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HARMONIC FUNCTIONS ON HOMOGENEOUS SPACES OF COMMUTATOR INDUCED EXTENSIONS OF COMPACT GROUPS

BY

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0. Introduction. In an earlier paper [6], we showed that the Hilbert transform C, as defined in the context of the 1-torus T by

$$Cx(t) = P\cdot V\cdot \int_T \operatorname{Cot} s/2\, x(t-s)\,ds,$$

can be characterized among a certain class of convolution operators K essentially by the requirement that there exists a composition operator $Gx(t) = x(g(t))$, induced by a homeomorphism of T satisfying certain smoothness conditions, for which the commutator $GK-KG$ is a rank-one operator with range in the constant functions. The operator $K = C$, in turn, determines the group $\mathcal{G}$ of all such operators and one finds that certain concepts associated with the potential theory and the harmonic analysis of the disk arise naturally as a result of this commutator condition. Indeed, as we shall show in a detailed example in section 7, the group $\mathrm{SU}(1, 1)/\{\pm I\}$ emerges naturally as isomorphic to this subgroup $\mathcal{G}$ of the semigroup of all composition operators almost commuting with C in the sense defined above. The unit disk appears as the homeomorph of the image $\{G^* 1 \mid G \in \mathcal{G}\}$ of $\mathcal{G}$ in $L^2(T)$ and as such is a homogeneous space of G. The Poisson kernel is $G^* 1$, where G^* is the Hilbert space operator adjoint to G and 1 is the constant function 1. That the Poisson kernel appears thus is, of course, well known. Perhaps less known is the fact that the conjugate Poisson kernel makes its appearance as the element $y(G)$ of $L^2(T)$ such that $GCx - CGx = (x, y(G))1$.

This paper is motivated by this example and provides a general Hilbert space setting for the study of the implications for harmonic analysis of the rank-one commutator condition. While there are developments in our formulation that are parallel to those of the theory of harmonic functions on symmetric spaces (see [1]–[3]), our point of view is fundamentally different. We start with a compact topological group B (which may be finite), a unitary representation and an intertwining operator. Using a rank-one commutator condition and a concept of harmonic function patterned after Godement's mean value characterization (see [3]–[5]), we obtain as a consequence: a group of operators containing the unitary group, homogeneous spaces, and what we call harmonic and conjugate harmonic functions which are, in a sense, "extensions" of "boundary elements". We exhibit several versions of the Poisson integral formula, discuss connections with harmonic analysis in certain invariant subspaces of $L^2(B)$ and touch on how boundary elements may be recovered. We conclude with the example that prompted this effort to find an appropriate abstract setting in which its significance may be further explored.

1. Harmonic functions of operators. Given a compact group B and a unitary representation U of B in a Hilbert space X, let us suppose that the representation U leaves some element y_0 of X fixed and that any element that is fixed by all U_s, $s \in B$, is necessarily a scalar multiple of y_0. We shall say that 1 is a *simple eigenvalue* for the representation U. Let $\mathscr{W}$ be the set of all bounded operators that leave y_0 fixed. It is evident that $\mathscr{W}$ is closed under complex affine combinations and that, multiplicatively, $\mathscr{W}$ is a subsemigroup of the multiplicative group of $\mathscr{B}(X)$, the Banach algebra of all bounded operators on X, and contains $\mathscr{U} = \{U_s | \ s \in B\}$. By a *complex affine combination* of two operators S_1, S_2 we mean the operator $z_1 S_1 + z_2 S_2$, where z_1 and z_2 are complex numbers such that $z_1 + z_2 = 1$.

1.1. Definition. Let $\mathscr{S}$ be a subsemigroup of $\mathscr{W}$ containing $\mathscr{U}$. We call a real- or complex-valued function F defined on $\mathscr{S}$ *harmonic* if $F(S' U_s S)$ is continuous as a function of s for each S and S' in $\mathscr{S}$ and

$$F(S) = \int_B F(S' U_s S)\, ds$$

for all S' in $\mathscr{S}$.

1.2. Theorem. *Let $\mathscr{S}$ be any subsemigroup of $\mathscr{W}$ that contains $\mathscr{U}$. The function F, defined by $F(S) = (Sx, y_0)$, is harmonic on $\mathscr{S}$ for all $x \in X$.*

This theorem is an immediate consequence of the following lemma:

1.3. Lemma. *Let $A \in \mathscr{B}(X)$ and let U be a unitary representation of B in X with the property that 1 is a simple eigenvalue of U and let y_0 be a corresponding normalized eigenvector. Then*

$$(Ay_0, y)(x, y_0) = \int_B (AU_s x, y)\, ds$$

for each x and y in X.

Proof. For each $y \in X$, the integral $\int_B (AU_s x, y)\, ds$ is a bounded linear functional in x. Thus, there exists an element z_y in X such that

$$(x, z_y) = \int_B (AU_s x, y)\, ds.$$

Upon replacing x by $U_t^{-1}x$, we find, for each $t \in B$ and all $x \in X$, that

$$(x, U_t z_y) = (U_t^{-1} x, z_y) = \int_B (AU_s U_{t^{-1}} x, y)\, ds$$

$$= \int_B (AU_{st^{-1}} x, y)\, ds = (x, z_y).$$

This last equality is a consequence of the invariance of the Haar measure ds. Since the eigenvalue 1 is simple, it follows that $z_y = (y, Ay_0)\, y_0$.

1.4. COROLLARY. *If $Ay_0 = y_0$, then*

$$(x, y_0)(y_0, y) = \int_B (AU_s x, y)\, ds.$$

The theorem now follows from the lemma and the corollary as applied in the chain of equalities:

$$\int_B F(S' U_s S)\, ds = \int_B (S' U_s Sx, y_0)\, ds = (Sx, y_0)(S' y_0, y_0) = (Sx, y_0) = F(S).$$

It is evident that any harmonic function F on $\mathscr{S}$ is left-invariant under translations by $\mathscr{U}$ in the sense that $F(U_t S) = F(S)$ for all $S \in \mathscr{S}$ and all $t \in B$.

Thus, harmonic functions on $\mathscr{S}$ are constant on the equivalence classes in $\mathscr{S}$ defined by the equivalence relation: $S' \sim S$ if $S' = U_t S$ for some $t \in B$. These classes will be called *right cosets* (mod $\mathscr{U}$).

2. The intertwining operator K as analog of the Hilbert transform. Let $K \in \mathscr{B}(X)$ be an intertwining operator for the representation U. Let (y_0) denote the one-dimensional subspace spanned by y_0. It will be understood in all that follows that $Ky_0 = 0$. Let

$$\mathscr{S} = \{S \in \mathscr{W} \mid SKx - KSx \in (y_0) \text{ for each } x \in X\}.$$

One verifies that $\mathscr{S}$ is an affine subspace of $\mathscr{B}(X)$, that is, $\mathscr{S}$ is closed under complex affine combinations and that, multiplicatively, $\mathscr{S}$ is a subsemigroup of $\mathscr{W}$. In fact, $\mathscr{S}$ is a translate by the identity I of a weakly closed subalgebra of $\mathscr{B}(X)$. Since all the operators concerned are bounded, there exists, for each $S \in \mathscr{S}$, a unique element $y(S)$ in X such that

$$SKx - KSx = (x, y(S))\, y_0$$

for all $x \in X$.

