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ON THE STRUCTURE OF EQUATIONALLY COMPLETE VARIETIES. I

BY

DON PIGOZZI (AMES, IOWA)

The equationally complete varieties of groups, rings, semigroups, and many other classical algebraic systems have been completely classified, and their structure is well understood. Recently, these results have been greatly improved so that we now have considerable insight into the structure of all complete varieties that are both locally finite and congruence-permutable. This is a consequence of the work of a number of different authors on para-primal varieties; see Caine [6], Clark and Krauss [7]-[9], MacDonald [17], McKenzie [20], [21], and Quackenbush [28]. Similar results for locally finite and congruence-distributive varieties had been obtained earlier by Day [10]. In contrast, very little is known about complete varieties that fail to be locally finite or have at least one of the two congruence properties; cf. Bol'bot [4], [5], Evans [11], and Pigozzi [22].

In this paper and its sequel [23] we shall show that, generally speaking, this situation cannot be improved. If a complete variety is not required to be locally finite or if it is so required, but is not influenced by some special property like congruence-permutability or congruence-distributivity, then its structure can, with some reservations of no great consequence, be as complex as that of an arbitrary incomplete variety of the same kind. This is demonstrated by constructing functorial isomorphisms between members of a wide class of varieties, treated as categories in the usual way, and subcategories of complete varieties. These functors turn out to preserve many important algebraic and metamathematical properties, in particular all Mal'cev conditions.

Our main result is contained in Theorem 1. We then apply it to solve two open problems that have appeared in the literature; see Corollaries 1 and 2 and the remarks preceding them.

In the following $\mathscr{K}$ represents an arbitrary variety and $\mathscr{L}$ an equationally complete variety. All unexplained terminology from category theory comes from Mac Lane [18].

A functor $\mathfrak{G}$ from $\mathscr{L}$ into $\mathscr{K}$ is *forgetful* if it coincides with the identity function on arrows, and, for all $\mathfrak{A} \in \mathscr{L}$, $\mathfrak{G}\mathfrak{A}$ is a reduct of $\mathfrak{A}$, i.e., $\mathfrak{G}\mathfrak{A}$ is obtained from $\mathfrak{A}$ by disregarding some of its fundamental operations. A functor $\mathfrak{F}$ from $\mathscr{K}$ to $\mathscr{L}$ will be called *polyinjective* whenever it satisfies the following three conditions:

(1) $\mathfrak{F}$ is injective both as an object and arrow function, i.e., it is an isomorphism between $\mathscr{K}$ and a subcategory of $\mathscr{L}$.

(2) $\mathfrak{F}$ as an arrow function preserves injections, i.e., for all $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ and every homomorphism $h : \mathfrak{A} \to \mathfrak{B}$, the homomorphism $\mathfrak{F}h : \mathfrak{F}\mathfrak{A} \to \mathfrak{A}\mathfrak{B}$ is one-one whenever h is.

(3) There exist a forgetful functor $\mathfrak{G}$ from $\mathscr{L}$ into $\mathscr{K}$ and a natural transformation η from the identity functor $\mathfrak{I}$ on $\mathscr{K}$ into $\mathfrak{G} \circ \mathfrak{F}$ such that $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{G}\mathfrak{F}\mathfrak{A}$ is an injection for every $\mathfrak{A} \in \mathscr{K}$.

A variety $\mathscr{K}$ is *variable-uniform*, or *regular*, if, for each identity $\tau = \sigma$ of $\mathscr{K}$, every variable occurring in τ also occurs in σ and vice versa.

THEOREM 1. *Let $\mathscr{K}$ be an arbitrary variety. Then each of the following two conditions is sufficient for $\mathscr{K}$ to be isomorphic to a subcategory of some equationally complete variety $\mathscr{L}$. Moreover, in both cases the functor that establishes the isomorphism can be taken to be polyinjective.*

(i) *$\mathscr{K}$ has a null object; i.e., every member of $\mathscr{K}$ has a unique one-element subalgebra.*

(ii) *$\mathscr{K}$ is variable-uniform; in this case, if $\mathscr{K}$ is generated by a finite algebra, then $\mathscr{L}$ can be taken to be locally finite.*

In [23] it is shown that the following third condition can be added to (i) and (ii):

(iii) *$\mathscr{K}$ has a certain weak form of the amalgamation property, called the flat amalgamation property by Bacsich* [1], *together with a corresponding weak form of the joint embedding property.*

In this case, if $\mathscr{K}$ is of countable similarity type, the complete variety $\mathscr{L}$ containing an isomorphic copy of $\mathscr{K}$ can in fact be taken to be a variety of groupoids, or quasigroups, or any one of a large class of non-associative algebras. Moreover, the functor can be taken to have a right adjoint, so that $\mathscr{K}$ is isomorphic to a coreflective subcategory of $\mathscr{L}$.

