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EVEN EQUATIONS AND AGASSIZ SUMS

BY

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For a term p and a variable x let $\mathrm{occ}(x, p)$ denote the number of occurrences of x in p. An equation $p = q$ is said to be an *even equation* if for each variable x, $\mathrm{occ}(x, p)$ is even iff $\mathrm{occ}(x, q)$ is even. $p = q$ is said to be *regular* (see [9]) if for each variable x, $\mathrm{occ}(x, p) \neq 0$ iff $\mathrm{occ}(x, q) \neq 0$.

For a given similarity type ϱ let $T(\varrho)$ denote the set of all terms of type ϱ with variables among $x_0, x_1, \ldots$. Let $N_\tau\colon T(\tau) \to T(2, 0)$ be a mapping defined as follows: $N_\tau(e) = 0$ for each constant e, $N_\tau(x_i) = x_i$ for each x_i, and $N_\tau(f(q_1, \ldots, q_n)) = \big(\ldots\big((N_\tau(q_1) + N_\tau(q_2)) + N_\tau(q_3)\big) + \ldots\big) + N_\tau(q_n)$ for each n-ary $(n \geqslant 1)$ operation f and terms $q_1, \ldots, q_n$, where 0 and + are 0- or 2-ary operation symbols, respectively.

$\mathfrak{S}_2$ and $\mathfrak{Z}_2$ denote, respectively, a 2-element join-semilattice with zero as a constant, and a 2-element semigroup with addition modulo 2 and 0 as a constant.

From Płonka [9] and [10] we know that if $\boldsymbol{K}$ is a variety of algebras of type τ in which an equation of the form $p(x_0, x_1) = x_0$ is valid, then the variety described by the set of all regular equations valid in $\boldsymbol{K}$ coincides with the class of sums of all fine Agassiz systems of algebras from $\boldsymbol{K}$, indexed by algebras from $\mathrm{HSP}(\mathfrak{S}_2)$ with N_τ as naming functor.

The main aim of this paper is to show that the same result holds for even equations after replacing $\mathrm{HSP}(\mathfrak{S}_2)$ by $\mathrm{HSP}(\mathfrak{Z}_2)$.

A triple $\mathscr{S} = \big(\mathfrak{B}, (\mathfrak{A}_b;\, b \in B), (h_{bc};\, (b, c) \in R)\big)$ is called a *fine Agassiz system* (for general case see [2], [4]) if the following conditions hold:

(i) $\mathfrak{B}$ is an algebra of type $\langle 2, 0\rangle$,

(ii) $(\mathfrak{A}_b;\, b \in B)$ is a family of algebras of type τ,

(iii) $R \subset B \times B$ is the least transitive relation such that for every n-ary $(n \geqslant 1)$ operation symbol f of type τ, if $b_1, \ldots, b_n \in B$ and $b = N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n)$ then $(b_i, b) \in R$ for all i,

(iv) $(h_{bc};\, (b, c) \in R)$ is a family such that h_{bc} is a homomorphism of $\mathfrak{A}_b$ into $\mathfrak{A}_c$, $h_{cd} \circ h_{bc} = h_{bd}$ whenever (b, c), (c, d) are in R, and for each $b \in B$ there exists $b_0 \in B$ such that h_{bb_0} is a monomorphism.

The *sum of an Agassiz system* $\mathcal{S}$ (see [2], [4]) is an algebra $\mathfrak{A}$ of the type τ with the carrier $\bigcup(A_b \times \{b\};\, b \in B)$ and operations defined as follows: for an operation f of type τ, if f is 0-ary then $f_{\mathfrak{A}} = (f_{\mathfrak{A}_{0_{\mathfrak{B}}}}, 0_{\mathfrak{B}})$, and if f is n-ary ($n \geqslant 1$) then for all $(a_1, b_1), \ldots, (a_n, b_n)$,

$$f_{\mathfrak{A}}\big((a_1, b_1), \ldots, (a_n, b_n)\big)$$
$$= \big(f_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)), b\big) \quad \text{where } b = N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n).$$

By $\lim(\boldsymbol{I}, \boldsymbol{K})$ we shall denote the class of all isomorphic copies of sums of fine Agassiz systems $\big(\mathfrak{B}, (\mathfrak{A}_b;\, b \in B), (h_{bc};\, (b, c) \in R)\big)$ where $\mathfrak{B}$ is an algebra from a class $\boldsymbol{I}$ of algebras of type $\langle 2, 0\rangle$, and all $\mathfrak{A}_b$'s are members of a class $\boldsymbol{K}$ of algebras of type τ.

