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ON GAME IDEALS

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0. Introduction. In this paper some ideals of subsets of the Cantor space and the Baire space of infinite sequence of natural numbers are considered. These ideals are defined in terms of infinite games of two players. It is natural to regard sets which can be omitted in the infinite game as small sets. Unfortunately, in the case of typical games such sets do not constitute an ideal – the union of two sets for which the second player has a winning strategy can be the whole space. But, if we consider a family of games with some restrictions on movements of the players, and then we take sets for which the second player has winning strategies in all games from this family, then we obtain a σ-ideal. Such a construction was made by Mycielski [M2]. We will investigate this ideal in Section 2. In Section 3 we will define three other ideals. Their properties are investigated in the following sections.

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1. Notation. We use the standard set theoretic notation. Continuum is denoted by $\mathfrak{c}$, $X^{<\omega}$ is the set of all finite sequences of elements of X, and $[X]^\omega$ is the family of all infinite countable subsets of X. The Baire space ω^ω of all infinite sequences of natural numbers and the Cantor space 2^ω are endowed with the topology determined by basic neighbourhoods of the form $[\sigma] = \{x \in X^\omega : \sigma \subseteq x\}$, where $\sigma \in X^{<\omega}$ ($X = \omega$ or $X = 2$, respectively). If $A \subseteq X^\omega$, $T \subseteq \omega$, then $A|T$ denotes the set

$$\{x \in X^T : (\exists y \in A)(x \subseteq y)\}.$$

$\exists^\infty$ and $\forall^\infty$ denote the quantifiers "there exist infinitely many" and "for all but finitely many", respectively. For any ideal $\mathscr{I}$ of subsets of X we define

$$\operatorname{add}(\mathscr{I}) = \min\{|\mathscr{H}| : \mathscr{H} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{H} \notin \mathscr{I}\},$$

$$\operatorname{cov}(\mathscr{I}) = \min\{|\mathscr{H}| : \mathscr{H} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{H} = X\},$$

$$\mathrm{non}(\mathscr{I}) = \min\{|H|:\ H \subseteq X \,\&\, H \notin \mathscr{I}\},$$

$$\mathrm{cof}(\mathscr{I}) = \min\{|\mathscr{H}|:\ \mathscr{H} \subseteq \mathscr{I} \,\&\, (\forall A \in \mathscr{I})(\exists B \in \mathscr{H})(A \subseteq B)\}.$$

If X is a group and $\mathscr{I}$ is invariant under translations, then

$$\mathrm{cov}_t(\mathscr{I}) = \min\{|T|:\ T \subseteq X \,\&\, (\exists A \in \mathscr{I})(\bigcup\{A+t:\ t \in T\} = X)\}.$$

An old result of Rothberger says that if $\mathscr{I}$ and $\mathscr{J}$ are orthogonal and invariant under translations, then $\mathrm{cov}_t(\mathscr{I}) \leqslant \mathrm{non}(\mathscr{J})$.

The set $A \subseteq X$ is a $\mathfrak{c}$-*Lusin set* for an ideal $\mathscr{I}$ if

$$|A| = \mathfrak{c} \quad \text{and} \quad (\forall B \in \mathscr{I})(|A \cap B| < \mathfrak{c}).$$

$\mathscr{I}^c$ is the filter dual to $\mathscr{I}$ and

$$\mathscr{I}^* = \{A \subseteq X:\ (\forall B \in \mathscr{I})(A + B \neq X)\}.$$

$\boldsymbol{K}$ and $\boldsymbol{L}$ denote the ideals of first category sets and of Lebesgue null sets, respectively. BOREL, BAIRE, MEASURE denote the families of Borel sets, sets with the Baire property and measurable sets, respectively. BOREL($\mathscr{I}$) is the σ-algebra generated by BOREL $\cup\, \mathscr{I}$.

2. Mycielski ideals. For $K \subseteq \omega$ and $S \subseteq X^\omega$ let $\Gamma_X(S, K)$ be the infinite game of two players in which both players choose the consecutive elements of a sequence $x \in X^\omega$. The choice $x(n)$ is done by the second player if $n \in K$ and by the first player if $n \notin K$. The first player wins if and only if $x \in S$. Now, let $\mathscr{K} = \{K_\sigma:\ \sigma \in 2^{<\omega}\}$ be a system of infinite subsets of ω such that

$$K_\sigma = K_{\sigma * \langle 0 \rangle} \cup K_{\sigma * \langle 1 \rangle} \quad \text{and} \quad K_{\sigma * \langle 0 \rangle} \cap K_{\sigma * \langle 1 \rangle} = \emptyset.$$

Put $\mathfrak{M}_{X,\mathscr{K}} = \{A \subseteq X^\omega$: for every $\sigma \in 2^{<\omega}$ the second player has a winning strategy in the game $\Gamma_X(A, K_\sigma)\}$.