Before turning to the proof of the theorem we give some applications; these results were announced in [19] and [25]. In [12] Fajtlowicz poses the problem (P 644) whether or not every equationally complete variety has the amalgamation property. Baldwin [2] proved that every variety that is categorical in some infinite power does have the property. In general however the answer turns out to be negative.

COROLLARY 1. *There exists an equationally complete variety $\mathscr{L}$ that fails to have the amalgamation property. $\mathscr{L}$ can be taken to be either congru-*

ence-permutable, congruence-distributive, or locally finite. However, if it has either of the two congruence properties, it cannot be locally finite.

The last part of the theorem follows from Theorem 4.5 in [10] and from Theorem 1 in [9].

Let $\mathfrak{F}$ be a polyinjective functor from $\mathscr{K}$ into $\mathscr{L}$. Let $\mathfrak{G}$ be the forgetful functor from $\mathscr{L}$ into $\mathscr{K}$ and η the natural transformation from $\mathfrak{I}$ to $\mathfrak{G} \circ \mathfrak{F}$ such that $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{G}\mathfrak{F}\mathfrak{A}$ is injective for every $\mathfrak{A} \in \mathscr{K}$.

Assume $\mathscr{L}$ has the amalgamation property. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathscr{K}$ and let $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{A} \to \mathfrak{C}$ be injections. Then, by (2), $\mathfrak{F}f : \mathfrak{F}\mathfrak{A} \to \mathfrak{F}\mathfrak{B}$ and $\mathfrak{F}g : \mathfrak{F}\mathfrak{A} \to \mathfrak{F}\mathfrak{C}$ are also injective. Since $\mathscr{L}$ is assumed to have the amalgamation property, there exist $\mathfrak{D} \in \mathscr{L}$ and injections $h : \mathfrak{F}\mathfrak{B} \to \mathfrak{D}$ and $k : \mathfrak{F}\mathfrak{C} \to \mathfrak{D}$ such that

$$h \circ \mathfrak{F}f = k \circ \mathfrak{F}g. \tag{4}$$

The homomorphisms

$$(h \circ \eta_{\mathfrak{B}}) : \mathfrak{B} \to \mathfrak{G}\mathfrak{D} \quad \text{and} \quad (k \circ \eta_{\mathfrak{C}}) : \mathfrak{C} \to \mathfrak{G}\mathfrak{D}$$

are injective, and using (4) and the fact η is a natural transformation we have

$$(h \circ \eta_{\mathfrak{B}}) \circ f = h \circ \mathfrak{F}f \circ \eta_{\mathfrak{A}} = k \circ \mathfrak{F}g \circ \eta_{\mathfrak{A}} = (k \circ \eta_{\mathfrak{C}}) \circ g.$$

Thus $\mathscr{K}$ has the amalgamation property whenever $\mathscr{L}$ does.

Since every member of $\mathscr{L}$ has a member of $\mathscr{K}$ as a reduct, it is clear that $\mathscr{L}$ satisfies every Mal'cev condition $\mathscr{K}$ does. In particular, $\mathscr{L}$ is congruence-permutable or congruence-distributive if $\mathscr{K}$ is.

Combining these results with part (i) of the theorem we see that in order to obtain an example of a complete variety $\mathscr{L}$ that is congruence-permutable and fails to have the amalgamation property it suffices to exhibit an arbitrary incomplete $\mathscr{K}$ with these properties which, in addition, has a null object. But such varieties are well known. For instance, we can take $\mathscr{K}$ to be the variety of rings without unit. We can also take various special varieties of groups (see [31] and [32]). Similarly, to obtain an example of a complete, congruence-distributive variety $\mathscr{L}$ without the amalgamation property we can take $\mathscr{K}$ to be any variety of modular lattices with a distinguished element that fails to satisfy the distributive law (see [13]).