For an algebra $\mathfrak{B}$ of type $\langle 2, 0\rangle$ let ${}^\tau\mathfrak{B}$ denote an algebra of type τ with the carrier B and operations defined as follows: for an operation symbol f of type τ, if f is 0-ary then $f_{\tau_{\mathfrak{B}}} = 0_{\mathfrak{B}}$, and if f is n-ary ($n \geqslant 1$) then for all $b_1, \ldots, b_n \in B$, $f_{\tau_{\mathfrak{B}}}(b_1, \ldots, b_n) = N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n)$.

LEMMA 1. *If the relation R in a fine Agassiz system $\mathcal{S}$ is full then each of h_{bc}'s is an isomorphism and the sum of $\mathcal{S}$ is isomorphic to any of $\mathfrak{A}_b \times {}^\tau\mathfrak{B}$ where $b \in B$.*

Proof. By (iv) of the definition of $\mathcal{S}$, for each $b \in B$ there exists $b_0 \in B$ such that h_{bb_0} is a monomorphism, and moreover $h_{bb_0} \circ h_{bb} = h_{bb_0}$ since R is full. Hence, $h_{bb} = 1_{A_b}$ for each b. Again by (iv), $h_{bc} \circ h_{cb} = h_{cc}$ and $h_{cb} \circ h_{bc} = h_{bb}$ for all b, c since R is full. Thus h_{bc} is an isomorphism for all $b, c \in B$.

For a fixed element $b^* \in B$ define $h\colon \bigcup(A_b \times \{b\};\, b \in B) \to A_{b^*} \times {}^\tau B$ as follows: $h(a, b) = \big(h_{bb^*}(a), b\big)$ for all $a \in A$ and $b \in B$. As each of h_{bb}'s is an isomorphism, h is a bijection. Let f be an n-ary operation symbol of type τ, $(a_1, b_1), \ldots, (a_n, b_n) \in \bigcup(A_b \times \{b\};\, b \in B)$ and let $b = N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n)$. If $n \geqslant 1$ then notice that

$$\begin{aligned} h\big(f_{\mathfrak{A}}((a_1, b_1), \ldots, (a_n, b_n))\big) &= h\big(f_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)), N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n)\big) \\ &= \big(h_{bb^*}(f_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n))), N_\tau(f)_{\mathfrak{B}}(b_1, \ldots, b_n)\big) \\ &= \big(f_{\mathfrak{A}_{b^*}}(h_{b_1 b^*}(a_1), \ldots, h_{b_n b^*}(a_n)), f_{\tau_{\mathfrak{B}}}(b_1, \ldots, b_n)\big) \\ &= f_{\mathfrak{A}_{b^*} \times \tau_{\mathfrak{B}}}\big(h(a_1, b_1), \ldots, h(a_n, b_n)\big), \end{aligned}$$

else

$$h(f_{\mathfrak{A}_{0_{\mathfrak{B}}}}, 0_{\mathfrak{B}}) = \big(h_{0_{\mathfrak{B}} b^*}(f_{\mathfrak{A}_{0_{\mathfrak{B}}}}), 0_{\mathfrak{B}}\big) = (f_{\mathfrak{A}_{b^*}}, 0_{\mathfrak{B}}).$$

Thus h preserves all operations, and consequently it is an isomorphism.

Notice that the above lemma is also true for an arbitrary fine Agassiz system.

Denote by $\mathrm{Even}(\tau)$, where τ is a similarity type, the set of all even equations of type τ. For a class $\boldsymbol{K}$ of algebras of type τ let $\mathrm{Eq}(\boldsymbol{K})$ denote the set of all equations valid in $\boldsymbol{K}$. Instead of $\mathrm{Eq}(\boldsymbol{K}) \cap \mathrm{Even}(\tau)$ we shall write $\mathrm{Even}(\boldsymbol{K})$.

PROPOSITION. *For a variety* $\boldsymbol{K}$ *of algebras of type* τ *the following conditions are equivalent*:

(i) $\mathrm{Eq}(\boldsymbol{K}) = \mathrm{Even}(\boldsymbol{K})$;

(ii) $\lim(\mathrm{HSP}(\mathfrak{Z}_2), \boldsymbol{K}) = \boldsymbol{K}$;

(iii) $\lim(\{\mathfrak{Z}_2\}, \boldsymbol{K}) \subseteq \boldsymbol{K}$;

(iv) ${}^{\tau}\mathfrak{Z}_2$ *is in* $\boldsymbol{K}$.

