 \in G$.

COROLLARY 4. *An ordered semigroup $(G; \cdot, \leqslant)$ has a preserving intersections isomorphism onto an inclusion ordered semigroup of reflexive, antisymmetric and extensive binary relations on a densely linearly ordered set $(A; \leqslant)$ without endpoints if and only if $(G; \cdot, \leqslant)$ is positively ordered.*

Proof. One can easily verify that every ordered semigroup isomorphic to an inclusion ordered semigroup of reflexive binary relations is positively ordered [13]. Conversely, let $(G; \cdot, \leqslant)$ be positively ordered, let P' be the isomorphism constructed in the proof of Corollary 1, and let $\mathsf{P}''(g)$ be the reflexive closure of $\mathsf{P}'(g)$, i.e.,

$$\mathsf{P}''(g) = \mathsf{P}'(g) \cup \Delta_A \quad \text{for all } g \in G.$$

Then

$$\begin{aligned} \mathsf{P}''(h) \circ \mathsf{P}''(g) &= \mathsf{P}'(h) \circ \mathsf{P}'(g) \cup \mathsf{P}'(g) \cup \mathsf{P}'(h) \cup \Delta_A \\ &= \mathsf{P}'(gh) \cup \mathsf{P}'(g) \cup \mathsf{P}'(h) \cup \Delta_A = \mathsf{P}'(gh) \cup \Delta_A = \mathsf{P}''(gh). \end{aligned}$$

Clearly, P'' satisfies all properties listed in Corollary 4. As a densely linearly ordered set one can take $(A; \leqslant)$, where $(A; <)$ has been constructed in the proof of Corollary 1.

COROLLARY 5. *An ordered semigroup $(G; \cdot, \leqslant)$ has an isomorphism satisfying all the properties listed in Corollary 4 and mapping every element of G onto a dense order relation on A if and only if $(G; \cdot, \leqslant)$ is an idempotent positively ordered semigroup or, equivalently, if and only if $(G; \cdot)$ is a semilattice (i.e., a commutative idempotent semigroup) and $\leqslant$ is the converse natural order relation of this semilattice, i.e., $g \leqslant h$ iff $gh = h$ for all $g, h \in G$.*

Proof. If $(G; \cdot, \leqslant)$ has an isomorphism satisfying all the properties mentioned in Corollary 5, then, by Corollaries 3 and 4, $(G; \cdot, \leqslant)$ is an idempotent positively ordered semigroup. If $(G; \cdot, \leqslant)$ is an idempotent positively ordered semigroup, then $g \leqslant gh$ and $h \leqslant gh$ for all $g, h \in G$. Multiplying these inequalities we obtain $hg \leqslant (gh)^2 = gh$. Interchanging the roles of g and h we obtain $gh \leqslant hg$, i.e., $gh = hg$. Thus $(G; \cdot)$ is a semilattice. If $g \leqslant h$, then $gh \leqslant h$. However, since the order is positive, $h \leqslant gh$. Therefore $gh = h$. Conversely, if $h = gh$, then $g \leqslant gh = h$. Thus $\leqslant$ is precisely the converse natural order of the semilattice $(G; \cdot)$. Now suppose that $(G; \cdot)$ is a semilattice and $g \leqslant h$ iff $gh = h$. Let P'' be the isomorphism constructed in the proof of Corollary 4. Then P'' has all the properties listed in Corollary 5, since $\leqslant$ is a positive order.

If $(G; \cdot, \leqslant)$ is a semilattice with the converse natural order, then gh is precisely the least upper bound of g and h in $(G; \leqslant)$, i.e., $gh = g \vee h$. Thus $(G; \vee)$ is an upper semilattice. Therefore, Corollary 5 can be formulated in the following equivalent form:

COROLLARY 6. *For every upper semilattice $(G; \vee)$ there exists a preserving intersections isomorphism onto a semigroup of extensive binary relations on a densely linearly ordered set $(A; \leqslant)$ without endpoints; this isomorphism maps every element of G onto a dense order relation on A.*

It is known [6] that every semilattice is isomorphic to a semigroup of binary relations whose elements are equivalence relations.