To get a complete variety $\mathscr{L}$ without the amalgamation property that is locally finite it suffices to find an incomplete variety $\mathscr{K}$ without the amalgamation property that is generated by a finite variable-uniform algebra. We are not aware of an example of such a variety explicitly occurring in the literature, but one can be obtained with little difficulty by a simple modification of a construction of Howie [14]; see also [15]. The key lemma required is the following:

LEMMA 1. *Assume $\mathfrak{A} = \langle A, \cdot \rangle$ is a finite semigroup and $a, \bar{a} \in A$ any pair of distinct elements such that a fails to divide $\bar{a}$ on the left in $\mathfrak{A}$, i.e., $ax \neq \bar{a}$ for all $x \in A$. Then $\mathfrak{A}$ can be embedded in a finite semigroup $\mathfrak{C}$ with an element c such that $ca = a$ but $c\bar{a} \neq \bar{a}$.*

To prove this assume $\mathfrak{A}$, a, and $\bar{a}$ satisfy the hypothesis. Without loss of generality we also assume that $\mathfrak{A}$ contains an identity element e. Take $\mathfrak{C}$ to be the semigroup of all transformations of A. The semigroup $\mathfrak{A}$ can be naturally identified with a subsemigroup of $\mathfrak{C}$ by identifying each $b \in A$ with the transformation that takes every x into bx. By hypothesis $\bar{a}$ is not in the range of (the transformation identified with) a. Thus there exists a member f of $\mathfrak{C}$ that leaves every element of the range of a fixed while it moves $\bar{a}$. Obviously, the conclusion of the lemma holds when c is taken to be this f.

Take $\mathfrak{A}$ and $\mathfrak{B}$ to be a pair of finite semigroups such that $\mathfrak{A}$ is a subsemigroup of $\mathfrak{B}$, and $\mathfrak{A}$ contains distinct elements a, $\bar{a}$ such that a divides $\bar{a}$ on the left in $\mathfrak{B}$ but fails to do so in $\mathfrak{A}$. For example, $\mathfrak{A}$ can be taken to be a constant semigroup with three elements a, $\bar{a}$, and d such that $xy = d$ for all x, y; $\mathfrak{B}$ can be the extension of $\mathfrak{A}$ by a single element b such that $ab = \bar{a}$ but $xb = bx = d$ in all other situations.

Let $\mathfrak{C}$ be a finite extension of $\mathfrak{A}$, and c an element of $\mathfrak{C}$ such that $\mathfrak{C}$ and c together satisfy the conclusion of the lemma. Then there exists no semigroup which is a common extension of both $\mathfrak{B}$ and $\mathfrak{C}$. For suppose one existed, then we would have $cab = ab = \bar{a}$ and $cab = c\bar{a} \neq \bar{a}$, a contradiction. Take $\mathscr{K}$ to be the variety of semigroups generated by $\mathfrak{B}$ and $\mathfrak{C}$ together with the two-element semilattice. $\mathscr{K}$ is obviously generated by a finite algebra and is variable-uniform since the semilattice is. Clearly, $\mathscr{K}$ fails to have the amalgamation property. This completes the proof of Corollary 1.

The complete variety $\mathscr{L}$ in the corollary can also be taken to be a variety of groupoids or quasigroups; this follows from the results of [23]. However, we know of no example of a complete, locally finite variety of groupoids that fails to have the amalgamation property. **(P 1233)**

Clark and Krauss ask in [7], Problem 2.6, if every complete, locally finite variety $\mathscr{L}$ has a plain subdirect Stone generator, i.e., if there exists an $\mathfrak{A} \in \mathscr{K}$ such that every member of $\mathscr{K}$ is isomorphic to a subdirect power of $\mathfrak{A}$. According to Taylor [29] a variety is *residually small* if its subdirectly irreducible members form a proper class. Obviously, any variety with a plain subdirect Stone generator is residually small, so the following corollary provides a negative answer to the problem of Clark and Krauss [1].

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[1] A solution to this problem was obtained independently by Brian McEvoy; both solutions are based on the constructions of [24].