The following two properties of $y(S)$ can also be readily verified:

(i) If $S \in \mathscr{S}$ and S^{-1} exists in $\mathscr{B}(X)$, then $S^{-1} \in \mathscr{S}$ and

$$(x, y(S^{-1})) = (S^{-1} x, y(S)).$$

(ii) For each S and S' in $\mathscr{S}$, we have

$$(x, y(SS')) = (S' x, y(S)) + (x, y(S')).$$

We remark in passing that this last relation is a logarithmic version of what has been termed the cross-homomorphism or 1-cocycle property. That is, if we define $\sigma(S, x) = e^{(x, y(S))}$, then

$$\sigma(SS', x) = \sigma(S, S'x)\sigma(S', x)$$

(see [2], p. 274, and [9], p. 202, line 9). No use will be made of this property in the present paper.

The set $\mathscr{G} = \{S \in \mathscr{S} | \ S^{-1} \in \mathscr{B}(X)\}$ is a subgroup of $\mathscr{S}$ and contains $\mathscr{U}$. However, $\mathscr{G}$ is no longer an affine subspace of $\mathscr{B}(X)$.

In the example of section 7, where K is the Hilbert transform on the torus, the element $y(S)$ turns out to be the conjugate Poisson kernel. In the present setting we have the following theorem:

2.1. THEOREM. *The function F, defined by $F(S) = (x, y(S))$, is harmonic on $\mathscr{S}$ for each $x \in X$.*

Proof. For each S, S' in $\mathscr{S}$, we have the relation

$$((SKx - KSx), y_0) = (x, y(S)) = \int_H (x, y(S' U_s S))\, ds - \int_H (U_s Sx, y(S'))\, ds.$$

From corollary 1.4 with $A = I$ and x replaced by Sx we see that this last integral equals $(y_0, y(S'))(Sx, y_0)$, which vanishes because of the vanishing of the first factor under the hypothesis that $Ky_0 = 0$.

Remark. We have proved theorem 2.1 under the hypothesis only that $Ky_0 = 0$. If we were to assume that $K^* y_0 = 0$ as well, then a more direct proof is available, which also has the virtue of exhibiting the connection between the analog K of the Hilbert transform and the analog $y(S)$ of the conjugate Poisson kernel in a more familiar way. That is, if we put $z = Kx$, then

$$(x, y(S)) = (SKx, y_0) - (KSx, y_0) = (Sz, y_0) - (Sx, K^* y_0) = (Sz, y_0),$$

which is harmonic according to theorem 1.2.

3. The algebraic structure of $\mathscr{S}$.

3.1. THEOREM. *If $K^* y_0 = 0$, then each S in $\mathscr{S}$ has the decomposition*

$$S = T + N = T(I + N),$$

where T and N are operators having the following properties:

(1) *T commutes with K and $T^* y_0 = y_0$.*

(2) *$N \in \mathscr{B}(X)$, $N^2 = 0$, $Ny_0 = 0$, and* range $N = (y_0)$.

(3) *If S is invertible, then so is T.*

(4) *If the null space of K^* is (y_0), then T and N are unique.*

Proof. Given $S \in \mathscr{S}$, set

$$T = (I - Q_0)S + Q_0 \quad \text{and} \quad N = Q_0(S - I),$$

where Q_0 is the projection of X on (y_0). The properties listed above are easily verified.

Let $\mathcal{N}$ denote the set of all bounded nilpotent operators of index 2 and range (y_0). For each y in the orthogonal complement of (y_0), define N_y by $N_y x = (x, y) y_0$. Let $\mathcal{G}$ be defined as in section 2.

3.2. THEOREM. *If $K^* y_0 = 0$, then the set $I + \mathcal{N}$ is a maximal abelian normal subgroup of $\mathcal{G}$. The mapping $y \to N_y$ is an isometric isomorphism of the additive group of the orthogonal complement of (y_0) onto $I + \mathcal{N}$, equipped with the metric*

$$d(I+N,\ I+N') = \|(I+N)-(I+N')\|$$

of the relative uniform operator topology.

The proof relies on the results of theorem 3.1 applied to elements of $\mathcal{G}$ and is a straightforward verification. We omit the details.

Let $\mathcal{T} = \{T \in \mathcal{G} |\ TK = KT\}$.

3.3. LEMMA. *If T and T^* are both in $\mathcal{T}$, then T has a unique polar decomposition $T = UP$, where U is unitary, P is positive, and U and P are in $\mathcal{T}$.*

Proof. Let $S = T^* T$ and let P be the unique positive square root of S. Since $Sy_0 = y_0$ and since P is the strong limit of polynomials in S, it follows that $Py_0 = \lambda y_0$ for some complex number λ. Since $P^2 y_0 = \lambda^2 y_0 = Sy_0 = y_0$, it follows that $\lambda^2 = 1$, and since P is positive, $\lambda = 1$. The limit P is also self-adjoint and commutes with K. Since T and T^* are supposed invertible, so is S. Hence P is invertible with inverse $P^{-1} = S^{-1} P$. Writing $U = TP^{-1}$, we obtain a unitary operator that commutes with K and leaves y_0 fixed.

3.4. THEOREM. *If the null space of K^* is (y_0) and if the group $\mathcal{T}$ is self-adjoint in the sense that $T \in \mathcal{T}$ implies $T^* \in \mathcal{T}$, then $\mathcal{G}$ is the semidirect product:*

$$\mathcal{G} = \mathcal{V}\mathcal{P}(I+\mathcal{N})$$

of the following subgroups of $\mathcal{G}$: $\mathcal{V}$ = all unitary operators in $\mathcal{T}$; $\mathcal{P}$ = all positive operators in $\mathcal{T}$; $I+\mathcal{N}$ = all operators of the form $I+N$, where N is nilpotent of index 2, vanishes on (y_0) and has range (y_0), provided also that $\mathcal{P}$ is abelian.

Proof. The theorem is a consequence of theorems 3.1, 3.2, lemma 3.3 and the following observation: if

$$G = VP\,(I+N) \qquad \text{and} \qquad G' = V'\,P'(I+N'),$$

then

$$\begin{aligned} GG' = VP(I+N)\,V'\,P'(I+N') &= VPV'\,P'(I+N'') \\ &= VV'(V'^{-1}\,PV')\,P'(I+N''), \end{aligned}$$

where $V'^{-1}\,PV'$ is positive.