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathrm{Eq}(\boldsymbol{K}) = \mathrm{Even}(\boldsymbol{K})$. Proposition 3 in [2] says that $p = q \in \mathrm{Eq}(\lim(\boldsymbol{I}, \boldsymbol{K}))$ iff $p = q \in \mathrm{Eq}(\boldsymbol{K})$ and $N_\tau(p) = N_\tau(q) \in \mathrm{Eq}(\boldsymbol{I})$ for each equation $p = q$ of type τ and a class $\boldsymbol{I}$ of algebras of type $\langle 2, 0\rangle$. Notice that $N_\tau(p) = N_\tau(q) \in \mathrm{Eq}(\mathrm{HSP}(\mathfrak{Z}_2))$ iff $p = q$ is even. Therefore, for each equation $p = q$ of type τ, $p = q \in \mathrm{Eq}(\lim(\mathrm{HSP}(\mathfrak{Z}_2), \boldsymbol{K}))$ iff $p = q \in \mathrm{Even}(\boldsymbol{K})$. Hence, $\lim(\mathrm{HSP}(\mathfrak{Z}_2), \boldsymbol{K}) \subseteq \boldsymbol{K}$. Thus (i) implies (ii).

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (iv) It immediately follows from the observation that ${}^{\tau}\mathfrak{Z}_2$ is isomorphic to the sum of the following fine Agassiz system $(\mathfrak{Z}_2, (\mathfrak{T}_a;\, a \in \{0, 1\}), (h_{ab};\, a, b \in \{0, 1\}))$ where $\mathfrak{T}_0$ and $\mathfrak{T}_1$ are trivial algebras of type τ.

(iv) $\Rightarrow$ (i) Assume that ${}^{\tau}\mathfrak{Z}_2$ is in $\boldsymbol{K}$. It suffices to show that $\mathrm{Eq}(\boldsymbol{K}) \subseteq \mathrm{Even}(\boldsymbol{K})$. Let $p = q \in \mathrm{Eq}(\boldsymbol{K})$ and suppose that $p = q$ is not even. Then $p = q \in \mathrm{Eq}({}^{\tau}\mathfrak{Z}_2) \setminus \mathrm{Even}(\boldsymbol{K})$. Hence there exists a variable x_i, occurring in p or in q, such that $\mathrm{occ}(x_i, p) - \mathrm{occ}(x_i, q)$ is odd. Then $v(p) \neq v(q)$, a contradiction, where v is an assignment in ${}^{\tau}\mathfrak{Z}_2$ such that $v(x_j) = 1$ when $j = i$, and $v(x_j) = 0$ otherwise.

For a set Σ of equations of type τ let Σ^* denote the class of all algebras of type τ in which every equation from Σ is valid. Let ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$ denote the class of all algebras of the form ${}^{\tau}\mathfrak{B}$, where $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$. Observe that ${}^{\tau}\mathfrak{B}$ becomes a well-known structure, namely an n-group, whenever τ consists of exactly one operation whose arity equals n (see [1]).

LEMMA 2. *Let* τ *be a similarity type having an operation with arity at least two. Then* ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$ *is a variety and* $\mathrm{Eq}({}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2))$ *consists of all even equations of type* τ. *Moreover* ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$ *is a minimal variety.*

Proof. To prove the first statement it suffices to show that ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2) = (\mathrm{Even}(\tau))^*$.

$\subseteq$: Since ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2) = \lim(\mathrm{HSP}(\mathfrak{Z}_2), \mathfrak{T})$ where $\mathfrak{T}$ is the trivial variety of type τ, then by Proposition 3 from [2] (see the proof of Proposition above), ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2) \subseteq (\mathrm{Even}(\tau))^*$.

$\supseteq$: Let $\mathfrak{A} \in (\mathrm{Even}(\tau))^*$. We need to show that $\mathfrak{A} = {}^\tau\mathfrak{B}$ for some $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$.

Case 1. There exists an even-ary operation of type τ.