COROLLARY 2. *There exists an equationally complete and locally finite variety that fails to be residually small.*

Let $\mathscr{K}, \mathscr{L}, \mathfrak{F}, \mathfrak{G}$, and η be as in the second paragraph of the proof of Corollary 1. Let $\mathfrak{A} \in \mathscr{K}$ be subdirectly irreducible. $\mathfrak{F}\mathfrak{A}$ is isomorphic to a subdirect product of a system $\mathfrak{B}_i$, $i \in I$, of subdirectly irreducible members of $\mathscr{L}$. Since $\mathfrak{G}$ is forgetful, it preserves direct products, so $\mathfrak{G}\mathfrak{F}\mathfrak{A}$ is isomorphic to a subalgebra of a direct product of the $\mathfrak{G}\mathfrak{B}_i$. Hence, as $\mathfrak{A}$ is subdirectly irreducible and isomorphic to a subalgebra of $\mathfrak{G}\mathfrak{F}\mathfrak{A}$, it must be isomorphically embeddable in some $\mathfrak{G}\mathfrak{B}_i$. In particular, $|A| \leqslant |B_i|$. This shows that, if $\mathscr{K}$ fails to be residually small, so does $\mathscr{L}$.

It is enough now to exhibit a finite variable-uniform algebra $\mathfrak{A}$ which generates a variety $\mathscr{K}$ that fails to be residually small. Actually, any finite algebra which generates such a variety will do since we can then take $\mathfrak{A}$ to be any finite variable-uniform extension of it (cf. [26]). Examples of finite algebras whose generated variety fails to be residually small are well known, for instance, either of the two 8-element non-Abelian group will do (see [30], Section 14.8). This completes the proof of Corollary 2.

By a result of Quackenbush [27] the variety of Corollary 2 must contain infinitely many pairwise non-isomorphic, finite, subdirectly irreducible algebras each of which generates the variety. But the functor $\mathfrak{F}$ we used to prove the corollary itself provides an explicit construction of such algebras which are, moreover, simple. This gives another solution to Problem 67 of Birkhoff [3].

Proof of Theorem 1. The two parts of the theorem require very different constructions, and the last part of Corollary 1 would seem to indicate that this difference is essential. We first prove part (i).

Let $\mathscr{K}$ be any variety with a null object. The desired equationally complete variety $\mathscr{L}$ will, loosely speaking, be built within $\mathscr{K}$ itself using the following simple idea. Consider an arbitrary equation

$$\tau = \sigma,$$

where $\tau = \tau(v_0, v_1, \ldots, v_{n-1})$ and $\sigma = \sigma(v_0, v_1, \ldots, v_{n-1})$ represent, as usual, terms or polynomial symbols of the formal language of $\mathscr{K}$; the list $v_0, v_1, \ldots, v_{n-1}$ of variable symbols is assumed to include all those which occur in either τ or σ. Let R be a new operation symbol of rank 4 and $P_0, \ldots, P_{n-1}$ new constant symbols. Consider the equations

$$(5) \qquad R\tau(P_0, \ldots, P_{n-1})\sigma(P_0, \ldots, P_{n-1})xy = x, \qquad Rzzxy = y.$$

Any such pair of equations where the symbols $R, P_0, \ldots, P_{n-1}$ are all distinct from the symbols of the language of $\mathscr{K}$ is called a (*local*) *discriminating system* for $\tau = \sigma$ relative to $\mathscr{K}$. We usually suppress explicit reference to the variety $\mathscr{K}$, relying on the context to make it clear. The symbols $R, P_0, \ldots, P_{n-1}$ are referred to as the *basic symbols* of the system (5).

Assume now that $\tau = \sigma$ fails to be an identity of $\mathscr{K}$. A sequence $a_0, \ldots, a_{n-1}$ of elements of an algebra $\mathfrak{A} \in \mathscr{K}$ is called a *system of witnesses* to $\tau = \sigma$ if

$$\tau^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \neq \sigma^{\mathfrak{A}}(a_0, \ldots, a_{n-1}),$$

where $\tau^{\mathfrak{A}}$ and $\sigma^{\mathfrak{A}}$ represent the polynomial operations over $\mathfrak{A}$ naturally associated with τ and σ. It is obvious that an algebra of $\mathscr{K}$ is a model of a discriminating system for $\tau = \sigma$ (or rather can be made into one by interpreting the basic symbols properly) iff it contains a system of witnesses to $\tau = \sigma$. Thus, by adjoining to the (equational) theory of $\mathscr{K}$ (that is, the set of all equations that are identically satisfied in all members of $\mathscr{K}$) a discriminating system for every non-identity, in effect we define a subvariety consisting of those members of $\mathscr{K}$ which contain witnesses to all non-identities. This subvariety turns out to include the essential part of $\mathscr{K}$ and, while not complete, constitutes the first step of a recursive process which ultimately leads to the desired variety $\mathscr{L}$. We now give the details.