4. Representation of a class of functions harmonic on $\mathscr{S}$. The theorems of the previous section on the structure of $\mathscr{S}$ enable us to prove a representation and uniqueness theorem for a class of harmonic functions on $\mathscr{S}$.

4.1. DEFINITION. A function F defined on the affine subspace $\mathscr{S}$ will be called *affine* if, for every S and S' in $\mathscr{S}$ and complex numbers a and a' such that $a+a' = 1$, the function F has the property

$$F(aS+a'S') = aF(S)+a'F(S').$$

We shall also say that the affine function F is *bounded* on $\mathscr{S}$ if

$$|F(S)-F(I)| \leqslant M$$

for some M and all S in $\mathscr{S}$ satisfying $\|S-I\| \leqslant 1$.

4.2. LEMMA. *If $K^* y_0 = 0$ and F is a harmonic and bounded affine function on $\mathscr{S}$, then there exist a z in X and a complex number c such that*

$$F(S) = (Sz, y_0)+c \quad \textit{for all } S \textit{ in } \mathscr{S}.$$

Proof. Since F is a bounded affine function on $\mathscr{S}$, the functional L defined by $L(S-I) = F(S)-F(I)$ on the linear subspace $\mathscr{S}-I$ of $\mathscr{B}(X)$ is a bounded linear functional there, and so by the Hahn–Banach theorem is extendable to $\mathscr{B}(X)$ (we shall continue to denote the extension by L). For each $x \in X$, we may write uniquely $x = (x, y_0)y_0+y$, where $y \in (y_0)^{\perp}$. We define a bounded linear functional r on X by setting

$$r(x) = \bar{L}((y_0, x)I+N_y),$$

where the bar denotes complex conjugation, and N_y is the operator previously defined. By the Riesz theorem, there exists a $z \in X$ such that

$$L((y_0, x)I+N_y) = (z, x)$$

for all $x \in X$ with $y = x-(x, y_0)y_0$. In particular, $L(I) = (z, y_0)$ and $L(N_y) = (z, y)$. We may now apply the results of the previous section.

Let $S \in \mathscr{S}$ and let $S = T+N_y$, where $y = N^* y_0$ and $T^* y_0 = y_0$. From the definition of L on $\mathscr{S}-I$ we know that

$$F(S) = L(S-I)+F(I) = L(S)-L(I)+F(I),$$

where we have made use of the linearity of the extension of L. Thus $F(S) = L(S)+c$. It remains to show that $L(S) = (Sz, y_0)$.

To this end, we first observe that

$$L(S) = L(T+N_y) = L(T)+L(N_y) = L(T)+(z, y).$$

Now, since F is harmonic, so is L. Accordingly,

$$L(S) = \int_B L(S' U_s S)\,ds \quad \text{for all } S' \text{ and } S \text{ in } \mathscr{S}.$$

In particular, taking $S' = Q_0$, the projection on (y_0), and $S = T$, we obtain

$$L(T) = \int_B L(Q_0 U_s T)\,ds = \int_B L(Q_0)\,ds = L(Q_0)$$

since $Q_0 U_s T = Q_0$. Furthermore

$$L(Q_0) = \int_B L(U_s Q_0)\,ds = \int_B L(Q_0 U_s)\,ds = \int_B L(U_s)\,ds = L(I).$$

Thus

$$L(S) = (z, y_0)+(z, y) = (z, y_0+y).$$

Since $T^* y_0 = y_0$, we have the equalities

$$(Sz, y_0) = \big(T(I+N_y)z, y_0\big) = \big(z,(I+N_y^*)y_0\big) = (z, y_0+y).$$

This completes the proof of the lemma.

If we put $x = z+cy_0$, we find that $(Sx, y_0) = F(S)$ for all $S \in \mathscr{S}$. It is easily seen that x is unique with this property. Indeed, if $(Sx, y_0) = 0$ for all $S \in \mathscr{S}$, then using the particular choices $S = I$ and $S = I+N_y$, we find that x is orthogonal to all elements of the Hilbert space X. The following theorem has now been proved:

4.3. Theorem. *If $K^* y_0 = 0$, then every bounded affine harmonic function F on $\mathscr{S}$ can be represented in the form*

$$F(S) = (Sx, y_0)$$

with a unique $x \in X$. Thus, there is a one-to-one correspondence between the Hilbert space X and the set of all bounded affine harmonic functions on $\mathscr{S}$.

In what follows, we shall suppose that $K^* y_0 = 0$ as well as $Ky_0 = 0$. The truth of the following assertions follows from the theory of Hilbert spaces and the general theory of topological groups.

The group $\mathscr{U}$ is compact in the strong and weak operator topology in $\mathscr{S}$. The group $\mathscr{T}$ is weakly closed. If $\mathscr{G}$ is given a topology in which it is a topological group, then $\mathscr{G}$ acts transitively on the right coset space $\mathscr{T}\backslash\mathscr{G}$ with action $p \cdot G = \mathscr{T} G' G$, where $p \in \mathscr{T}\backslash\mathscr{G}$ and $G' \in p$. In particular, $\mathscr{G}$ is a topological group in the relative uniform operator topology.

With the help of theorems 3.1 and 3.2, the following additional facts about $\mathscr{G}$ in the relative uniform topology can be proved:

(1) $\mathscr{T}$ is topologically isomorphic to the factor group $\mathscr{G}/(I+\mathscr{N})$.

(2) The quotient space $\mathscr{T}\backslash\mathscr{G}$ of right cosets (mod $\mathscr{T}$) is homeomorphic to a subspace of $I+\mathscr{N}$ in the relative uniform topology, and therefore to a topological subspace of the orthogonal complement of (y_0), which, because of the decomposition $\mathscr{G} = \mathscr{T}(I+\mathscr{N})$, is precisely the set $\{(G^*-I)y_0 \mid G \in \mathscr{G}\}$. The proof of the first of these is standard. We shall only state and prove a version of the second of these assertions.

4.4. THEOREM. *Let $\mathscr{G}$ have the relative uniform topology. The quotient space $\mathscr{T}\backslash\mathscr{G}$ is homeomorphic to the topological subspace $(\mathscr{G}^*-I)\,y_0$ of $(y_0)^{\perp}$.*

Proof. We define the mapping

$$\alpha\colon (\mathscr{T}\backslash\mathscr{G})\to((\mathscr{G}^*-I)\,y_0)$$

by $\alpha(p)=(G^*-I)\,y_0$, where $p\in\mathscr{T}\backslash\mathscr{G}$ and $G\in p$. This mapping is well defined, for if $G=T(I+N)$ and $T'\in\mathscr{T}$, then

$$((T'G)^*-I)\,y_0=(G^*\,T'^*-I)\,y_0=(G^*-I)\,y_0.$$

Now, if $(G_1^*-I)\,y_0=(G_2^*-I)\,y_0$, then if

$$G_1=T_1(I+N_1)\quad\text{and}\quad G_2=T_2(I+N_2),$$

it follows that $N_1^*\,y_0=N_2^*\,y_0$ and $N_1=N_2$. Thus $G_1=T_1\,T_2^{-1}\,G_2$.

