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ON REDUCTIVE AND PROJECTIVE CLASSES

BY

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The notion of a projective class was introduced by Mal'cev [5]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be axiomatic classes and let K be a set of first order sentences formulated in a language which contains the languages L and M for $\mathfrak{A}$ and $\mathfrak{B}$, respectively. K describes an axiomatic relation Γ between the models $\boldsymbol{A} \in \mathfrak{A}$ and $\boldsymbol{B} \in \mathfrak{B}$. The models of K are all disjoint unions $\boldsymbol{A} \dot{\cup} \boldsymbol{B}$ which admit an interpretation of the set T of relational symbols occurring in K and not belonging to L or M such that the enlarged structure $(\boldsymbol{A} \dot{\cup} \boldsymbol{B})^{\hat{}}$ becomes a model of K. The domain of the axiomatic relation Γ is called a *projective subclass* $\mathscr{P} = \mathfrak{A}\Gamma\mathfrak{B}$ of $\mathfrak{A}$: $\boldsymbol{A} \in \mathscr{P}$ iff $(\boldsymbol{A} \dot{\cup} \boldsymbol{B})^{\hat{}} \models K$ for some $\boldsymbol{B} \in \mathfrak{B}$ and an interpretation of T iff $\boldsymbol{A}$ acts on $\boldsymbol{B}$ in the sense of Γ (see [1] and [2]).

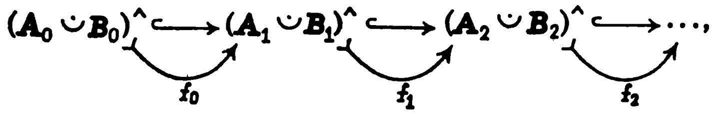
A class $\mathfrak{A}$ of relational structures of type τ is called *pseudo-axiomatic* or a *PC_Δ-class* if it is the class of τ-reducts of an axiomatic class of extended type $\varrho \supseteq \tau$. This notion is due to Tarski. We use here the term reductive instead of pseudo-axiomatic. Clearly, reductive classes are projective.

In this note we first point out that a projective class $\mathscr{P}$ contains a "big" reductive class. But a projective class is also the intersection of reductive classes containing it. The latter statement is a consequence of Keisler's ingenious limit ultraproduct theorem.

THEOREM 1. *Let $\mathscr{P}$ be a projective subclass of the axiomatic class $\mathfrak{A}$ and assume that $\mathscr{P}$ contains only infinite models. Then $\mathscr{P}$ contains a reductive class $\mathfrak{R}$ such that any $\boldsymbol{A} \in \mathscr{P}$ can be elementarily embedded into a model $\boldsymbol{A}^{\hat{}} \in \mathfrak{R}$.*

Proof. Let the projective class $\mathscr{P}$ be defined by the axiomatic relation Γ between $\mathfrak{A}$ and the axiomatic class $\mathfrak{B}$. We select in $\mathscr{P}$ the class $\mathfrak{R}$ of all $\boldsymbol{A} \in \mathscr{P}$ which admit a realization of Γ by a model $\boldsymbol{B} \in \mathfrak{B}$ with $\operatorname{card}(\boldsymbol{A}) \geqslant \operatorname{card}(\boldsymbol{B})$. We first show that $\mathfrak{R}$ is reductive. We extend for this purpose the language L_τ for $\mathfrak{A}$ by the set of relational symbols in M, by the set T of the connecting relations for Γ and by one unary relational symbol R. Let the set $K^{\hat{}}$ say that a τ-reduct $\boldsymbol{A}$ of a model $\boldsymbol{A}^{\hat{}}$ of $K^{\hat{}}$ is a model in $\mathfrak{A}$, the subset defined by R is a model $\boldsymbol{B}$ in $\mathfrak{B}$ and that $\boldsymbol{A}$ acts on $\boldsymbol{B}$ in the sense of Γ. Then $\boldsymbol{A}^\wedge \in \mathrm{Md}\, K^\wedge$ iff $\boldsymbol{A} \in \mathfrak{R}$.

For $\boldsymbol{A}_0 \in \mathscr{P}$, we have a $\boldsymbol{B}_0 \in \mathfrak{B}$ such that $(\boldsymbol{A}_0 \dot\cup \boldsymbol{B}_0)^\wedge \models K$. We can find an elementary extension $(\boldsymbol{A}_1 \dot\cup \boldsymbol{B}_1)^\wedge$ of $(\boldsymbol{A}_0 \dot\cup \boldsymbol{B}_0)^\wedge$ together with an injective map $f_0\colon \boldsymbol{B}_0 \rightarrowtail \boldsymbol{A}_1$ such that $A_1 - \mathrm{im}(f_0)$ is infinite. There is an elementary extension $(\boldsymbol{A}_2 \dot\cup \boldsymbol{B}_2)^\wedge$ of $(\boldsymbol{A}_1 \dot\cup \boldsymbol{B}_1)^\wedge$ together with an injective map f_1: $\boldsymbol{B}_1 \rightarrowtail \boldsymbol{A}_2$, where f_1 extends f_0 and where $A_2 - \mathrm{im}(f_1)$ is infinite. Continuing in this way, we get an elementary chain of extensions and a chain of injections:



$$f_0 \subset f_1 \subset \ldots$$

By the well-known Tarski-Vaught theorem, the union $\boldsymbol{A}_\omega$ of the $\boldsymbol{A}_i$ acts on the union $\boldsymbol{B}_\omega$ of the $\boldsymbol{B}_i$ in the sense of Γ and, moreover, we have an injection

$$f_\omega\colon \boldsymbol{B}_\omega \rightarrowtail \boldsymbol{A}_\omega, \qquad f_\omega = \bigcup_{i=1}^{\infty} f_i.$$