In every member of $\mathscr{K}$ the element of the unique one-element subalgebra is denoted by the same symbol 0. Let Φ_0 be the theory of $\mathscr{K}$ and for every $n < \omega$ let Φ_{n+1} be the theory axiomatized by the set of equations obtained by adjoining to Φ_n a discriminating system for each equation not contained in Φ_n (but in the language of Φ_n). We must make sure of course that distinct systems have no basic symbols in common; observe that the language of Φ_{n+1} is the extension of the language of Φ_n by all these basic symbols. Finally, let $\mathscr{L}$ be the class of models of the theory

$$\Phi = \bigcup_{n<\omega} \Phi_n.$$

If Φ_n is consistent, so is Φ_{n+1}. For as observed above any model of Φ_n, such as the free algebra, which contains witnesses for all non-identities of Φ_n, can be made into a non-trivial model of Φ_{n+1}. Thus every Φ_n is consistent, and therefore Φ must be consistent. Φ is also complete. For suppose $\tau = \sigma$ is an equation in the language of Φ but is not contained in Φ. Then, for some $n < \omega$, $\tau = \sigma$ is in the language of Φ_n and $\tau = \sigma \notin \Phi_n$, so Φ_{n+1} and hence Φ itself contains a discriminating system for $\tau = \sigma$. Thus no consistent extension of Φ can contain $\tau = \sigma$; hence every consistent extension is conservative. This shows that $\mathscr{L}$ is an equationally complete variety.

Our next task is to define the polyinjective functor $\mathfrak{F}$ from $\mathscr{K}$ into $\mathscr{L}$. In the following we let I be the set of operation symbols of the language of $\mathscr{K}$, and J the corresponding set for $\mathscr{L}$. Notice that $I \subset J$, and $J \sim I$

consists of all the basic symbols of the discriminating systems that were adjoined to the theory of $\mathscr{K}$ to define $\mathscr{L}$.

Choose an arbitrary non-trivial member $\mathfrak{A}$ of $\mathscr{L}$. For each $\mathfrak{B} \in \mathscr{K}$ let $\mathfrak{F}\mathfrak{B}$ be defined in the following way. The universe of $\mathfrak{F}\mathfrak{B}$ is the Cartesian product $B \times A$ of the universes of $\mathfrak{B}$ and $\mathfrak{A}$. For each operation symbol $Q \in I$ we define its interpretation $Q^{\mathfrak{F}\mathfrak{B}}$ coordinatewise in the usual way: Let n be the rank of Q. For all $b_0, \ldots, b_{n-1} \in B$ and $a_0, \ldots, a_{n-1} \in A$,

$$Q^{\mathfrak{F}\mathfrak{B}}(\langle b_0, a_0\rangle, \ldots, \langle b_{n-1}, a_{n-1}\rangle) = \langle Q^{\mathfrak{B}}(b_0, \ldots, b_{n-1}), Q^{\mathfrak{A}}(a_0, \ldots, a_{n-1})\rangle.$$

This makes sense of course since $I \subset J$. This definition can be summarized succinctly by the equality

$$\mathfrak{G}\mathfrak{F}\mathfrak{B} = \mathfrak{B} \times \mathfrak{G}\mathfrak{A}, \tag{6}$$

where $\mathfrak{G}$ is the forgetful functor from $\mathscr{L}$ to $\mathscr{K}$. We must now define the interpretations of the extra symbols in $J \sim I$. Let

$$\tau(v_0, \ldots, v_{n-1}) = \sigma(v_0, \ldots, v_{n-1})$$

be any equation of type J that is not an identity of $\mathscr{L}$, and let (5) be the discriminating system associated with $\tau = \sigma$. Take

$$P_k^{\mathfrak{F}\mathfrak{B}} = \langle 0, P_k^{\mathfrak{A}}\rangle \quad \text{for each } k < n, \tag{7}$$

and for all $\langle b_i, a_i\rangle \in B \times A$ with $i = 0, 1, 2, 3$ define

$$R^{\mathfrak{F}\mathfrak{B}}(\langle b_0, a_0\rangle, \ldots, \langle b_3, a_3\rangle) = \langle b_i, R^{\mathfrak{A}}(a_0, a_1, a_2, a_3)\rangle, \tag{8}$$

where

$$i = \begin{cases} 2 & \text{if } a_0 = \tau^{\mathfrak{A}}(P_0^{\mathfrak{A}}, \ldots, P_{n-1}^{\mathfrak{A}}) \text{ and } a_1 = \sigma^{\mathfrak{A}}(P_0^{\mathfrak{A}}, \ldots, P_{n-1}^{\mathfrak{A}}), \\ 3 & \text{otherwise.} \end{cases}$$

Because of (6), $\mathfrak{F}\mathfrak{B}$ obviously satisfies all the identities of $\mathscr{K}$. Thus to prove that $\mathfrak{F}\mathfrak{B} \in \mathscr{L}$ we must show that the identities (5) hold.