Since $\mathscr{G}$ has the uniform operator topology, the mapping $G\to(G^*-I)\,y_0$ is continuous. It follows that the mapping α is also continuous.

It remains to show that the mapping α is open on its range. That is, if v is an open set in $\mathscr{T}\backslash\mathscr{G}$, then $\alpha(v)$ is open in $\alpha(\mathscr{T}\backslash\mathscr{G})\subset X$. This requirement is equivalent to the following: for every $G\in\mathscr{G}$ and $\varepsilon>0$, there is a $\delta>0$ such that if $G'\in\mathscr{G}$ and

$$\|G'^*\,y_0-G^*\,y_0\|<\delta,$$

then there exists $T''\in\mathscr{T}$ such that

$$\|T''\,G'-G\|<\varepsilon.$$

We show now that this criterion is met.

For a given $G=T(I+N)\in\mathscr{G}$ and $\varepsilon>0$, let $\delta=\varepsilon/\|T\|$. Let $G'=T'(I+N')$ and suppose $\|G'^*\,y_0-G^*y_0\|<\delta$. It follows from theorem 3.2 that

$$\|(I+N')-(I+N)\|=\|N'^*\,y_0-N^*\,y_0\|<\varepsilon/\|T\|.$$

Thus

$$\begin{aligned}\|T(T')^{-1}\,G'-G\|&=\|T(I+N')-T(I+N)\|\\&\leqslant\|T\|\,\|(I+N')-(I+N)\|<\varepsilon.\end{aligned}$$

5. Subgroups of $\mathscr{G}$ and Poisson spaces.

5.1. DEFINITION. A pair $(\mathscr{H},R)$, where $\mathscr{H}$ is a subgroup of $\mathscr{G}$, and R is a subset of $y_0+(y_0)^{\perp}$, will be called a *Poisson pair* if the following three conditions are satisfied:

(1) $y_0\in R$;

(2) $H^*R\subset R$ for all $H\in\mathscr{H}$;

(3) for each $x\in R$, there is an $H\in\mathscr{H}$ such that $H^*\,y_0=x$.

It is evident that if $(\mathscr{H},R)$ is a Poisson pair, then $R=\mathscr{H}^*\,y_0$ and that each subgroup $\mathscr{H}$ of $\mathscr{G}$ is a first element of some Poisson pair. The second element R of a Poisson pair will be called the *Poisson set of $\mathscr{H}$ in the Hilbert space X*. The set R is not necessarily an affine subset of the affine subspace $y_0+(y_0)^\perp$, and R may be a Poisson subset for more than one subgroup $\mathscr{H}$. One verifies that the mapping $(x, H) \to H^* x$ is jointly continuous with respect to the norm topology of X and the relative uniform topology of $\mathscr{H} \subset \mathscr{B}(X)$, and defines a transitive action of $\mathscr{H}$ on R. We write $x \cdot H = H^* x$. According to theorem 3.1, each $H \in \mathscr{H}$ has the unique decomposition

$$H = T + N_y = T(I+N_y),$$

where $T^* y_0 = y_0$ and $N_y x = (x, y) y_0$, with $y = (H^* - I) y_0$. However, there is no guarantee that either T or $I+N_y$ belongs to $\mathscr{H}$. Of course, if either belongs, then so does the other.

5.2. Definition. A triple $(M, \pi, \mathscr{H})$ will be called a *Poisson space* if M is a topological space, $\mathscr{H}$ a subgroup of $\mathscr{G}$ containing $\mathscr{U}$, and π a homeomorphism of M onto the Poisson set $\mathscr{H}^* y_0$ of $\mathscr{H}$. The Hilbert space valued function π will be called the *Poisson kernel* of the Poisson space.

It will be understood that, unless the contrary is stated, the topology of the Poisson set $\mathscr{H}^* y_0$ is the relative norm topology in X.

One verifies that $\mathscr{H}$ acts transitively on M with respect to the jointly continuous mapping

$$(p, H) \to p \cdot H = \pi^{-1}\big(H^* \pi(p)\big),$$

so that M is a homogeneous space for $\mathscr{H}$. There is an element $p_0 \in M$ such that $\pi(p_0) = y_0$ and the stability group in $\mathscr{H}$ of p_0 is $\mathscr{T} \cap \mathscr{H}$.

We will presently give a few general examples of subgroups $\mathscr{H}$, Poisson sets and Poisson spaces. First we remark that the chief distinction being made through the use of the words *set* and *space* is that the Poisson *set* is defined for any subgroup $\mathscr{H}$ of $\mathscr{G}$ and is a subset of the Hilbert space X, while a Poisson *space* involves a subgroup $\mathscr{H}$ containing $\mathscr{U}$ and need only be a homeomorph of the corresponding Poisson *set*. Secondly, if $(M, \pi, \mathscr{H})$ is a Poisson space, then the right coset space $\mathscr{T} \cap \mathscr{H} \backslash \mathscr{H}$ in the quotient topology is a homogeneous space for $\mathscr{H}$ and is in one-to-one correspondence with the homogeneous space M, and therefore with $\mathscr{H}^* y_0$, and while the mapping β, defined by

$$\beta\big((\mathscr{T} \cap \mathscr{H}) H\big) = H^* y_0,$$

of $\mathscr{T} \cap \mathscr{H} \backslash \mathscr{H}$ onto $\mathscr{H}^* y_0 \subset X$ is continuous and one-to-one, yet if it is not open onto its range, the coset space will not be homeomorphic to the Poisson set $\mathscr{H}^* y_0$. Thus, if $\mathscr{H}$ is a subgroup of $\mathscr{G}$ and contains $\mathscr{U}$, then the triple $(T \cap \mathscr{H} \backslash \mathscr{H}, \beta, \mathscr{H})$ will be a Poisson space if and only if the mapping $H \to H^* y_0$ is open (see [8], p. 65).

5.3. EXAMPLES. (1) The triple $(\mathcal{T}\backslash\mathcal{G}, \alpha, \mathcal{G})$ is a Poisson space. This is a direct consequence of theorem 4.4, since $\mathcal{G}^* y_0$ is homeomorphic to $(\mathcal{G}^*-I)y_0$.

(2) If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and contains $\mathcal{U}$, then $(\mathcal{H}^* y_0$, identity, $\mathcal{H})$ is a Poisson space.

(3) The Poisson set of $\mathcal{G}$ is $y_0+(y_0)^\perp$. This is also the Poisson set of $I+\mathcal{N}$ as well as of other subgroups of $\mathcal{G}$, e.g., of $\mathcal{H} = \mathcal{U}(I+\mathcal{N})$.

