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ON THE LATTICE PACKING-COVERING RATIO OF FINITE DIMENSIONAL NORMED SPACES

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Let X be an n-dimensional normed space, $n = 1, 2, \ldots$ Let B be the closed unit ball and d the distance in X. Let L be a lattice in X, i.e., an additive subgroup generated by n linearly independent vectors. We define the packing and covering numbers of L:

$$\begin{aligned} p(L) &= \tfrac{1}{2}\min\{\|x-y\|:\ x, y \in L,\ x \neq y\} \\ &= \tfrac{1}{2}\min\{\|x\|:\ x \in L\setminus\{0\}\} \\ &= \sup\{t > 0:\ \text{the sets } x+tB,\ x \in L, \text{ are pairwise disjoint}\}; \\ c(L) &= \max\{d(x, L):\ x \in X\} \\ &= \min\{t > 0:\ \text{the sets } x+tB,\ x \in L, \text{ form a covering of } X\}. \end{aligned}$$

Next, we define the lattice packing-covering ratio of X:

$$r(X) = \inf\left\{\frac{c(L)}{p(L)}:\ L \text{ is a lattice in } X\right\}.$$

The problem is to estimate $r(X)$ from above. In other words, we try to find a lattice L which is simultaneously as "discrete" and as "dense" as possible. Let us write

$$r_n = \sup\{r(X):\ X \text{ is an } n\text{-dimensional normed space}\}$$

for $n = 1, 2, \ldots$ It is clear that $r_n < \infty$ for every n. Furthermore, it turns out that

$$\sup_n r_n < \infty.$$

This problem was investigated by Butler [2] and Bourgain [1]. Their methods, however, are rather complicated. In this paper we present a simple proof of the following statement:

THEOREM. *For each finite dimensional normed space X, one has $r(X) < 3$.*

This estimate is not the best possible. Butler [2] proved that $r_n < 2+o(1)$ as $n \to \infty$. For an estimate from below, see Remark 1.

The above theorem is an immediate consequence of the following fact:

PROPOSITION. *Let L be a lattice in a finite dimensional normed space X. Then there exists a lattice L' in X with*

$$L \subset L' \quad \textit{and} \quad c(L') < 3p(L') = 3p(L).$$

Proof. Let B be the closed unit ball and d the distance in X. If $c(L) < 3p(L)$, we may take $L' = L$. So, assume that

$$c(L) \geqslant 3p(L). \tag{1}$$

Write

$$s = \min\{t > 0\colon\ \tfrac{1}{3}L \subset L + tB\}.$$

We have $\frac{1}{3}L \subset L + sB$, whence

$$3^{-i-1}L \subset 3^{-i}L + 3^{-i}sB \quad (i = 0, 1, 2, \ldots).$$

So, for each $m = 1, 2, \ldots$, we obtain

$$\begin{aligned} 3^{-m-1}L &\subset 3^{-m}L + 3^{-m}sB \subset 3^{-m+1}L + 3^{-m+1}sB + 3^{-m}sB \subset \ldots \\ &\subset L + (1 + 3^{-1} + \ldots + 3^{-m})sB = L + \tfrac{3}{2}(1 - 3^{-m-1})sB. \end{aligned}$$

Evidently, this implies that $X \subset L + \frac{3}{2}sB$. Thus $c(L) \leqslant \frac{3}{2}s$. Applying (1), we get $2p(L) \leqslant s$. So, there is some $x_0 \in \frac{1}{3}L$ such that

$$d(x_0, L) \geqslant 2p(L). \tag{2}$$

Let L_1 be the lattice generated by L and x_0, i.e.,

$$L_1 = L \cup (L + x_0) \cup (L + 2x_0).$$

From (2) it follows easily that $p(L_1) = p(L)$. If $c(L_1) < 3p(L)$, we take $L' = L_1$. In the other case, we may repeat the above argument to obtain a lattice $L_2 \supset L_1$ with $L_2 \neq L_1$ and $p(L_2) = p(L)$, and so on. After a finite number of steps we shall obtain a lattice with the desired properties.

Remark 1. Let l_2^n denote the n-dimensional euclidean space and let V_n be the volume of the unit ball in l_2^n. It follows directly from the definitions of $r(l_2^n)$ and of Hermite's constant γ_n (see [4], p. 386) that

$$r(l_2^n) > 2\gamma_n^{-1/2} V_n^{-1/n}.$$

The results of Kabatyanskiĭ and Levenshteĭn [5] imply that

$$\gamma_n \leqslant 0.872 \frac{n}{\pi e}(1 + o(1)) \quad \text{as } n \to \infty$$

(see also [7]). Hence, after simple calculations, we obtain $r(l_2^n) > 1.514$ for large n.

Remark 2. Let S be a centrally symmetric compact convex body in $\boldsymbol{R}^n$. By a *lattice packing* of S we mean a family $\{S + u\}_{u \in L}$, where L is a lattice in $\boldsymbol{R}^n$ and the sets $u + \operatorname{Int} S$, $u \in L$, are pairwise disjoint. The *density* of the lattice packing

$\{S+u\}_{u\in L}$ is defined as $V(S)/d(L)$, where $V(S)$ is the volume of S and $d(L)$ denotes the determinant of L (see [4], p. 23). From the Minkowski–Hlawka theorem it follows that there is a lattice packing of S with density greater than 2^{-n} (cf. [4], p. 525). However, constructive methods allow one only to find lattice packings with density δ such that

$$\liminf_{n\to\infty} \log_2 \sqrt[n]{\delta}$$

is between -1.694 and -2.218 (see [3], Chapter 8, Section 7.5), assuming the generalized Riemann hypothesis (recently, Rush [6] found constructions which give density δ such that

$$\liminf_{n\to\infty} \log_2 \sqrt[n]{\delta} \geqslant -1,$$

provided that S is symmetric through each of the coordinate hyperplanes). The argument applied in the proof of the above theorem allows one to find in a finite number of steps a set of generators of a lattice L such that $\{S+u\}_{u\in L}$ is a lattice packing of S with density $\delta > 3^{-n}$; then

$$\log_2 \sqrt[n]{\delta} > -\log_2 3 > -1.585.$$

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