2.1. PROPOSITION. *There are systems $\mathscr{K}$, $\mathscr{K}'$ such that the ideals $\mathfrak{M}_{X,\mathscr{K}}$ and $\mathfrak{M}_{X,\mathscr{K}'}$ are orthogonal.*

Proof. Let $\mathscr{K} = \{K_\sigma:\ \sigma \in 2^{<\omega}\}$ and $\mathscr{K}' = \{K'_\sigma:\ \sigma \in 2^{<\omega}\}$ be such that

$$K_{\sigma_1} \cap K'_{\sigma_2} \neq \emptyset \quad \text{for every } \sigma_1, \sigma_2 \in 2^{<\omega}.$$

Let

$$A = \{x \in 2^\omega:\ (\forall \sigma \in 2^{<\omega})(\exists i \in K_\sigma)(x(i) = 0)\}.$$

Then $A \in \mathfrak{M}_{2,\mathscr{K}}$ and $2^\omega \setminus A \in \mathfrak{M}_{2,\mathscr{K}'}$. Indeed, winning strategies for the second player in the games $\Gamma_2(A, K_\sigma)$ may be described as "put always 1" and in the games $\Gamma_2(2^\omega \setminus A, K'_\sigma)$ as "put always 0".

For the rest of the paper we assume that the system $\mathscr{K}$ is fixed and we write $\mathfrak{M}_X$ instead of $\mathfrak{M}_{X,\mathscr{K}}$.

2.2. THEOREM (J. Mycielski, cf. [M2]). *If $X = \omega$ or $X = 2$, then $\mathfrak{M}_X$ is a σ-ideal such that*:

(a) $\mathfrak{M}_X$ *is orthogonal to* $\boldsymbol{K} \cap \boldsymbol{L}$;
(b) *if* $A \in \mathfrak{M}_X$, *then there exists* $B \in \mathfrak{M}_X \cap \Pi_2^0(X^\omega)$ *such that* $A \subseteq B$;
(c) *there exist* $\mathfrak{c}$ *disjoint closed subsets of* X^ω *that do not belong to* $\mathfrak{M}_X$;
(d) $\mathfrak{M}_X$ *is invariant under translations.*

In the next theorem we describe cardinal invariants of Mycielski ideals.

2.3. THEOREM. *If* $X = \omega$ *or* $X = 2$, *then*
(a) $\mathrm{cov}_t(\mathfrak{M}_X) = \omega_1$ (*hence* $\mathrm{cov}(\mathfrak{M}_X) = \mathrm{add}(\mathfrak{M}_X) = \omega_1$);
(b) $\mathrm{non}(\mathfrak{M}_X) = \mathfrak{c}$ (*hence* $\mathrm{cof}(\mathfrak{M}_X) = \mathfrak{c}$).

Proof. (a) For any function $x \in X^\omega$ the set

$$\{y \in X^\omega\colon (\exists\sigma)(x|K_\sigma = y|K_\sigma)\}$$

is of first category. So there exists a sequence $\{x_\alpha\colon \alpha < \omega_1\}$ of elements of X^ω such that $(\forall\alpha \neq \beta)(\forall\sigma)(x_\alpha|K_\sigma \neq x_\beta|K_\sigma)$. Let

$$A = \{x \in X^\omega\colon (\forall\sigma)(\exists i \in K_\sigma)(x(i) \neq 0)\}.$$

Clearly, $A \in \mathfrak{M}_X$. Note that if $x \notin \bigcup\{A + x_\alpha\colon \alpha < \omega_1\}$, then

$$(\forall\alpha < \omega_1)(\exists\sigma)(\forall i \in K_\sigma)(x_\alpha(i) = x(i)).$$

Since there are countably many σ's and uncountably many α's, there are $\alpha_1 \neq \alpha_2$ and σ such that $x|K_\sigma = x_{\alpha_1}|K_\sigma = x_{\alpha_2}|K_\sigma$. This gives a contradiction. Hence $\bigcup\{A + x_\alpha\colon \alpha < \omega_1\} = X^\omega$.

(b) It is sufficient to note that if $A \subseteq X^\omega$ and $|A| < \mathfrak{c}$, then $A_\sigma|K \neq X^{K_\sigma}$

Mycielski asked in [M2] several questions about $\mathfrak{M}_2$. The affirmative answer to Problem P 649 was given by A. Iwanik (cf. [I]). The next result solves both Problems P 647 and P 649.

2.4. THEOREM. *There exists an analytic set* $Z \subseteq 2^\omega$ *such that if* $B \supseteq Z$ *is a Borel subset of* 2^ω, *then there is a Borel set* $B_1 \subseteq B \setminus Z$ *with* $B_1 \notin \mathfrak{M}_2$. *Consequently,* $Z \notin \mathrm{BOREL}(\mathfrak{M}_2)$ *and* $\mathrm{BOREL}(\mathfrak{M}_2)$ *is not closed under the Suslin operation* $\mathscr{A}$.

Proof. Let $X \in (\Sigma_1^1 \setminus \Pi_1^1)(2^{K_1})$. Let $Z = X \times 2^{\omega \setminus K_1} \in \Sigma_1^1 \setminus \Pi_1^1$. Now, if $B \supseteq Z$ is a Borel subset of 2^ω, then the set

$$A = \{x \in 2^{K_1}\colon (\exists y \in 2^{\omega \setminus K_1})(x, y) \notin B\}$$

is analytic and disjoint from X. Hence there exists $x \in 2^{K_1} \setminus (X \cup A)$. Then, obviously, $B_1 = \{x\} \times 2^{\omega \setminus K_1} \subseteq B$ and $B_1 \notin \mathfrak{M}_2$, $B_1 \cap Z = \emptyset$.