Using (6)-(8) it is an easy matter to prove by induction on the length of terms that, for any term $\pi = \pi(v_0, \ldots, v_{n-1})$ of type J and for all $\langle b_0, a_0\rangle, \ldots, \langle b_{n-1}, a_{n-1}\rangle \in B \times A$, we have

$$\pi^{\mathfrak{F}\mathfrak{B}}(\langle b_0, a_0\rangle, \ldots, \langle b_{n-1}, a_{n-1}\rangle) = \langle c, \pi^{\mathfrak{A}}(a_0, \ldots, a_{n-1})\rangle$$

for some c contained in the subalgebra of $\mathfrak{B}$ generated by 0 and $b_0, \ldots, b_{n-1}$. Hence by (7) we have

$$\tau^{\mathfrak{F}\mathfrak{B}}(\bar{P}^{\mathfrak{F}\mathfrak{B}}) = \langle 0, \tau^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}})\rangle, \qquad \sigma^{\mathfrak{F}\mathfrak{B}}(\bar{P}^{\mathfrak{F}\mathfrak{B}}) = \langle 0, \sigma^{\mathfrak{B}}(\bar{P}^{\mathfrak{B}})\rangle, \tag{9}$$

where $\bar{P}^{\mathfrak{F}\mathfrak{B}} = \langle P_0^{\mathfrak{F}\mathfrak{B}}, \ldots, P_{n-1}^{\mathfrak{F}\mathfrak{B}}\rangle$ and $\bar{P}^{\mathfrak{X}} = \langle P_0^{\mathfrak{X}}, \ldots, P_{n-1}^{\mathfrak{X}}\rangle$. Take π to be the left-hand side of the first equation of (5). Consequently, for all

$\langle x_0, x_1\rangle$, $\langle y_0, y_1\rangle \in B\times A$ we get

$$\begin{aligned}
&\pi^{\mathfrak{F}\mathfrak{B}}(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle)\\
&\quad = R^{\mathfrak{F}\mathfrak{B}}\big(\langle 0, \tau^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}})\rangle, \langle 0, \sigma^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}})\rangle, \langle x_0, x_1\rangle, \langle y_0, y_1\rangle\big) \quad \text{by (9)}\\
&\quad = \big\langle x_0, R^{\mathfrak{A}}\big(\tau^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}}), \sigma^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}}), x_1, y_1\big)\big\rangle \quad \text{by (8)}\\
&\quad = \langle x_0, x_1\rangle.
\end{aligned}$$

The last equality is a consequence of the fact that $\mathfrak{A}$, as a member of $\mathscr{L}$, identically satisfies the equations of (5). We conclude therefore that $\mathfrak{F}\mathfrak{B}$ identically satisfies the first equation. To see the second equation is also identically satisfied observe that, since $\mathfrak{A}$ satisfies both members of (5) and is non-trivial by assumption,

$$\tau^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}}) \neq \sigma^{\mathfrak{A}}(\bar{P}^{\mathfrak{A}}).$$

Hence, when $\langle b_0, a_0\rangle = \langle b_1, a_1\rangle$, the first alternative of (8) can never hold. Thus, for all $\langle z_0, z_1\rangle$, $\langle x_0, x_1\rangle$, $\langle y_0, y_1\rangle \in B\times A$,

$$\begin{aligned}
R^{\mathfrak{F}\mathfrak{B}}(\langle z_0, z_1\rangle, \langle z_0, z_1\rangle, \langle x_0, x_1\rangle, \langle y_0, y_1\rangle)&\\
= \langle y_0, R^{\mathfrak{A}}(z_1, z_1, x_1, y_1)\rangle &= \langle y_0, y_1\rangle.
\end{aligned}$$

Thus $\mathfrak{F}\mathfrak{B}$ identically satisfies both members of (5), and we infer that $\mathfrak{F}\mathfrak{B} \in \mathscr{L}$ for all $\mathfrak{B} \in \mathscr{K}$.