(4) If X_1 is any subspace of $(y_0)^\perp$ that is $\mathcal{U}$-invariant, then

$$\mathcal{H} = \{U_s+N_y|\ y\in X_1,\ s\in B\}$$

is a subgroup of $\mathcal{G}$ with Poisson set y_0+X_1. The action of $\mathcal{H}$ on y_0+X_1 is given by

$$\begin{aligned}(y_0+x)\cdot H_{(s,y)} &= H^*_{(s,y)}(y_0+x) = (U_{s^{-1}}x+N_y^*)(y_0+x)\\ &= y_0+(U_{s^{-1}}x+y), \quad \text{where } H_{(s,y)} = U_s+N_y.\end{aligned}$$

5.4. DEFINITION. If $(M, \pi, \mathcal{H})$ is a Poisson space and F is a complex-valued function defined on M, then F is said to be *harmonic* on $(M, \pi, \mathcal{H})$ if for each $p\in M$ and each $H\in\mathcal{H}$:

(1) $F(p\cdot U_s H)$ is continuous as a function of s;

(2) $F(p_0\cdot H) = \int_B F(p\cdot U_s H)\,ds$.

The following proposition is evident:

5.5. PROPOSITION. *If $(M, \pi, \mathcal{H})$ is a Poisson space, then for each $x\in X$ the function $F_x(\cdot) = (x, \pi(\cdot))$ is harmonic on $(M, \pi, \mathcal{H})$ and satisfies*

$$F_x(p\cdot H) = F_{Hx}(p).$$

Not every function F harmonic on a Poisson space $(M, \pi, \mathcal{H})$ need have a representation $F(p) = (x, \pi(p))$ for some $x\in X$, as the following example shows.

5.6. EXAMPLE. Let B be the 1-torus. Let X be the Hardy space $H^2(B)$. Let U be the regular representation $U_s x(t) = x(t+s)$ in $H^2(B)$. Let

$$e_n(t) = e^{int} \quad \text{and} \quad Kx = -i\big(x-(x, e_0)e_0\big).$$

The operator K is an intertwining operator for the representation U in $H^2(B)$. Let m be a positive integer. For $w\in C$, $s\in B$, let $H_{(s,w)}$ be the operator defined by the equations

$$H_{(s,w)}e_n = \begin{cases}\sum_{k=0}^{n}\binom{n}{k}e^{iks}w^{n-k}e_k & \text{for } n\leqslant m,\\ e^{ins}e_n & \text{for } n>m.\end{cases}$$

Let $\mathcal{H} = \{H_{(s,w)}|\ s\in B,\ w\in C\}$, where C denotes the complex numbers.

One sees that

$$H_{(s,w)}Kx - KH_{(s,w)}x = -i(x, \bar{w}e_1 + \bar{w}^2 e_2 + \ldots + \bar{w}^m e_m)e_0.$$

Thus $\mathscr{H}$ is a subgroup of

$$\mathscr{G} = \{G \in \mathscr{B}(H^2(B)) |\; G^{-1} \in \mathscr{B}(H^2(B)),\; Ge_0 = e_0,\; GKx - KGx \in (e_0)\}$$

and contains $\mathscr{U}$, since $U_s = H_{(s,0)}$. The Poisson set of $\mathscr{H}$ is the set of all elements in $H^2(B)$ of the form

$$\pi(z) = e_0 + \bar{z}e_1 + \bar{z}^2 e_2 + \ldots + \bar{z}^m e_m, \quad z \in C.$$

Also,

$$H^*_{(s,w)}\pi(z) = \pi(ze^{is} + w).$$

Thus, if we take $M = C$, the triple $(C, \pi, \mathscr{H})$ is a Poisson space, and the action of $\mathscr{H}$ on C is given by

$$z \cdot H_{(s,w)} = \pi^{-1}(H^*_{(s,w)}\pi(z)) = ze^{is} + w.$$

In particular, $0 \cdot H_{(s,w)} = w$.

If F is harmonic on $(C, \pi, \mathscr{H})$, then for all $s \in B$, z and w in C

$$F(0 \cdot H_{(s,w)}) = \int_B F(z \cdot U_t H_{(s,w)})\, dt.$$

Since

$$F(z \cdot U_t H_{(s,w)}) = F(ze^{it} \cdot H_{(s,w)}) = F(ze^{i(t+s)} + w),$$

it follows from translation invariance of the integral that F is harmonic on $(C, \pi, \mathscr{H})$ if and only if F has the ordinary mean value property

$$F(w) = \int_B F(ze^{it} + w)\, dt,$$

whenever the integrand is continuous in t. In particular, any entire function has this property. Thus if $F(w) = w^{m+1}$, for example, then F is harmonic on $(C, \pi, \mathscr{H})$ but there does not exist an $x \in H^2(B)$ such that $F(w) = (x, \pi(w))$. However, every polynomial of degree $\leqslant m$ has this representation.

6. An invariant subspace of $L^2(B)$ and boundary values. In all that follows, we shall assume that the Hilbert space X is separable and that the representation U is the direct sum, *without repetitions*, of countably many irreducible subrepresentations $\{U^{(k)}\}$; that is, we suppose that $U^{(j)}$ is not equivalent to $U^{(k)}$ for $j \neq k$. This is true, in particular, if B is abelian and U is the right regular representation:

$$R_t f(s) = f(st).$$

Let $X = \oplus X^{(k)}$ denote the corresponding direct sum decomposition of X

with $X^{(0)} = (y_0)$. For each $k \geqslant 0$ select, once and for all, an orthonormal basis $(y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)})$ and let

$$w^{(k)} = y_1^{(k)} + y_2^{(k)} + \ldots + y_{n_k}^{(k)}.$$

For each $x \in X$, define

$$f_x^{(k)}(s) = (U_s x, w^{(k)}) = (U_s^{(k)} x_k, w^{(k)}),$$

where x_k denotes the component of x in $X^{(k)}$.

With the aid of the orthogonality relations

$$\int_B (U_s^{(j)} x, y)\overline{(U_s^{(k)} u, v)}\, ds = \frac{1}{\sqrt{n_j n_k}}(x, u)\overline{(y, v)}\,\delta_{jk}$$

of the theory of group representations, one may prove the following proposition identifying X with a subspace of $L^2(B)$:

6.1. Proposition. *For each $x, y \in X$, we have the following relations:*

(1) $\int_B f_x^{(j)}(s)\overline{f_y^{(k)}(s)}\, ds = (x_j, x_k)$ *for all* $j \geqslant 0$, $k \geqslant 0$.

(2) $\sum_{k=0}^{\infty} f_x^{(k)}$ *converges to an element* f_x *in* $L^2(B)$.

(3) $\int_B f_x(s)\overline{f_y(s)}\, ds = (x, y)$.

The mapping $x \to f_x$ is a linear isometry of X onto a subspace of $L^2(B)$.