For a given system $\mathscr{K} = \{K_\sigma\colon \sigma \in 2^{<\omega}\}$ one can consider the following σ-ideal: $\mathfrak{M}'_{2,\mathscr{K}} = \{A \subseteq 2^\omega\colon (\forall\sigma)(A|K_\sigma \neq 2^{K_\sigma})\}$. This ideal is a proper subset of $\mathfrak{M}_{2,\mathscr{K}}$. In the case of $\mathfrak{M}'_{X,\mathscr{K}}$ it is easy to give an affirmative partial answer to Problem P 648:

2.5. PROPOSITION. *If* $A \in \mathfrak{M}'_2$, *then there exists a perfect set* $P \subseteq 2^\omega$ *such that* $P + P \cap A \subseteq \{0\}$.

Proof. Let $A \in \mathfrak{M}_2$. For each $i < \omega$ take an incompatible σ_i and choose $x_i \in 2^{K_{\sigma_i}} \setminus A|2^{K_{\sigma_i}}$ such that $x_i(n) = 1$ for some $n \in K_{\sigma_i}$. Put

$$P = \{y \in 2^\omega\colon (\forall i \in \omega)(y|K_{\sigma_i} = x_i \text{ or } y|K_{\sigma_i} = 0)\}.$$

Unfortunately, we do not know whether Proposition 2.5 is true for $\mathfrak{M}_2$, so Problem P 648 remains open. Note that all properties of $\mathfrak{M}_2$ stated above are true also for $\mathfrak{M}'_2$. So it is natural to ask: Are these ideals Borel isomorphic?

3. Other ideals. We shall study the following ideals of subsets of X^ω:

$$\mathfrak{C}_X = \{A \subseteq X^\omega\colon (\forall K \in [\omega]^\omega) \text{ (the second player has a winning strategy in the game } \Gamma_X(A, K))\},$$

$$\mathfrak{P}_X = \{A \subseteq X^\omega\colon (\forall T \in [\omega]^\omega)(A|T \neq X^T)\},$$

$$\mathfrak{D}_X = \{A \subseteq X^\omega\colon (\exists f\colon X^{<\omega} \to X)(\forall x \in A)(\forall^\infty n)(x(n) \neq f(x|n))\}.$$

Note that $A \in \mathfrak{P}_X$ if and only if for every infinite $K \subseteq \omega$ the second player has in the game $\Gamma_X(A, K)$ a winning strategy which does not depend on movements of the first player. Moreover, $\mathfrak{P}_X$ is the intersection of all ideals $\mathfrak{M}'_{X,\mathscr{K}}$ and $\mathfrak{C}_X$ is the intersection of all Mycielski ideals $\mathfrak{M}_{X,\mathscr{K}}$. The ideal $\mathfrak{D}_X$ is the set of those $A \subseteq X^\omega$ such that the second player has a winning strategy in the game on A in which the first player plays finite sequences of elements of X and the second plays single elements. The ideal $\mathfrak{D}_\omega$ appeared implicitly in Mycielski's proof of the determinacy of analytic sets in nonsymmetric games on ω (cf. [M1]).

3.1. Proposition. *The ideals $\mathfrak{D}_X$, $\mathfrak{P}_X$, and $\mathfrak{C}_X$ are translation invariant σ-ideals and $\mathfrak{D}_X \cup \mathfrak{P}_X \subseteq \mathfrak{C}_X \subseteq \mathfrak{M}_X$.*

3.2. Theorem. $\mathfrak{D}_\omega \setminus \mathfrak{P}_\omega \neq \emptyset$.

Proof. Let $A = \{x \in \omega^\omega\colon (\forall^\infty n)(x(n) \neq \max(x|n))\}$. Obviously, $A \in \mathfrak{D}_\omega$. To prove that $A \notin \mathfrak{P}_\omega$ put $T = \{2n\colon n \in \omega\}$. For $y \in \omega^T$ take x such that

$$x|T = y, \quad x(2n+1) = 1 + y(2n+2) + \max(x|2n+1).$$

Then $x \in A$ (and $y \in A|T$).