Then there exists a term $p(x_0, x_1)$ of type τ such that $\mathrm{occ}(x_0, p) = \mathrm{occ}(x_1, p)$ and $\mathrm{occ}(x_0, p)$ is odd.

As $p(x_0, x_0) = p(x_1, x_1)$ is even, we may define an algebra $\mathfrak{B} = (A, +, 0)$ where $0 = p_{\mathfrak{A}}(a, a)$ for a certain $a \in A$, and $a_1 + a_2 = p_{\mathfrak{A}}(a_1, a_2)$ for all $a_1, a_2 \in A$. Since the following equations: $p(p(x_1, x_2), x_3) = p(x_1, p(x_2, x_3))$, $p(x_1, x_2) = p(x_2, x_1)$, $p(x_1, p(x_2, x_2)) = x_1$, $p(x_1, x_1) = p(x_2, x_2)$ are even, then by the fact that

$$\text{(I)} \quad \mathrm{HSP}(\mathfrak{Z}_2) = \{(x_1+x_2)+x_3 = x_1+(x_2+x_3),\ x_1+x_2 = x_2+x_1,\ x_1+0 = x_1,\ x_1+x_1 = 0\}^*,$$

we have $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$. It remains to show that $\mathfrak{A} = {}^\tau\mathfrak{B}$. Since $f(x_1, \ldots, x_n) = p(\ldots p(p(x_1, x_2), x_3), \ldots, x_n)$ is even for each n-ary ($n \geqslant 1$) operation f of type τ,

$$f_A(a_1, \ldots, a_n) = p_{\mathfrak{A}}(\ldots p_{\mathfrak{A}}(p_{\mathfrak{A}}(a_1, a_2), a_3), \ldots, a_n) = N_\tau(f)_{\mathfrak{B}}(a_1, \ldots, a_n) = f_{\tau_{\mathfrak{B}}}(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n \in A$. Since $f = p(x_1, x_1)$ is even for each 0-ary operation f of type τ, $f_{\mathfrak{A}} = p_{\mathfrak{A}}(a, a) = 0_{\mathfrak{B}} = f_{\tau_{\mathfrak{B}}}$. Then $\mathfrak{A} = {}^\tau\mathfrak{B}$.

Case 2. Each operation of type τ is odd-ary.

Then there exists a ternary term $p(x_0, x_1, x_2)$ of type τ such that $\mathrm{occ}(x_0, p) = \mathrm{occ}(x_2, p) = 1$ and $\mathrm{occ}(x_1, p)$ is odd.

Let $\mathfrak{B} = (A, +, 0)$ be an algebra where 0 is a fixed element of A and $a_1 + a_2 = p_{\mathfrak{A}}(a_1, 0, a_2)$ for all $a_1, a_2 \in A$. Since the following equations:

$$p(p(x_1, x_4, x_2), x_4, x_3) = p(x_1, x_4, p(x_2, x_4, x_3)),$$

$$p(x_1, x_3, x_2) = p(x_2, x_3, x_1), \quad p(x_1, x_2, x_2) = x_1, \quad p(x_1, x_2, x_1) = x_2$$

are even, then by the assumption that $\mathfrak{A}$ is in $(\mathrm{Even}(\tau))^*$ we have

$$p_{\mathfrak{A}}(p_{\mathfrak{A}}(a_1, 0, a_2), 0, a_3) = p_{\mathfrak{A}}(a_1, 0, p_{\mathfrak{A}}(a_2, 0, a_3)),$$

$$p_{\mathfrak{A}}(a_1, 0, a_2) = p_{\mathfrak{A}}(a_2, 0, a_1), \quad p_{\mathfrak{A}}(a_1, 0, 0) = a_1, \quad p_{\mathfrak{A}}(a_1, 0, a_1) = 0$$

for all $a_1, a_2, a_3 \in A$. So, $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$, by (I). But

$$f_{\mathfrak{A}}(a_1, \ldots, a_n) = p_{\mathfrak{A}}(\ldots p_{\mathfrak{A}}(p_{\mathfrak{A}}(a_1, 0, a_2), 0, a_3), \ldots, 0, a_n) = N_\tau(f)_{\mathfrak{B}}(a_1, \ldots, a_n) = f_{\tau_{\mathfrak{B}}}(a_1, \ldots, a_n),$$

for all $a_1, \ldots, a_n \in A$, because

$$f(x_1, \ldots, x_n) = p(\ldots p(p(x_1, x_{n+1}, x_2), x_{n+1}, x_3), \ldots, x_{n+1}, x_n)$$

is even for each n-ary ($n \geqslant 1$) operation f of type τ, and so $\mathfrak{A} = {}^\tau\mathfrak{B}$.