If $\mathfrak{B}$ contains only infinite models, then we can interchange in the last proof the roles of $\boldsymbol{A}$ and $\boldsymbol{B}$ successively, to get an elementary embedding

$$(\boldsymbol{A}_0 \dot\cup \boldsymbol{B}_0)^\wedge \hookrightarrow (\overline{\boldsymbol{A}} \dot\cup \overline{\boldsymbol{B}})^\wedge, \qquad \text{where } \mathrm{card}(\overline{\boldsymbol{A}}) = \mathrm{card}(\overline{\boldsymbol{B}}).$$

Call a realization of Γ *balanced* if $\boldsymbol{A}$ acts on itself, $\boldsymbol{A}\Gamma\boldsymbol{B}$, the set B a double of A. Then we have the following

COROLLARY. *Let Γ be an axiomatic relation between the axiomatic classes $\mathfrak{A}$ and $\mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ contain only infinite models. Let $\mathfrak{R}$ be the reductive subclass of $\mathscr{P} = \mathfrak{A}\Gamma\mathfrak{B}$ given by all models which admit a balanced realization of Γ. Then any $\boldsymbol{A} \in \mathfrak{A}$ can be elementarily embedded into a model $\overline{\boldsymbol{A}} \in \mathfrak{R}$.*

By Theorem 1, any projective class of infinite models contains a reductive class dense with respect to the axiomatic closure. Denote by $\mathscr{C}_{\mathscr{P}}$ the topological closure defined by the projective classes [3], and by $\mathscr{C}_{\mathfrak{R}}$ the topological closure given by the reductive classes.

THEOREM 2 (Keisler [4]). *Let $\mathfrak{A}$ be a class of relational structures of type τ. Then $\boldsymbol{A} \in \mathscr{C}_{\mathfrak{R}}\mathfrak{A}$ iff $\boldsymbol{A}$ is isomorphic to a limit ultraproduct with factors in $\mathfrak{A}$.*

Recall that, for a family $(\boldsymbol{A}_i)_{i \in I}$, ultrafilter $\mathfrak{u}$ over $I \times J$ and filter $\mathfrak{G}$ over $I \times J \times J$, the limit ultraproduct

$$\boldsymbol{A} = \mathop{\mathsf{X}}_{(i,j) \in I \times J} \boldsymbol{A}_{ij}/\mathfrak{u}\,|\,\mathfrak{G}, \qquad \boldsymbol{A}_{ij} = \boldsymbol{A}_i,$$

is defined as the subset of all equivalence classes a of $\mathop{\mathsf{X}}_{i,j} A_{ij}/\mathfrak{u}$ which contain a representative f such that

$$\mathrm{eq}(f) = \{(i, j, j') \mid f(i, j) = f(i, j')\} \in \mathfrak{G}.$$

THEOREM 3. *A projective class $\mathscr{P}$ is closed under limit ultraproducts.*

Proof (cf. [1]). Let $(A_i)_{i\in I}$ be a family of models in $\mathscr{P}$, where $\mathscr{P}$ is a projective class defined by the relation Γ between the axiomatic class $\mathfrak{A}$ and the axiomatic class $\mathfrak{B}$. We then have $B_i \in \mathfrak{B}$ such that $(A_i \dot\cup B_i)^{\wedge} \models K$, K describing Γ. It is easy to see that there is a canonically given isomorphism

$$\mathop{\mathsf{X}}_{i,j} (A_{ij} \dot\cup B_{ij})/\mathfrak{u} \mid \mathfrak{G} \cong \mathop{\mathsf{X}}_{i,j} A_{ij}/\mathfrak{u} \mid \mathfrak{G} \dot\cup \mathop{\mathsf{X}}_{i,j} B_{ij}/\mathfrak{u} \mid \mathfrak{G},$$

and this isomorphism can be made to one of the extended structures, showing that the limit ultraproduct

$$A = \mathop{\mathsf{X}}_{i,j} A_{ij}/\mathfrak{u} \mid \mathfrak{G} \in \mathfrak{A}$$

acts on

$$B = \mathop{\mathsf{X}}_{i,j} B_{ij}/\mathfrak{u} \mid \mathfrak{G} \in \mathfrak{B}$$

in the sense of Γ.

By Keisler's theorem we now have the following

COROLLARY. *Let $\mathscr{P}$ be a projective class and let $A \notin \mathscr{P}$. Then there is a reductive class $\mathfrak{R} \supseteq \mathscr{P}$ with $A \notin \mathfrak{R}$, $\mathscr{C}_{\mathfrak{R}} = \mathscr{C}_{\mathscr{P}}$.*

In [3] we proved that $A' \in \mathscr{C}_{\mathscr{P}}\mathfrak{A}$ iff there is an ultraproduct

$$A = \mathop{\mathsf{X}}_{i\in I} A_i/\mathfrak{u}, \quad A_i \in \mathfrak{A},$$

such that any projective class containing not only A, but also almost all factors, contains also A'. If equation (*) in Keisler [4], p. 219, holds for $\mathfrak{A}$, then one can dispense with the restriction for $\mathscr{P}$ to contain almost all factors; but the truth of (*), in general, is unsettled.

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