A *semilattice ordered semigroup* is an algebra of the form $(G; \cdot, \wedge)$, where $(G; \cdot)$ is a semigroup, $(G; \wedge)$ is a semilattice whose natural order relation is stable on $(G; \cdot)$ (the latter condition is equivalent to the identities $x(y\wedge z) = x(y\wedge z)\wedge xy$ and $(x\wedge y)z = (x\wedge y)z\wedge yz$).

COROLLARY 7. *Every semilattice ordered semigroup is isomorphic to an algebra of binary relations of the form* $(\Phi; \circ, \cap)$, *i.e., to a semigroup* $(\Phi; \circ)$ *of binary relations closed under the binary intersection* $\cap$.

Of course, in Corollary 7 we could demand that all greatest lower bounds of subsets of G existing in $(G; \leqslant)$ were mapped onto intersections of corresponding binary relations from Φ.

The next corollary to Theorem 1 is formulated as a theorem.

THEOREM 2. *An algebra of binary relations of the form* $(\Phi; \circ, \cap)$, *where* Φ *is a set of quasi-order relations on a set* A, *is a lattice. Conversely, every abstract lattice* $(G; \vee, \wedge)$ *is isomorphic to an algebra of binary relations of the form* $(\Phi; \circ, \cap)$, *where all the elements of* Φ *are quasi-order relations. Moreover, one can choose* $(\Phi; \circ, \cap)$ *in such a way that all the elements of* Φ *are extensive binary relations on a densely linearly ordered set* $(A; \leqslant)$ *without endpoints and they are dense order relations on* A; *the isomorphism can be chosen to preserve intersections.*

Proof. We need to prove only the first sentence. Suppose that $(\Phi; \circ, \cap)$ is an algebra of quasi-order relations and $\varphi, \psi \in \Phi$. Clearly, $\varphi, \psi \subset \psi\circ\varphi$, since φ and ψ are reflexive. Therefore, $(\Phi; \circ, \cap)$ is positively ordered. Moreover, all the elements of Φ are idempotent. By Corollary 5, $(\Phi; \circ)$ is a semilattice. It follows that $(\Phi; \circ, \cap)$ is a lattice. Thus Theorem 2 is proved.

A semigroup $(\Phi; \circ)$ of binary relations is called 1-*fold* [7] if, for any two distinct binary relations φ and ψ from Φ, $\varphi\cap\psi = \emptyset$.

COROLLARY 8. *Every semigroup is isomorphic to a* 1-*fold semigroup of binary relations.*

Proof. Let $(G; \cdot)$ be a semigroup. Suppose that G is ordered by Δ_G. We can add to G the greatest element ∞ which is the multiplicative zero as it has been done in the proof of Theorem 1. We obtain an ordered semigroup $(G^\infty; \cdot, \leqslant)$. Now we add a new zero element 0 to the semigroup $(G^\infty; \cdot)$. This element is defined to be the smallest element of the set $G^{\infty 0} = G^\infty \cup \{0\}$. In particular, $0\infty = \infty 0 = 0$. Thus 0 is the smallest and ∞ the largest elements of $G^{\infty 0}$, the remaining elements being pairwise incomparable. Consider the representation P constructed in the proof of Theorem 1. Then $\mathsf{P}(0) = \emptyset$. Now, P is a representation of the semigroup $G^0 = G\cup\{0\}$ which preserves intersections. For any two distinct elements $g, h \in G$ their greatest lower bound is 0. Therefore,

$$\mathsf{P}(g)\cap\mathsf{P}(h) = \mathsf{P}(g\wedge h) = \mathsf{P}(0) = \emptyset,$$

i.e., P is an isomorphism of the semigroup $(G; \cdot)$ onto a 1-fold semigroup of binary relations. Moreover, all the relations $\mathsf{P}(g)$ are effective and asymmetric.