Now we show that ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ is a minimal variety.

Let $p'(x_1, \ldots, x_n) = q'(x_1, \ldots, x_n) \notin \mathrm{Even}(\tau)$ and assume that $\mathrm{occ}(x_1, p')$

is odd and $\operatorname{occ}(x_1, q')$ is even. Hence $p(x_1, x_2) = q(x_1, x_2) \notin \operatorname{Even}(\tau)$, where $p(x_1, x_2)$ and $q(x_1, x_2)$ denote terms $p'(x_1, x_2, \ldots, x_2)$ and $q'(x_1, x_2, \ldots, x_2)$, respectively. We claim that $x_1 = x_2$ is a consequence of the set $\Sigma = \operatorname{Even}(\tau) \cup \{p(x_1, x_2) = q(x_1, x_2)\}$, which implies that $x_1 = x_2$ is a consequence of the set $\operatorname{Even}(\tau) \cup \{p'(x_1, \ldots, x_n) = q'(x_1, \ldots, x_n)\}$. When both $\operatorname{occ}(x_2, p)$ and $\operatorname{occ}(x_2, q)$ are even, then $x_1 = p(x_1, x_2) = q(x_1, x_2) = q(x_2, x_1) = p(x_2, x_1) = x_2$ are consequences of Σ. If $\operatorname{occ}(x_2, p)$ is even and $\operatorname{occ}(x_2, q)$ is odd then $x_1 = p(x_1, x_2) = q(x_1, x_2) = x_2$, and so, in this case our claim is true. If $\operatorname{occ}(x_2, p)$ is odd and $\operatorname{occ}(x_2, q)$ is even then, as before, we have

$$x_1 = p(x_1, q(x_1, x_2)) = p(x_1, p(x_1, x_2)) = x_2.$$

In the remaining case notice that

$$x_1 = q(x_2, x_1) = p(x_2, x_1) = p(x_2, q(x_1, x_1)) = p(x_2, p(x_1, x_1)) = x_2,$$

completing the proof of the claim. From the claim it follows immediately that $(\operatorname{Even}(\tau))^*$ is a minimal variety, and hence such is also ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$.

For a variety K of type $\langle 2, 0\rangle$, ${}^{\tau}K$ need not be a variety. In fact, the class ${}^{\langle 3\rangle}\mathrm{HSP}(\mathfrak{Z}_8)$ is not closed under subalgebras since the subalgebra of ${}^{\langle 3\rangle}\mathfrak{Z}_8$ generated by the set $\{2, 6\}$ is not of the form ${}^{\langle 3\rangle}\mathfrak{B}$ for any $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_8)$.

For varieties K and L of type τ let $K \times L$ denote the class of all isomorphic copies of algebras of the form $\mathfrak{A} \times \mathfrak{B}$ where $\mathfrak{A} \in K$ and $\mathfrak{B} \in L$, and let $K \vee L$ denote the join of K and L in the lattice of all varieties of type τ.

THEOREM. *If K is a variety of algebras of type τ in which an equation of the form $p(x_0, x_1) = x_0$ is valid ($p(x_0, x_1)$ is a term on two variables) then $(\operatorname{Even}(K))^* = \lim(\mathrm{HSP}(\mathfrak{Z}_2), K)$.*

Proof. It suffices to prove that $(\operatorname{Even}(K))^* \subseteq \lim(\mathrm{HSP}(\mathfrak{Z}_2), K)$ since in view of Proposition 3 from [2] (see the proof of Proposition above) we have $\lim(\mathrm{HSP}(\mathfrak{Z}_2), K) \subseteq (\operatorname{Even}(K))^*$. Moreover, we may assume that $\operatorname{Eq}(K) \neq \operatorname{Even}(K)$ since otherwise, by Proposition, the Theorem is true.