3.3. Theorem. *There are perfect sets P and Q such that $P \in \mathfrak{P}_2$ and $Q \in \mathfrak{C}_2 \setminus \mathfrak{P}_2$.*

Proof. Let $T_1, T_2 \subseteq 2^{<\omega}$ be two infinite trees in which each finite branch ramifies and on each level exactly one branch ramifies. In T_1, 1 appears only at points of ramification, in T_2 the value is the same between two successive points of ramification. Let $P = [T_1]$ and $Q = [T_2]$. Then P and Q are perfect sets. Moreover, $P \in \mathfrak{P}_2$ and $Q \in \mathfrak{C}_2 \setminus \mathfrak{P}_2$. In fact, take $K = \{k_0, k_1, \ldots\}$. If there exists $\sigma \in T_1$, $\operatorname{lh} \sigma > k_2$ such that $\sigma(k_0) = \sigma(k_1) = \sigma(k_2) = 1$, then we put $f(k_0) = 1$, $f(k_1) = 0$ and $f(k_i) = 1$ for $i > 1$. If such a σ does not exist, then we put $f(k_i) = 1$ for $i \geqslant 0$. Obviously, $f \notin P|K$, hence $P \in \mathfrak{P}_2$. The winning strategy for the second player in $\Gamma_2(Q, K)$ may be described as follows. If the first player plays σ, $\operatorname{lh}\sigma = k_0$, then the second in his first move puts an i such that no extension of $\sigma * i$ in T_2 ramifies at k_1. In the second move, having σ' of length k_1 the second player plays i' such that $\sigma' * i' \notin T_2$. Hence $Q \in \mathfrak{C}_2$. To see that $Q \notin \mathfrak{P}_2$ we consider $K = \{k_0, k_1, \ldots\}$ such that all branches of length $k_i + 1$ ramify before k_{i+1}. Then $Q|K = 2^K$.

3.4. PROPOSITION. *If $A \subseteq X^\omega$ is a $\mathfrak{c}$-Lusin set for $\boldsymbol{K}$ or for $\boldsymbol{L}$, then $A \in \mathfrak{P}_X$.*

Proof. Let $K \in [\omega]^\omega$ and let K_1, K_2 be two disjoint infinite parts of K. Then $Z = \{x: x|K_1 = 0\} \in \boldsymbol{K} \cap \boldsymbol{L}$, and therefore $|Z \cap A| < \mathfrak{c}$. Hence there is a $y' \in X^{K_2}$ such that $y' \notin (Z \cap A)|K_2$. Putting $y|K_2 = y'$ and $y|K_1 = 0$ we obtain $y \notin A|K$.

3.5. PROPOSITION. $\boldsymbol{K}^*(X^\omega) \cup \boldsymbol{L}^*(X^\omega) \subseteq \mathfrak{P}_X$ $(X = \omega, 2)$.

Proof. This is a consequence of the fact that if $A \in \mathscr{I}^*(X^\omega)$, then $A|T \in \mathscr{I}^*(X^T)$, where $T \in [\omega]^\omega$, $\mathscr{I} = \boldsymbol{K}$ or $\mathscr{I} = \boldsymbol{L}$.

4. Properties of $\mathfrak{D}_X$.

4.1. THEOREM. (a) *The ideal $\mathfrak{D}_X$ has a Borel basis, $\mathfrak{D}_X \subseteq \boldsymbol{K}(X^\omega)$ and* $\mathfrak{D}_2 = [2^\omega]^{\leqslant\omega}$.

(b) $\operatorname{cov}(\mathfrak{D}_\omega) = \operatorname{cov}(\boldsymbol{K})$ *and* $\operatorname{non}(\mathfrak{D}_\omega) = \operatorname{non}(\boldsymbol{K})$.

(c) *$\mathfrak{D}_\omega$ is orthogonal to $\boldsymbol{L}$.*

Proof. (a) For $f: X^{<\omega} \to X$ let

$$A_f = \{x \in X^\omega: (\forall^\infty n)(x(n) \neq f(x|n))\}.$$

Then the family of all sets A_f constitutes a basis of the ideal $\mathfrak{D}_X$. Clearly, $A_f \in \boldsymbol{K}$.

(b) Bartoszyński (see [B]) proved that

(1) $\quad \operatorname{cov}(\boldsymbol{K}) = \min\{|X|: X \subseteq \omega^\omega \,\&\, (\forall y \in \omega^\omega)(\exists x \in X)(\forall^\infty n)(y(n) \neq x(n))\}$,

(2) $\quad \operatorname{non}(\boldsymbol{K}) = \min\{|Y|: Y \subseteq \omega^\omega \,\&\, (\forall x \in \omega^\omega)(\exists y \in Y)(\exists^\infty n)(y(n) = x(n))\}$.

Now, if $X = \{x_\alpha: \alpha < \operatorname{cov}(\boldsymbol{K})\}$ is a subset of ω^ω realizing the minimum in (1), then we put

$$A_\alpha = \{x \in \omega^\omega: (\forall^\infty n)(x(n) \neq x_\alpha(n))\}.$$

Obviously, $A_\alpha \in \mathfrak{D}_\omega$ and $\bigcup\{A_\alpha: \alpha < \operatorname{cov}(\boldsymbol{K})\} = \omega^\omega$. This shows that $\operatorname{cov}(\mathfrak{D}_\omega) \leqslant \operatorname{cov}(\boldsymbol{K})$.

If $Y \notin \mathfrak{D}_\omega$, $Y \subseteq \omega^\omega$, then obviously

$$(\forall x \in \omega^\omega)(\exists y \in Y)(\exists^\infty n)(y(n) = x(n)).$$

Hence $\operatorname{non}(\mathfrak{D}_\omega) \geqslant \operatorname{non}(\boldsymbol{K})$.