The proofs of these assertions rely on the orthogonality relations and completeness; they are entirely computational and are omitted.

The image $X' = \{f_x \mid x \in X\}$ of X in $L^2(B)$ is an invariant subspace of the right regular representation R of B in $L^2(B)$. Indeed, given $f_x \in X'$, we see that

$$R_t f_x = \sum_{k=0}^{\infty} R_t f_x^{(k)} = \sum_{k=0}^{\infty} f_{U_t x}^{(k)} = f_{U_t x}.$$

We also observe that, in general, if A is any bounded operator on X, then the formula $\hat{A} f_x = f_{Ax}$ defines an operator $\hat{A}$ on X' and $\|\hat{A}\| = \|A\|$.

Let $(M, \pi, \mathscr{H})$ be a Poisson space. The formula

$$F(p) = (Hx, y_0) = (x, H^* y_0) = \big(x, \pi(p)\big)$$

has its counterpart

$$F(p) = \int_B f_{Hx}(s)\overline{f_{y_0}(s)}\, ds = \int_B f_{Hx}(s)\, ds = \int_B f_x(s)\overline{f_{H^* y_0}(s)}\, ds$$

$$= \int_B f_x(s)\overline{f_{\pi(p)}(s)}\, ds,$$

where use has been made of the fact that $f_{y_0}(s) = 1$ for all $s \in B$.

6.2. Definition. The (in general) complex-valued function P defined on $M \times B$ by

$$P(p, s) = \overline{f_{\pi(p)}(s)}$$

is called the *Poisson kernel* in X' of the Poisson space $(M, \pi, \mathscr{H})$.

We thus obtain the familiar looking Poisson integral formula

$$F(p) = \int_B f_x(s) P(p, s)\, ds.$$

In particular, for $x = y_0$, we get

$$1 = \int_B P(p, s)\, ds \qquad \text{for all } p \in M.$$

We now show that under certain hypotheses the Poisson kernel of a Poisson space is harmonic. If we set

$$v^{(n)} = w^{(0)} + w^{(1)} + \ldots + w^{(n)},$$

we may conveniently restate formula (2) of proposition 6.1, as applied to $x = \pi(p)$, as follows:

$$\lim_{n \to \infty} (U_{s^{-1}} v^{(n)}, \pi(p)) = P(p, s) \qquad \text{for each } p \in M$$

in the sense of $L^2(B)$.

That $(U_{s^{-1}} v^{(n)}, \pi(p))$ is harmonic on $(M, \pi, \mathscr{H})$ for each $s \in B$ and each $n \geqslant 0$ is clear.

6.3. Theorem. *The Poisson kernel $P(p, s)$ is harmonic on $(M, \pi, \mathscr{H})$ for each $s \in B$ for which the sequence $\{(U_{s^{-1}} v^{(n)}, \pi(p))\}$ converges uniformly to $P(p, s)$ on every compact subset of M.*

Proof. For any given $H \in \mathscr{H}$ and $p \in M$, the set $\{p \cdot U_t H \mid t \in B\}$ is a continuous image of B in M, and so is compact. If $s \in B$ is as in the hypothesis, then the sequence $\{(U_{s^{-1}} v^{(n)}, \pi(p \cdot U_t H))\}$, for fixed p, s and H, converges uniformly to $P(p \cdot U_t H, s)$ which is, consequently, continuous in t. It follows that

$$\lim_{n \to \infty} \int_B (U_{s^{-1}} v^{(n)}, \pi(p \cdot U_t H))\, dt = \int_B P(p \cdot U_t H, s)\, dt.$$

Since $(U_{s^{-1}} v^{(n)}, \pi(p))$ is harmonic, we have

$$\int_B (U_{s^{-1}} v^{(n)}, \pi(p \cdot U_t H))\, dt = (U_{s^{-1}} v^{(n)}, \pi(p_0 \cdot H)),$$

and since

$$\lim_{n \to \infty} (U_{s^{-1}} v^{(n)}, \pi(p_0 \cdot H)) = P(p_0 \cdot H, s)$$

for this particular $s \in B$, we arrive at

$$P(p_0 \cdot H, s) = \int_B P(p \cdot U_t H, s)\, dt.$$

6.4. COROLLARY. *Under the same hypotheses, the Poisson integral of any $f \in L^2(B)$ is harmonic.*

The proof consists in applying the definitions, theorem 6.3 and interchange of the order of integration.

In the theory of harmonic functions on symmetric spaces (see [4] and [5]), questions concerning representation of harmonic functions as Poisson integrals are formulated in terms of boundaries, as are questions concerning recovery of boundary values. We may fit our formulation into this setting, in a manner suggested by the situation in the motivating example. A class of problems thus arises: if B is a compact group, U a representation satisfying the conditions imposed in the present section, K an intertwining operator, and $\mathscr{H}$ a subgroup of $\mathscr{G}$ consisting of composition operators induced by homeomorphisms of B onto itself, does there exist an element $H \in \mathscr{H}$ such that the sequence $\{H^n\}$ of iterates converges and provides a "pointwise" solution to the recovery problem? Such is the case in the motivating example [6], p. 125. Beyond this, so far as I know, this is an unsolved problem.

The following definitions are modelled after those in [5].

6.5. DEFINITION. Let $(M, \pi, \mathscr{H})$ be a Poisson space. A homogeneous space M' for $\mathscr{H}$ is said to be a *weak boundary* for $(M, \pi, \mathscr{H})$ if B acts transitively on M' through U; that is, if for each q, $q' \in M'$ there exists an $s \in B$ such that $q' = q * U_s$, where $(q, H) \to q * H$ is the action of $\mathscr{H}$ on M'.

6.6. DEFINITION. A weak boundary for $(M, \pi, \mathscr{H})$ is called a *boundary* if there exist a sequence $\{H_n\}$ in $\mathscr{H}$ and a $q_1 \in M'$ such that

$$\lim_{n \to \infty} \int_B f(q * U_s H_n)\, ds = f(q_1)$$

for each continuous function f on M' and $q \in M'$.

If M' is a weak boundary for $(M, \pi, \mathscr{H})$, then there exists a unique normalized U-invariant measure m defined on M' such that

$$\int_{M'} f(q)\, dm(q) = \int_B f(q_0 * U_s)\, ds$$

for all $f \in L^1(M')$ and arbitrary $q_0 \in M'$. Still following [5], one says that a function F defined on M is a *Poisson integral* of f: $M' \to C$ if

$$F(p_0 \cdot H) = \int_{M'} f(q * H)\, dm(q).$$

It is easy to show that if F is a Poisson integral of a continuous f: $M' \to C$ in the above sense, then F is harmonic on $(M, \pi, \mathscr{H})$ in the sense of definition

5.4. The proof depends on the definitions, transitivity of the action under $\mathscr{U}$ and $\mathscr{H}$ and interchange of the order of integration.