To define the arrow-function part of $\mathfrak{F}$ consider any $\mathfrak{B}$, $\mathfrak{C} \in \mathscr{K}$ and $h : \mathfrak{B} \to \mathfrak{C}$. Define a function $\mathfrak{F}h$ from $B\times A$ to $C\times A$ by

$$\mathfrak{F}h(\langle b, a\rangle) = \langle hb, a\rangle \quad \text{for all } \langle b, a\rangle \in B\times A.$$

From (6) we immediately see that $\mathfrak{F}h$ preserves interpretations of the operation symbols of I, and it is an easy matter to check it also preserves those of the basic symbols of the discriminating systems; in the case of the constant symbols $P_0, \ldots, P_{n-1}$ we use (7) and the fact that we must have $h0 = 0$ because of the assumption that $\mathfrak{B}$ and $\mathfrak{C}$ have unique one-element subalgebras. Therefore

$$\mathfrak{F}h : \mathfrak{F}\mathfrak{B} \to \mathfrak{F}\mathfrak{C}.$$

Since $\mathfrak{F}$ as an arrow function obviously preserves identities and compositions, $\mathfrak{F}$ is indeed a functor from $\mathscr{K}$ into $\mathscr{L}$.

It is an easy matter to check that $\mathfrak{F}$ is polyinjective. Conditions (1) and (2) are immediate consequences of the definition of $\mathfrak{F}$. For each $\mathfrak{B} \in \mathscr{K}$ let $\eta_{\mathfrak{B}}$ be the function from B into $B\times A$ such that $\eta_B(b) = \langle b, 0\rangle$ for every $b \in B$. Obviously, η satisfies (3). The proof of part (i) is complete.

The proof of part (ii) of the theorem relies heavily on the results of [24], and we begin by describing briefly the constructions from this paper that are important for our purposes. In order to simplify notation we shall assume $\mathscr{K}$ is a variety of groupoids, that is, algebras with a single binary operation.

We assume throughout that $\mathfrak{A} = \langle A, +\rangle$ is a fixed member of $\mathscr{K}$ and $\mathfrak{C} = \langle C, +\rangle$ is any groupoid that includes $\mathfrak{A}$ as a subgroupoid; in particular, $\mathfrak{C}$ may be $\mathfrak{A}$ itself. $\mathfrak{A}$ is called a *Thomas Wolfe subalgebra* of $\mathfrak{C}$ if, for all $x, y \in C$, $x+y \in A$ only if both x and y belong to A; this terminology was introduced by Jónsson and Nelson [16].

We take $\mathfrak{C}_{\propto\infty}$ to be the groupoid obtained from $\mathfrak{C}$ by adjoining two *infinity elements* $\propto$ and ∞ with $\propto$ adjoined first,

$$\mathfrak{C}_{\propto\infty} = \langle C\cup\{\propto, \infty\}, +\rangle,$$

where $x+\propto\; = \;\propto+x = \;\propto$ for all $x \in C\cup\{\propto\}$ and $y+\infty = \infty+y = \infty$ for all $y \in C\cup\{\propto, \infty\}$.

In [24] the structure

$$\mathfrak{H}_{\mathfrak{A}}\mathfrak{C} = \langle C\cup\{\propto, \infty\}, +, Q_a, a\rangle_{a\in A\cup\{\propto\}}$$

is defined by adjoining to $\mathfrak{C}_{\propto\infty}$, for each $a \in A\cup\{\propto\}$, a new nullary operation representing a and a new binary operation Q_a defined by

$$Q_a(x, y) = \begin{cases} y & \text{if } x = a, \\ \infty & \text{otherwise.} \end{cases}$$

For the purposes of the present paper this construction has two important features:

(10) $\mathfrak{H}_{\mathfrak{A}}\mathfrak{A}$ is equationally complete.

This is proved in Theorem 8 of [24].

(11) If $\mathfrak{C}$ is contained in the variety generated by $\mathfrak{A}$ and if $\mathfrak{A}$ is a Thomas Wolfe subalgebra of $\mathfrak{C}$, then $\mathfrak{H}_{\mathfrak{A}}\mathfrak{C}$ is in the variety generated by $\mathfrak{H}_{\mathfrak{A}}\mathfrak{A}$.