The inequalities $\operatorname{non}(\mathfrak{D}_\omega) \leqslant \operatorname{non}(\boldsymbol{K})$ and $\operatorname{cov}(\mathfrak{D}_\omega) \leqslant \operatorname{cov}(\boldsymbol{K})$ follow from (a).

(c) If λ_0 is the measure on ω defined by $\lambda_0(\{n\}) = 2^{-(n+1)}$ and λ is the corresponding product measure on ω^ω, then L is the ideal of sets of λ-measure zero. Let

$$A_n = \{x \in \omega^\omega\colon (\exists k)(x(k) \doteq k+n)\} \quad \text{and} \quad X = \bigcap \{A_n\colon n \in \omega\}.$$

Then $\lambda(A_n) \leqslant 2^{-n}$ and $\lambda(X) = 0$. We shall prove that $\omega^\omega \setminus X \in \mathfrak{D}_\omega$. The second player can win if in his n-th move he puts $k+n$, where k is the length of the sequence defined so far. Note that $X \in \mathfrak{P}_\omega$.

Note that since $\mathfrak{D}_\omega$ does not satisfy ccc (see 4.1 (c)), the ideals $\mathfrak{D}_\omega$ and $\boldsymbol{K}$ are not Borel isomorphic. Moreover, $\mathrm{BOREL}(\mathfrak{D}_\omega)$ is not closed under the Suslin operation $\mathscr{A}$ (the example for $\mathfrak{M}_2$ works).

We do not know any reasonable estimation of $\mathrm{add}(\mathfrak{D}_\omega)$ and $\mathrm{cof}(\mathfrak{D}_\omega)$.

5. Properties of $\mathfrak{C}_X$ and $\mathfrak{P}_X$.

5.1. THEOREM. (a) $\mathrm{non}(\mathfrak{C}_X) = \mathrm{non}(\mathfrak{P}_X) = |X|^\omega$.

(b) $\mathrm{add}(\mathfrak{P}_X) = \mathrm{cov}(\mathfrak{P}_X)$.

(c) *If* $|X| \geqslant \omega_1$, *then* $\mathrm{cov}(\mathfrak{C}_X) = \mathrm{cov}(\mathfrak{P}_X) = \omega_1$.

Proof. (c) Let $|X| \geqslant \omega_1$. We choose an increasing sequence $\{X_\alpha\colon \alpha < \omega_1\}$ whose union is X and we put $A_\alpha = X_\alpha^\omega$. Then

$$A_\alpha \in \mathfrak{P}_X \quad \text{and} \quad \bigcup \{A_\alpha\colon \alpha < \omega_1\} = X^\omega.$$

5.2. THEOREM. *If* $X = \omega$ *or* $X = 2$ *and* $\mathrm{cov}(\boldsymbol{K}) = \mathfrak{c}$, *then* $\mathrm{cof}(\mathfrak{C}_X) > \mathfrak{c}$ *and* $\mathrm{cof}(\mathfrak{P}_X) > \mathfrak{c}$.

Proof. Since proofs of both cases are similar, we give the proof only for the ideal $\mathfrak{P}_X$. Let us assume that $\mathrm{cov}(\boldsymbol{K}) = \mathfrak{c}$.

CLAIM. *If* $A \in \mathfrak{P}_X$, $\xi < \mathfrak{c}$, $K_\alpha \in [\omega]^\omega$ *and* $f_\alpha\colon K_\alpha \to X$ *for* $\alpha < \xi$, *then*

$$A \cup \bigcup \{\{f_\alpha\} \times X^{\omega \setminus K_\alpha}\colon \alpha < \xi\} \neq X^\omega.$$

Let $K \in [\omega]^\omega$ be such that $(\forall \alpha < \xi)(|K_\alpha \setminus K| = \omega)$ and $|\omega \setminus K| = \omega$. Let $f \in X^K \setminus A|K$. For $\alpha < \xi$ we put

$$Z_\alpha = (\{f_\alpha\} \times X^{\omega \setminus K_\alpha}) \cap (\{f\} \times X^{\omega \setminus K}).$$

Then Z_α are closed nowhere dense subsets of $\{f\} \times X^{\omega \setminus K}$. Hence

$$(\{f\} \times X^{\omega \setminus K}) \setminus \bigcup \{Z_\alpha\colon \alpha < \xi\} \neq \emptyset \quad \text{and} \quad A \cup \bigcup \{\{f_\alpha\} \times X^{\omega \setminus K_\alpha}\colon \alpha < \xi\} \neq X^\omega,$$

so the Claim is proved.