The following problem is well known (see [3] and [1]): which lattices are isomorphically embeddable in lattices of all topologies on sets? The problem has been solved in the affirmative for all lattices (see [11], [4], and [10]). Embeddability of arbitrary lattices in lattices of topologies follows from the Whitman representation theorem. It follows from our Theorem 2 as well.

COROLLARY 9. *Every lattice is isomorphically embeddable in the lattice of T_0-topologies on a suitable set.*

Remark. Clearly, T_0-topologies form a sublattice in the lattice of all topologies on a set.

Proof. Suppose that $(L; \vee, \wedge)$ is a lattice and R is a dual isomorphism of this lattice onto a lattice $(\Phi; \circ, \cap)$ of order relations on a set A. A subset $B \subset A$ is called *stable relatively to* $\varrho \subset A \times A$ (or *ϱ-stable*) if $\varrho(B) \subset B$, i.e., if $b \in B$ and $(b, a) \in \varrho$ imply $a \in B$. Let $\mathrm{St}\,\varrho$ be the set of all ϱ-stable subsets of A. Clearly, $\mathrm{St}\varrho$ is a topology. Moreover, this is a T_0-topology for every order relation ϱ and the intersection of any family of open sets is open (these two properties characterize topologies $\mathrm{St}\varrho$ corresponding to order relations ϱ). The mapping $\varphi \mapsto \mathrm{St}\varphi$ is a dual isomorphism of $(\Phi; \circ, \cap)$ into the lattice of all topologies on A, i.e.,

$$\mathrm{St}(\psi\circ\varphi) = \mathrm{St}\varphi \cap \mathrm{St}\psi \quad \text{and} \quad \mathrm{St}(\varphi\cap\psi) = \mathrm{St}\varphi \vee \mathrm{St}\psi.$$

In effect, if $B \in \mathrm{St}\varphi \cap \mathrm{St}\psi$, then $\varphi(B) \subset B$ and $\psi(B) \subset B$, whence

$$\psi\circ\varphi(B) = \psi\big(\varphi(B)\big) \subset \psi(B) \subset B, \quad \text{i.e.,} \quad B \in \mathrm{St}(\psi\circ\varphi).$$

On the other hand, if $C \in \mathrm{St}(\psi\circ\varphi)$, i.e., $\psi\circ\varphi(C) \subset C$, then, using the reflexivity of φ, we obtain $\varphi(C) \supset C$, whence

$$\psi(C) \subset \psi\big(\varphi(C)\big) = \psi\circ\varphi(C) \subset C \quad \text{and} \quad C \in \mathrm{St}\psi.$$

Using the reflexivity of ψ we obtain

$$\varphi(C) \subset \psi\big(\varphi(C)\big) = \psi\circ\varphi(C) \subset C \quad \text{and} \quad C \in \mathrm{St}\varphi.$$

If $B \in \mathrm{St}\varphi$, then

$$(\varphi\cap\psi)(B) \subset \varphi(B) \subset B \quad \text{and} \quad B \in \mathrm{St}(\varphi\cap\psi).$$

Analogously, $\mathrm{St}\psi \subset \mathrm{St}(\varphi\cap\psi)$, whence

$$\mathrm{St}\varphi \vee \mathrm{St}\psi \subset \mathrm{St}(\varphi\cap\psi).$$

Let $C \in \mathrm{St}(\varphi \cap \psi)$. Then

$$(\varphi \cap \psi)(C) \subset C$$

and

$$C = (\varphi \cap \psi)(C) = \bigcup_{b \in B} (\varphi \cap \psi)\langle b \rangle = \bigcup_{b \in B} (\varphi\langle b \rangle \cap \psi\langle b \rangle) \in \mathrm{St}\varphi \vee \mathrm{St}\psi,$$

since $\varphi\langle b \rangle \in \mathrm{St}\varphi$ and $\psi\langle b \rangle \in \mathrm{St}\psi$. Here $\varrho\langle b \rangle = \{a: (b, a) \in \varrho\}$.

Thus the mapping R_1 such that $R_1(x) = \mathrm{St}R(x)$ is an isomorphism of $(L; \vee, \wedge)$ into the lattice of T_0-topologies on A.

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