CLAIM 1. *There exists a term $q(x_0, x_1)$ of type τ such that $q(x_0, x_1) = x_0 \in \operatorname{Eq}(K) \setminus \operatorname{Even}(K)$.*

Proof. If $p(x_0, x_1) = x_0$ is noneven then as $q(x_0, x_1)$ we take $p(x_0, x_1)$. So, let $p(x_0, x_1) = x_0$ be even. Denote by $p'(x_0, x_1, x_2)$ a term obtained from $p(x_0, x_1)$ by replacing a fixed occurrence of x_1 in p by x_2. Let $r(x_{i_1}, \ldots, x_{i_n}) = s(x_{j_1}, \ldots, x_{j_m})$ be a fixed element of $\operatorname{Eq}(K) \setminus \operatorname{Even}(K)$ such that $\operatorname{occ}(x_0, r) = \operatorname{occ}(x_0, s) = 0$. Then

$$p'(x_0, r(x_{i_1}, \ldots, x_{i_n}), s(x_{j_1}, \ldots, x_{j_m})) = x_0 \in \operatorname{Eq}(K) \setminus \operatorname{Even}(K).$$

If $n = m = 1$ and $i_1 = j_1$ then as $q(x_0, x_1)$ we take $p'(x_0, r(x_{i_1}/x_1), s(x_{j_1}/x_1))$, else $\operatorname{occ}(x_k, p'(x_0, r(x_{i_1}, \ldots, x_{i_n}), s(x_{j_1}, \ldots, x_{j_m})))$ is odd for some $k \in \{i_1, \ldots, i_n, j_1, \ldots, j_m\}$, since $\operatorname{occ}(x_0, p'(x_0, r(x_{i_1}, \ldots, x_{i_n}), s(x_{j_1}, \ldots, x_{j_m})))$ is odd and $p'(x_0, r(x_{i_1}, \ldots, x_{i_n}), s(x_{j_1}, \ldots, x_{j_m})) = x_0$ is noneven. Therefore, as the term $q(x_0, x_1)$ one may take a term which we obtain from $p'(x_0, r(x_{i_1}, \ldots, x_{i_n}), s(x_{j_1}, \ldots, x_{j_m}))$ by replacing x_k by x_0 and each variable different both from x_k and x_0 by x_1.

CLAIM 2. $(\operatorname{Even}(\boldsymbol{K}))^* = \boldsymbol{K} \times {}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$.

Proof. Let $q(x_0, x_1) = x_0$ be a noneven equation valid in $\boldsymbol{K}$ (see Claim 1). Since $q(x_0, x_1) = x_0$ is noneven, either $\operatorname{occ}(x_0, q)$ is even and $\operatorname{occ}(x_1, q)$ is odd, or both of them are odd, or both of them are even. Let $t(x_0, x_1)$ denote $q(x_0, x_1)$ whenever $\operatorname{occ}(x_0, q)$ is even and $\operatorname{occ}(x_1, q)$ is odd, $q(x_0, q(x_0/x_1, x_1/x_0))$ whenever both of $\operatorname{occ}(x_0, q)$ and $\operatorname{occ}(x_1, q)$ are odd, and $q'(x_0, q(x_0/x_1, x_1/x_0), x_1)$ otherwise, where $q'(x_0, x_1, x_2)$ is the term obtained from $q(x_0, x_1)$ by replacing a fixed occurrence of x_1 in $q(x_0, x_1)$ by x_2. Notice that $\operatorname{occ}(x_0, t)$ is even and $\operatorname{occ}(x_1, t)$ is odd, that is $t(x_0, x_1) = x_1 \in \operatorname{Even}(\tau)$. So, $t(x_0, x_1) = x_1 \in \operatorname{Eq}({}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2))$, by Lemma 2. Obviously, $t(x_0, x_1) = x_0 \in \operatorname{Eq}(\boldsymbol{K})$. Therefore the varieties $\boldsymbol{K}$ and ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$ are independent and $\boldsymbol{K} \vee {}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2) = \boldsymbol{K} \times {}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$, by Theorem 1 from [3]. But $\boldsymbol{K} \vee {}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2) = (\operatorname{Even}(\boldsymbol{K}))^*$, by Lemma 2, which proves the Claim.