Let us assume that $\{A_\xi\colon \xi < \mathfrak{c}\}$ is a family of sets from $\mathfrak{P}_X$. We will show that there exists a set $B \in \mathfrak{P}_X$ such that $(\forall \xi < \mathfrak{c})(B \setminus A_\xi \neq \emptyset)$. Let $\{K_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$. Inductively we define $b_\xi \in X^\omega$ and $f_\xi \in X^{K_\xi}$ for $\xi < \mathfrak{c}$. Having defined b_ξ and f_ξ for $\xi < \zeta < \mathfrak{c}$, using the Claim we choose

$$b_\zeta \in X^\omega \setminus (A_\zeta \cup \bigcup \{\{f_\xi\} \times X^{\omega \setminus K_\xi}\colon \xi < \zeta\}) \quad \text{and} \quad f_\zeta \in X^{K_\zeta} \setminus \{b_\xi | K_\zeta\colon \xi \leqslant \zeta\}.$$

We put $B = \{b_\xi\colon \xi < \mathfrak{c}\}$. By construction, $B \setminus A_\xi \neq \emptyset$ and $f_\xi \notin B|K_\xi$. Hence $B \in \mathfrak{P}_X$.

Remark. Note that in the case of $\mathfrak{P}_\omega$ the assumption $\mathrm{cov}(\boldsymbol{K}) = \mathfrak{c}$ can be omitted. In the case of $\mathfrak{P}_2$ the same assumption can be slightly weakened.

5.3. THEOREM. (a) *$\mathfrak{C}_\omega$ and $\mathfrak{P}_\omega$ are orthogonal to $\boldsymbol{K} \cap \boldsymbol{L}$.*

(b) (L. Newelski) $\mathrm{cov}(\mathfrak{C}_\omega) = \mathrm{cov}(\mathfrak{P}_\omega) = \mathrm{add}(\mathfrak{P}_\omega) = \omega_1$.

Proof. (a) From Theorem 4.1 (c) it follows that $\mathfrak{C}_\omega$ and $\mathfrak{P}_\omega$ are orthogonal to $\boldsymbol{L}$. Now we put

$$H_{n+1} = \{\sigma * \underset{\text{ktimes}}{0\ldots0}\colon\ \sigma \in \omega^{<\omega} \,\&\, (k = 1+n+\max\sigma) \,\&\, (\exists k < \mathrm{lh}\,\sigma)(\sigma|k \in H_n)\},$$

$$G_n = \bigcup\{[\sigma]\colon\ \sigma \in H_n\}.$$

Obviously, G_n are open dense subsets of ω^ω, hence $G = \{G_n\colon\ n\in\omega\} \in \boldsymbol{K}^c$. We shall prove that $G \in \mathfrak{C}_\omega$. Let $K = \{k_1, k_2, \ldots\} \in [\omega]^\omega$. For $\sigma\colon\ k_i \to \omega$ we put

$$\phi(\sigma) = 1000 * (k_{i+1} - k_i) * k_i.$$

Then ϕ is a winning strategy for the second player in the game $\Gamma_\omega(G, K)$. Note that, in fact, $G \in \mathfrak{P}_\omega$.

(b) Let $\{K_{\xi,\alpha}\colon\ \xi < \mathfrak{c},\ \alpha < \omega_1\} \subseteq [\omega]^\omega$ be such that

(i) if $(\xi, \alpha) \neq (\zeta, \beta)$, then $K_{\xi,\alpha} \neq K_{\zeta,\beta}$;

(ii) $(\forall K \in [\omega]^\omega)(\forall \alpha < \omega_1)(\exists \xi < \mathfrak{c})(K_{\xi,\alpha} \subseteq K)$.

Let $f_{\xi,\alpha}\colon\ K_{\xi,\alpha} \to \omega$ ($\xi < \mathfrak{c}$, $\alpha < \omega_1$) be a family of functions such that, for $n \in K$, $f_{\xi,\alpha}(n)$ is the first element of $K_{\xi,\alpha}$ greater than n. For every $\alpha < \omega_1$ we put

$$A_\alpha = \{x \in 2^\omega\colon\ \neg(\exists \xi < \mathfrak{c})(f_{\xi,\alpha} \subseteq x)\}.$$

Since for all $K \in [\omega]^\omega$ there is $\xi < \mathfrak{c}$ such that $K_{\xi,\alpha} \subseteq K$, and no extension of $f_{\xi,\alpha}$ belongs to A_α, we have $A_\alpha \in \mathfrak{P}_\omega$. Moreover, $\bigcup\{A_\alpha\colon\ \alpha < \omega_1\} = \omega^\omega$. Indeed, let us assume that

$$x \notin \bigcup\{A_\alpha\colon\ \alpha < \omega_1\}.$$

Then $(\forall \alpha < \omega_1)(\exists \xi < \mathfrak{c})(f_{\xi,\alpha} \subseteq x)$. Since there are uncountably many α's, we can find $\alpha < \beta < \mathfrak{c}$ and $\xi, \zeta < \mathfrak{c}$ such that $f_{\xi,\alpha} \cup f_{\zeta,\beta} \subseteq x$ and $K_{\xi,\alpha}$ and $K_{\zeta,\beta}$ have the same first element. Let n be the first element of $K_{\xi,\alpha} \cap K_{\zeta,\beta}$ for which the next element of one set does not belong to the other set. Then we have $f_{\xi,\alpha}(n) = f(n) = f_{\zeta,\beta}(n)$, a contradiction.