This result is an immediate consequence of Lemma 10 of [24] where the identities of $\mathfrak{H}_{\mathfrak{A}}\mathfrak{C}$ are completely characterized in terms of those of $\mathfrak{H}_{\mathfrak{A}}\mathfrak{A}$ and $\mathfrak{C}$.

Consider any groupoid $\mathfrak{B} = \langle B, +\rangle$ whose universe B is disjoint from that of $\mathfrak{A}$. Let l be a new element not contained in either A or B. Define

$$\mathfrak{A}\oplus\mathfrak{B} = \langle A\oplus B, +\rangle,$$

where $A\oplus B = A\cup B\cup\{l\}$, $+$ coincides with the operations of $\mathfrak{A}$ and $\mathfrak{B}$ on their respective universes, and $x+y = l$ whenever x and y are not both in A or both in B. $\mathfrak{A}\oplus\mathfrak{B}$ is a simple example of a *direct sum* of a directed system of algebras (see [26]). It is easy to see that $\mathfrak{A}$ is a Thomas Wolfe subalgebra of $\mathfrak{A}\oplus\mathfrak{B}$. The following result is implied directly by Theorem I of [26]:

(12) If the variety $\mathscr{K}$ is variable-uniform, then $\mathfrak{A}\oplus\mathfrak{B} \in \mathscr{K}$ whenever $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$.

We are now ready to prove part (ii) of the theorem. Let $\mathscr{K}$ be an arbitrary variable-uniform variety. Let $\mathfrak{A}$ be a member of $\mathscr{K}$ which generates $\mathscr{K}$; if $\mathscr{K}$ happens to be generated by a finite algebra, we assume $\mathfrak{A}$ itself is finite. Let $\mathscr{L}$ be the variety generated by $\mathfrak{H}_{\mathfrak{A}}\mathfrak{A}$. By (10), $\mathscr{L}$ is equationally complete, and $\mathscr{L}$ is obviously locally finite if $\mathscr{K}$ is generated by a finite algebra.

The functor $\mathfrak{F}$ is defined as follows. For each $\mathfrak{B} \in \mathscr{K}$ let

$$\mathfrak{F}\mathfrak{B} = \mathfrak{H}_{\mathfrak{A}}(\mathfrak{A}\oplus\mathfrak{B}).$$

(We assume $\mathscr{K}$ has been replaced by an isomorphic full subcategory so that each $\mathfrak{B} \in \mathscr{K}$ is disjoint from $\mathfrak{A}$.) By (12), $\mathfrak{A}\oplus\mathfrak{B}$ is in the variety generated by $\mathfrak{A}$, so taking $\mathfrak{C}$ to be $\mathfrak{A}\oplus\mathfrak{B}$ in (11), we get $\mathfrak{F}\mathfrak{B} \in \mathscr{L}$ for every $\mathfrak{B} \in \mathscr{K}$. For all $\mathfrak{B}, \mathfrak{B}' \in \mathscr{K}$ and homomorphisms $h : \mathfrak{B} \to \mathfrak{B}'$, define $\mathfrak{F}h$ to be the function from $A\oplus B \cup \{\propto, \infty\}$ into $A\oplus B' \cup \{\propto, \infty\}$ such that $(\mathfrak{F}h)(x)$ equals $h(x)$ if $x \in B$, and x otherwise. It is clear $\mathfrak{F}h$ is a homomorphism from $\mathfrak{F}\mathfrak{B}$ into $\mathfrak{F}\mathfrak{B}'$ and $\mathfrak{F}$ is a functor from $\mathscr{K}$ into $\mathscr{L}$. It is also clear conditions (1) and (2) are satisfied. The forgetful functor $\mathfrak{G}$ which takes each algebra of $\mathscr{K}$ into its groupoid reduct clearly maps $\mathscr{L}$ into $\mathscr{K}$; moreover, since $\mathfrak{G}\mathfrak{F}\mathfrak{B} = (\mathfrak{A}\oplus\mathfrak{B})_{\propto\infty}$ for each $\mathfrak{B} \in \mathscr{K}$, it is easily seen that the identity embedding of $\mathfrak{B}$ into $(\mathfrak{A}\oplus\mathfrak{B})_{\propto\infty}$ for all $\mathfrak{B} \in \mathscr{K}$ constitutes a natural transformation satisfying condition (3). Therefore, $\mathfrak{F}$ is a polyinjective functor from $\mathscr{K}$ into $\mathscr{L}$, and the proof of Theorem 1 is complete.

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IOWA STATE UNIVERSITY
AMES, IOWA

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