The underlying topological space of B is a weak boundary for $(M, \pi, \mathscr{H})$ in the special case where there is an action $(s, H) \to s * H$ of $\mathscr{H}$ on B with the following two properties:

(1) $s * U_t = st$ for all $s, t \in B$.

(2) For each $x \in X, f_{Hx}(s) = f_x(s * H)$ for almost all $s \in B$, where f_x is as in proposition 6.1 (2).

The space B will be a boundary if, in addition, there exists an $H \in \mathscr{H}$ such that the sequence of iterates $\{H^n\}$ has the property that

$$\lim_{n \to \infty} (H^n x, y_0) = f_x(s_0)$$

for some $s_0 \in B$ and for each $x \in X$ for which f_x is continuous. We may assume in such a case that $s_0 = e$, the identity in B, since we may replace H by $U_{s_0} H U_{s_0^{-1}}$. Thus, under these hypotheses, for any $x \in X$ for which f_x is continuous we have

$$\begin{aligned} \lim_{n \to \infty} \int_B f_x(s * H^n)\, ds &= \lim_{n \to \infty} \int_B f_{H^n x}(s)\, ds \\ &= \lim_{n \to \infty} (H^n x, y_0) = f_x(e). \end{aligned}$$

Furthermore, since $P(p \cdot U_t, s) = P(p, st^{-1})$ for all $p \in M$ and $t, s \in B$, and since $f_x(t) = f_{U_t x}(e)$ and

$$\int_B f_x(s) P(p_0 \cdot H^n, s)\, ds = \int_B f_x(s * H^n)\, ds,$$

we have the equivalent formulas

$$\begin{aligned} f_x(t) &= \lim_{n \to \infty} \int_B f_x(st) P(p_0 \cdot H^n, s)\, ds \\ &= \lim_{n \to \infty} \int_B f_x(s) P\big(p_0 \cdot (U_{t^{-1}} H U_t)^n, s\big)\, ds \\ &= \lim_{n \to \infty} \int_B f_x(s) P(p_0 \cdot H^n, st^{-1})\, ds \end{aligned}$$

for each $t \in S$.

7. The motivating example. Let B be the 1-torus, $X = L^2(B)$ and $U_t x(s) = x(t+s)$. Let K be the convolution operator defined by

$$Kx(t) = P \cdot V \cdot \int_B k(t-s)\, x(s)\, ds,$$

where the kernel $k(t)$ is real valued and satisfies the following four conditions:

(1) $k'(t)$ is continuous on $(0, 2\pi)$;

(2) $k'(t) > 0$ for all $t \in (0, 2\pi)$;

(3) there exist an $a > 0$ and an integrable function φ on $(0, 2\pi)$ such that

$$k(t) = \varphi(t)/t^a(2\pi - t)^a \quad \text{for all } t \in (0, 2\pi);$$

(4) $P \cdot V \cdot \int_B k(t)\,dt$ exists.

Let y_0 be the constant function 1 and let

$$\mathcal{S} = \{S \in \mathcal{B}(X) |\ SKx - KSx = (x, y(S))y_0 \text{ for some } y(S) \in X\}.$$

Let $\mathcal{H}_c$ be the group of all composition operators H defined by

$$Hx(t) = x(h(t)),$$

induced by the set of all mappings h: $\boldsymbol{R} \to \boldsymbol{R}$ satisfying the following three conditions for all $t \in \boldsymbol{R}$:

(i) $h(t + 2\pi) = h(t) + 2\pi$;

(ii) $h'(t) > 0$;

(iii) $h''(t)$ exists.

Let $\mathcal{H} = \mathcal{S} \cap \mathcal{H}_c$. Note that $\mathcal{U} \subset \mathcal{H}$.

Essentially, the following theorem was proved in [6] under hypotheses somewhat weaker than those listed above.

7.1. THEOREM. *$\mathcal{U}$ is a proper subgroup of $\mathcal{H}$ if and only if*

$$k(t) = A\operatorname{Cot} t/2 + B \quad \textit{for some real numbers } A < 0 \textit{ and } B.$$

For the case $A = -1$ and $B = 0$, the operator K has the property $Ky_0 = 0$.

7.2. *Determination of $\mathcal{H}$ for the case $k(t) = -\operatorname{Cot} t/2$.* We begin by showing that for each $H \in \mathcal{H}$ the associated mapping h: $\boldsymbol{R} \to \boldsymbol{R}$ is of the form

$$h(t) = \theta + h_r(t - \varphi)$$

for some θ, φ, $r \in \boldsymbol{R}$, where $-1 < r < 1$ and

$$h_r(t) = \int_0^t \frac{1 - r^2}{1 + 2r\cos s + r^2}\,ds.$$

For, suppose that $k(t) = -\operatorname{Cot} t/2$ and that $H \in \mathcal{H}$. Since $\mathcal{U} \subset \mathcal{H}$, effecting a suitable translation if necessary we may assume that $h(0) = 0$.

By an argument similar to that given in [6], pp. 119–121, one can show that there exists a 2π-periodic function $G_h(s)$ which is continuous for all $s \in \boldsymbol{R}$ and is such that

$$h^{-1\prime}(s)\operatorname{Cot}\frac{h^{-1}(t) - h^{-1}(s)}{2} - \operatorname{Cot}\frac{t - s}{2} = G_h(s) \tag{1}$$

for all t, $s \in \boldsymbol{R}$, with $t \neq s \pmod{2\pi}$. (The function G_h is, in fact, $y(H)$.)

One may now deduce, by interchanging s and t, subtracting, and letting $t \to s$, that

$$\frac{h^{-1''}(s)}{h^{-1'}(s)} = G_h(s) \tag{2}$$

for all $s \in [0, 2\pi]$. Furthermore, since $h^{-1'}(0) = h^{-1'}(2\pi)$, there exists an $s_0 \in (0, 2\pi)$ such that $h^{-1''}(s_0) = 0$. Putting

$$g^{-1}(s) = h^{-1}(s+s_0) - h^{-1}(s_0),$$

one finds that $g^{-1}(0) = 0 = g^{-1''}(0)$ and that the function g^{-1} satisfies equations (1) and (2).

In particular, for $s = 0$, we have

$$g^{-1'}(0)\,\mathrm{Cot}\frac{g^{-1}(t)}{2} = \mathrm{Cot}\frac{t}{2} \tag{3}$$

for all $t \neq 0 \pmod{2\pi}$.