5.4. LEMMA. *If $G \in \Pi^0_2(2^\omega)$ is nonmeagre, then $G \notin \mathfrak{C}_2$.*

Proof. Let us assume that $G = \bigcap G_n \subseteq [\sigma]$ is dense in $[\sigma]$, and G_n are open decreasing subsets of $[\sigma]$. Let $k_0 = \mathrm{lh}\,\sigma$. We can choose σ_1^i ($i = 0, 1$) in such a way that $\mathrm{lh}\,\sigma_1^0 = \mathrm{lh}\,\sigma_1^1$, $[\sigma_1^i] \subseteq G_1$, $\sigma * i \subseteq \sigma_1^i$. We put $k_1 = \mathrm{lh}\,\sigma_1^0$. Having defined $\sigma_n^{i_1\ldots i_n}$ for all $\boldsymbol{i} = (i_1 \ldots i_n) \in 2^n$ we choose $\sigma_{n+1}^{\boldsymbol{i}*i}$ ($i = 0, 1$) all of the same length and such that

$$[\sigma_{n+1}^{\boldsymbol{i}*i}] \subseteq G_{n+1}, \qquad \sigma_n^{\boldsymbol{i}} * i \subseteq \sigma_{n+1}^{\boldsymbol{i}*i}.$$

We put $k_n = \mathrm{lh}\sigma_{n+1}^{i_{n+1}}$. Then sequences $\sigma_n^{i_1 \dots i_n}$ determine a winning strategy for the first player in the game $\Gamma_2(G, K)$, where $K = \{k_1, k_2, \dots\}$. Hence $G \notin \mathfrak{C}_2$.

5.5. COROLLARY. $\mathrm{BAIRE}(2^\omega) \cap \mathfrak{C}_2 \subseteq \boldsymbol{K}$ *and* $\mathrm{BAIRE}(2^\omega) \cap \mathfrak{P}_2 \subseteq \boldsymbol{K}$.

5.6. LEMMA. *If* $F \in \Pi_1^0(2^\omega)$, $\lambda(F) > 0$, *then* $F \notin \mathfrak{C}_2$.

Proof. Let λ be a product measure on 2^ω. Let us assume that F is a closed subset of 2^ω of positive λ-measure.

CLAIM. *If* $\sigma \in 2^{<\omega}$, $\lambda(F \cap [\sigma]) > \lambda([\sigma])/2$, *then there exist* σ_0, σ_1 *such that*

$$\sigma * i \subseteq \sigma_i, \quad \mathrm{lh}\sigma_0 = \mathrm{lh}\sigma_1 \quad \text{and} \quad \lambda(F \cap [\sigma_i]) > \lambda([\sigma_i])/2.$$

Indeed, since $\lambda(F \cap [\sigma * i]) > 0$ $(i = 0, 1)$, we can choose points $x_i \in F \cap [\sigma * i]$ at which the set F has density 1. Now we take $n \in \omega$ so large that

$$\lambda(F \cap [x_i|n]) > \lambda([x_i|n])/2 \quad (i = 0, 1)$$

and we put $\sigma_i = x_i|n$.

Now we iterate the Claim to obtain a set $K \in [\omega]^\omega$ such that the first player has a winning strategy in the game $\Gamma_2(F, K)$ (at the first step we take $x \in F$ at which F has density 1 and choose n such that $\lambda(F \cap [x|n]) > \lambda([x|n])/2$).

5.7. COROLLARY. $\mathrm{MEASURE} \cap \mathfrak{C}_2 \subseteq \boldsymbol{L}$ *and* $\mathrm{MEASURE} \cap \mathfrak{P}_2 \subseteq \boldsymbol{L}$.

For $A \in \mathfrak{C}_2$ and $K \in [\omega]^\omega$ let $\mathrm{STR}_K(A)$ be the set of all winning strategies for the second player in the game $\Gamma_2(A, K)$, i.e.,

$$\mathrm{STR}_K(A) = \{f \in 2^{2^{<\omega}}: (\forall x \in A)(\exists n \in K)(x(n) \neq f(x|n))\}.$$

5.8. LEMMA. *If* $A \in \mathfrak{C}_2$, $K \in [\omega]^\omega$ *and* $\mathrm{STR}_K(A)$ *has the Baire property, then* $\mathrm{STR}_K(A) \in \boldsymbol{K}^c$.

Proof. It is enough to prove that if $G \subseteq 2^{2^{<\omega}}$ is nonmeagre and $G \in \Pi_2^0$, then $G \cap \mathrm{STR}_K(A) \neq \emptyset$. Let us assume that $G = \bigcap G_n$ is dense in $[\sigma]$, where σ is a function from $2^{<m}$ into 2. We may assume that $m = k_0 \in K$. Then for any function $f: 2^m \to 2$ we find $n = n(f) > m$ and a function $\sigma_f^0: 2^{<n} \to 2$ such that $\sigma * f \subseteq \sigma_f^0$ and $[\sigma_f^0] \subseteq G_0$. Let $k_1 \in K$ be greater than all $n(f)$'s. We may assume that $n(f)$'s are equal to k_1. Assume that we have defined $k_0, \dots, k_i$ and $\sigma_{f_0 \dots f_r}^r: 2^{<m} \to 2$ $(m = k_{r+1})$ for all $r < i$ and $f_j: 2^m \to 2$ $(m = k_j)$, $j < r$. We can choose $k_{i+1} \in K$ and sequences $\sigma_{f_0 \dots f_i}^i: 2^{<m} \to 2$ $(m = k_{i+1})$ for $f_j: 2^m \to 2$ $(m = k_j)$, $j \leqslant i$, in such a way that

$$[\sigma_{f_0 \dots f_i}^i] \subseteq G_i \quad \text{and} \quad \sigma_{f_0 \dots f_{i-1}}^{i-1} * f_i \subseteq \sigma_{f_0 \dots f_i}^i.$$