In view of Claim 2 and Lemma 1, to complete the proof it is enough to show that the relation R from a given fine Agassiz system $\mathscr{S} = (\mathfrak{B}, (\mathfrak{A}_b;\, b \in B), (h_{bc};\, (b, c) \in R))$, where $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$ and $\mathfrak{A}_b \in \boldsymbol{K}$ for all b, is full. Since $p(x_0, x_1)$ is a term of type τ, there must exist an operation f of type τ with arity at least two. So, there exists a term $q(x_0, x_1)$ of type τ such that $\operatorname{occ}(x_0, q) = 1$ and $\operatorname{occ}(x_1, q)$ is even. Then $N_\tau(q)(x_0, x_1) = x_0$ is even, which implies that $N_\tau(q)(x_0, x_1) = x_0$ is valid in $\mathrm{HSP}(\mathfrak{Z}_2)$. Hence, $N_\tau(q)_{\mathfrak{B}}(a, b) = a$, for each algebra $\mathfrak{B} \in \mathrm{HSP}(\mathfrak{Z}_2)$ and all its elements a, b. Therefore, by (iii) of the definition of fine Agassiz system, R is full.

Let $\mathscr{L}(\boldsymbol{K})$ denote the lattice of all subvarieties of variety $\boldsymbol{K}$.

COROLLARY 1. *Suppose that* $\boldsymbol{K}$ *satisfies the assumption of the above Theorem and suppose that* $\operatorname{Eq}(\boldsymbol{K}) \neq \operatorname{Even}(\boldsymbol{K})$. *Then* $\mathscr{L}((\operatorname{Even}(\boldsymbol{K}))^*) \cong \mathscr{L}(\boldsymbol{K}) \times \mathscr{2}$ *where* $\mathscr{2}$ *is a 2-element chain.*

Proof. Since $\boldsymbol{K}$ and ${}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)$ are independent (see the proof of Theorem), then by Corollary 2.8 from [5], we have $\mathscr{L}((\operatorname{Even}(\boldsymbol{K}))^*) \cong \mathscr{L}(\boldsymbol{K}) \times \mathscr{L}({}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2))$. But, in virtue of Lemma 2, $\mathscr{L}({}^{\tau}\mathrm{HSP}(\mathfrak{Z}_2)) \cong \mathscr{2}$. So, $\mathscr{L}((\operatorname{Even}(\boldsymbol{K}))^*) = \mathscr{L}(\boldsymbol{K}) \times \mathscr{2}$.

COROLLARY 2. *Suppose that* $\boldsymbol{K}$ *satisfies the assumption of the above*

Theorem and let τ be finite. Then $\boldsymbol{K}$ *is finitely based iff* $(\mathrm{Even}(\boldsymbol{K}))^*$ *is finitely based.*

Proof. $\Rightarrow$ Since in the term p from the assumption concerning $\boldsymbol{K}$ there are two distinct variables, τ must have an operation with arity at least two. Therefore one may construct a ternary term $q(x_0, x_1, x_2)$ such that $\mathrm{occ}(x_i, q)$ is odd for all $i \in \{0, 1, 2\}$. Hence $q(x_0, x_1/x_0, x_2) = x_2$ and $q(x_0, x_1/x_2, x_2) = x_0$ are even, so by Lemma 2, each of them is valid in ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$. Then ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ is a congruence permutable variety.

${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ is a minimal variety (see Lemma 2) and it contains the finite algebra ${}^\tau\mathfrak{Z}_2$. So, ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ is locally finite.

Due to McKenzie [7] we know that any minimal locally finite variety of finite type with permutable congruences is finitely based. Hence ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ is finitely based.

In the proof of the Claim 2 we show that $\boldsymbol{K}$ and ${}^\tau\mathrm{HSP}(\mathfrak{Z}_2)$ are independent. Therefore, by Corollary 1 from [6] and Claim 2, if $\boldsymbol{K}$ is finitely based then $(\mathrm{Even}(\boldsymbol{K}))^*$ is finitely based.

$\Leftarrow$ It immediately follows from Corollary 1.

COROLLARY 3. *There exists a not finitely based finite algebra of finite type whose all equations are even.*

Proof. From [8] we know that there exists a not finitely based finite algebra $\mathfrak{A}$ of type $\langle 2 \rangle$ in which equation of the form $p(x_0, x_1) = x_0$ is valid. Then the variety $\mathrm{HSP}(\mathfrak{A})$ satisfies assumptions of our Theorem. By Claim 2, $(\mathrm{Even}(\mathrm{HSP}(\mathfrak{A})))^*$ is generated by $\mathfrak{A} \times^{\langle 2 \rangle} \mathfrak{Z}_2$. Obviously, each equation valid in $\mathfrak{A} \times^{\langle 2 \rangle} \mathfrak{Z}_2$ is even. So, Corollary 2 completes the proof.

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