We note that if $g^{-1'}(0) = 0$, then one can deduce from (3) and the continuity of g that $g(t) = t$ for all $t \in \boldsymbol{R}$. In any case, there exists a number r $(-1 < r < 1)$ such that

$$g^{-1'}(0) = \frac{1+r}{1-r}.$$

We thus obtain

$$g^{-1}(t) = 2\,\mathrm{Cot}^{-1}\frac{1-r}{1+r}\mathrm{Cot}\frac{t}{2} = \int_0^t \frac{1-r^2}{1-2r\cos s + r^2}\,ds. \tag{4}$$

Setting

$$h_r(t) = g(t) = \int_0^t \frac{1-r^2}{1+2r\cos s+r^2}\,ds,$$

we may write

$$h(s) = s_0 + h_r(s - h^{-1}(s_0)) \quad \text{and} \quad h^{-1}(s) = h^{-1}(s_0) + h_{-r}(s - s_0).$$

Conversely, for every mapping $h\colon \boldsymbol{R} \to \boldsymbol{R}$ of the form

$$h(t) = \theta + h_r(t-\varphi) \quad \text{with } -1 < r < 1,$$

the associated operator $H \in \mathscr{B}(X)$ defined by $Hx(t) = x(h(t))$ belongs to $\mathscr{H}$.

7.3. *Harmonic and conjugate harmonic functions on* $\mathscr{H}$. If

$$Hx(t) = x(h(t)) = x(\theta + h_r(t-\varphi)),$$

then

$$F(H) = (Hx, 1) = \frac{1}{2\pi}\int_B x(h(t))\,dt = \frac{1}{2\pi}\int_B x(\theta + h_r(t))\,dt = (x, H^* 1)$$

$$= \frac{1}{2\pi}\int_B x(t)\,h_r^{-1\prime}(t-\theta)\,dt = \frac{1}{2\pi}\int_B x(t)\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\,dt.$$

Also

$$(x, y(H)) = \frac{1}{2\pi}\int_B x(t)\,G_h(t)\,dt$$

$$= \frac{1}{2\pi}\int_B x(t)\frac{h^{-1\prime\prime}(t)}{h^{-1\prime}(t)}\,dt = \frac{1}{2\pi}\int_B x(t)\frac{h''_{-r}(t-\theta)}{h'_{-r}(t-\theta)}\,dt$$

$$= \frac{1}{2\pi}\int_B x(t)\frac{2r\sin(\theta - t)}{1-2r\cos(t-\theta)+r^2}\,dt.$$

7.4. *$\mathscr{H}$ is algebraically isomorphic to* $\mathrm{SU}(1, 1)/\{\pm I\}$.

The group $\mathscr{H}$ is algebraically isomorphic to the group Γ of mappings of the unit circle $C = \{\xi \mid |\xi| = 1\}$ onto itself under the correspondence defined as follows: If $h(t) = \theta + h_r(t-\varphi)$ with t, θ, φ in B and $0 \leqslant r \leqslant 1$, define the mapping $g_{u,a}\colon C \to C$ by the equation

$$g_{u,a}(\xi) = u\frac{\xi + a}{1+\bar{a}\xi}, \quad \text{where } u = e^{i(\theta-\varphi)} \text{ and } a = re^{i\varphi}. \tag{1}$$

Conversely, for any pair of complex numbers (u, a) with $|u| = 1$ and $|a| < 1$, there is a unique operator $H_{u,a}$ in $\mathscr{H}$ defined by the equation $H_{u,a}x(t) = x(h_{u,a}(t))$, where the mapping $h_{u,a}$ is well defined by the equation

$$e^{ih_{u,a}(t)} = g_{u,a}(e^{it}).$$

It is, of course, well known that the group of mappings

$$\Gamma = \{g_{u,a} \mid |u| = 1,\ |a| < 1\}$$

is isomorphic to $\mathrm{SU}(1, 1)/\{\pm I\}$.

We note that the parameter relations in $g_{u,a} \circ g_{v,b} = g_{w,c}$ are

$$w = \bar{v}u\frac{v + a\bar{b}}{\bar{v}+\bar{a}b} \quad \text{and} \quad c = \frac{a+bv}{a\bar{b}+v}$$

and that $H_{v,b}H_{u,a} = H_{w,c}$.

7.5. *$\mathscr{T} \cap \mathscr{H} = \mathscr{U}$ and $M = \mathscr{T} \cap \mathscr{H} \setminus \mathscr{H}$, as a point set, can be identified with the open disk $D = \{z \mid |z| < 1\}$.*

Two operators H_1 and H_2 are *equivalent* (mod $\mathscr{U}$) if there exists an $s \in B$ such that $H_2 = H_1 U_s$; in terms of the mappings, this is equivalent to the existence of $v \in C$ such that

$$g_{u_2,a_2}(\xi) = g_{u_1,a_1}(v\xi) \quad \text{for all } \xi \in C.$$

This, in turn, is equivalent to the condition $u_1 a_1 = u_2 a_2$. Thus each right coset in $\mathscr{H}$ (mod $\mathscr{U}$) is characterized by the complex number $z = ua$. As can be seen from 7.4 (1), $z = ua = re^{i\theta}$.

Now, suppose that $T \in \mathscr{T} \cap \mathscr{H}$, that is to say, T is a composition operator in $\mathscr{H}$ that commutes with K. The associated mapping τ: $\boldsymbol{R} \to \boldsymbol{R}$ can be written as $\tau(t) = h(t) + b$, where $h(0) = 0$ and the associated operator H is also in $\mathscr{T} \cap \mathscr{H}$. Since H commutes with K, the associated G_h has the property that $G_h(s) = 0$ for all $s \in \boldsymbol{R}$ since $G_h = y(H)$. Now, from 7.2 (2) it follows that $h''(s) = 0$ for all $s \in \boldsymbol{R}$, and so h is the identity mapping, and τ is a translation.

7.6. *The triple $(D, \pi, \mathscr{H})$, where D is the unit disk in its usual topology, π is the homeomorphism defined by $\pi(z) = H^*_{1,z}1$, and $\mathscr{H}$ is as in the preceding sections, is a Poisson space. The action of $\mathscr{H}$ on D turns out to be by linear fractional transformations.*

Indeed, the mapping π is given explicitly by the following: if $z = re^{i\theta}$, then

$$H_{1,z}x(t) = x(\theta + h_r(t-\theta)),$$

so that $H^*_{1,z}1 = h_r^{-1\prime}(t-\theta)$.

While it can be readily verified that the mapping π is one-to-one, the proof of its bicontinuity is more intricate and will not be given here.

The action $z \cdot H_{u,a}$ of $\mathscr{H}$ on D as defined in section 5.2 is obtained explicitly as follows:

$$\begin{aligned} z \cdot H_{u,a} &= \pi^{-1}(H^*_{u,a} H^*_{1,z} 1) = \pi^{-1}((H_{1,z} H_{u,a})^* 1) \\ &= \pi^{-1}(H^*_{w,c} 1) = \pi^{-1}(H^*_{1,wc} 1) = wc = u\frac{z+a}{1+\bar{a}z}, \end{aligned}$$

since

$$w = u\frac{1+a\bar{z}}{1+\bar{a}z} \quad \text{and} \quad c = \frac{a+z}{a\bar{z}+1}.$$

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