We put $K' = \{k_0, k_1, \dots\}$. Let $\phi \in \mathrm{STR}_{K'}(A)$. We can modify ϕ so that if $f_i = \phi|2^{k_i}$, then $\sigma_{f_0 \dots f_i}^i \subseteq \phi$ for all $i \in \omega$. Then obviously $\phi \in \mathrm{STR}_K(A) \cap G$.

5.9. COROLLARY. *If* $\{A_\alpha: \alpha < \varkappa\} \subseteq \mathrm{BOREL} \cap \mathfrak{C}_2$, $\varkappa < \mathrm{cov}(\boldsymbol{K})$, *then*

$$\bigcup \{A_\alpha: \alpha < \varkappa\} \in \mathfrak{C}_2.$$

Proof. If A_α are Borel, then $\mathrm{STR}_K(A_\alpha) \in \Pi_1^1 \subseteq \mathrm{BAIRE}$.

By similar arguments one can prove the following

5.10. COROLLARY. *The union of less than* $\operatorname{cov}(K)$ *sets from* $\mathrm{BOREL} \cap \mathfrak{P}_2$ *belongs to* $\mathfrak{P}_2$.

Corollary 5.9 could suggest that $\mathfrak{C}_2|\mathrm{BOREL}$ is $\operatorname{cov}(K)$-additive, but this is not true. If $\operatorname{non}(K) < \operatorname{cov}(K)$, and $A \notin K$, $|A| = \operatorname{non}(K)$, then for every Borel $B \supseteq A$ we have $B \notin \mathfrak{C}_2$ (because of Corollary 5.5) but A is a union of $\operatorname{non}(K)$ singletons.

In the case of $\mathfrak{C}_2$ and $\mathfrak{P}_2$ we have no estimates of additivity and covering. The facts above suggest that they can be large. Indeed, coverings can be greater than ω_1 (see Theorem 5.12 below). But an open problem is: Are there any relations between covering or additivity of $\mathfrak{C}_2$ ($\mathfrak{P}_2$) and cardinal coefficients for measure or category?

5.11. LEMMA. (a) *If* $A \in \mathfrak{C}_X$, $K \in [\omega]^\omega$, *then there exists a strategy* $\phi \in \mathrm{STR}_K(A)$ *with* $(\forall x \in A)(\exists^\infty n \in K)(x(n) \neq \phi(x|n))$.

(b) *If* $A \in \mathfrak{P}_X$, $K \in [\omega]^\omega$, *then there exists* $y \in X^K$ *such that*

$$(\forall x \in A)(\exists^\infty n \in K)(x(n) \neq y(n)).$$

In [S], pp. 57–67, Shelah considered the Uniformization Property (UP) for almost disjoint families of subsets of ω:

$\{A_\alpha\colon \alpha < \varkappa\}$ has UP if for every system of functions $f_\alpha\colon A_\alpha \to 2$ there is a function $f\colon \bigcup\{A_\alpha\colon \alpha < \varkappa\} \to 2$ such that for every $\alpha < \varkappa$ we have $f_\alpha \subseteq^* f$.

He built a model of ZFC in which there exists an almost disjoint family of power ω_1 with UP. Obviously, in this model we have $\operatorname{add}(\mathfrak{P}_2) > \omega_1$. Hence

5.12. THEOREM. $\mathrm{CON}(\mathrm{ZFC} + \omega_1 < \operatorname{add}(\mathfrak{P}_2))$.

Remark. By a slight modification of Shelah's forcing, one can obtain a model in which there is an almost disjoint family $\{A_\alpha\colon \alpha < \omega_1\}$ of subsets of ω with the following Strong Uniformization Property:

If $f_\alpha\colon A_\alpha \to \omega$ are functions such that $f_\alpha(n) < 2^{2^n}$ for $n \in A_\alpha$, then there exists a function $f\colon \bigcup\{A_\alpha\colon \alpha < \omega_1\} \to \omega$ such that for each $\alpha < \omega_1$ we have $f_\alpha \subseteq^* f$.

Since winning strategies for the second player in the games $\Gamma_2(A, K)$ may be regarded as functions from K into ω with the property $f(n) < 2^{2^n}$, by Lemma 5.11 we have a model with $\operatorname{cov}(\mathfrak{C}_2) > \omega_1$.

I. Recław has recently observed that the Proper Forcing Axiom implies $\operatorname{cov}(\mathfrak{P}_2) = \mathfrak{c}$.

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