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PARTIAL PARALLEL CLASSES IN STEINER SYSTEMS*

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1. Introduction. An $S(k, t, v)$ *Steiner system* is a pair (S, B), where S is a finite set containing v elements (*points*) and B is a collection of t-element subsets of S (*blocks*) such that every k-element subset of S is contained in *exactly* one block of B. The number $|S| = v$ is called the *order* of the Steiner system (S, B). An $S(2, 3, v)$ Steiner system is called a *triple system* (STS(v)), an $S(3, 4, v)$ Steiner system is called a *quadruple system* (SQS(v)). It is well known [2] that an STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, and in 1960 Hanani [1] proved that an SQS(v) exists if and only if $v \equiv 2$ or $4 \pmod 6$. Very little is known about $S(k, k+1, v)$ Steiner systems for $k \geqslant 4$.

By a *partial parallel class* of blocks of the Steiner system (S, T) is meant a collection π of pairwise disjoint blocks of T. If the blocks of π partition S, then π is called a *parallel class* of blocks.

In [4] Lindner and Phelps proved that any $S(k, k+1, v)$ Steiner system, with $v \geqslant k^4+3k^3+k^2+1$, has a partial parallel class containing at least $(v-k+1)/(k+2)$ blocks. For $k = 3$, it follows that any SQS(v) of order $v \geqslant 172$ has a partial parallel class containing at least $(v-2)/5$ quadruples. Further, they proved that any STS(v) of order $v \geqslant 9$ has a partial parallel class containing at least $(v-1)/4$ triples (except possibly for $v = 19$ and 27).

The purpose of this paper is to prove that any SQS(v) (for every admissible v) has a partial parallel class containing at least $[(v+2)/6]$ quadruples and that any STS(19) has a partial parallel class containing at least 5 triples. In what follows, given a number r, we denote by $[r]$ the largest integer s such that $s \leqslant r$.

2. Parallel classes in SQS(v). Let $\pi(v)$ be the largest number such that every SQS(v) has a partial parallel class of size $\pi(v)$. We have

$$\begin{array}{rcccccc} v = 4 & 8 & 10 & 14 & 16 & 20 & \ldots, \\ \pi(v) = 1 & 2 & 2 & 3 & ? & ? & \ldots \end{array}$$

THEOREM 2.1. *In a Steiner quadruple system of order v there are at least $[(v+2)/6]$ pairwise disjoint blocks.*

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Proof. Let (S, q) be an SQS(v) and let π be a partial parallel class of quadruples of q of size t such that if $P = \bigcup_{b\in\pi} b$, then $|S-P| \geqslant 2t+4$. It follows that $v \geqslant 6t+4$. For every x, $y \in S-P$, $x \neq y$, let

$$M_{xy} = \{u \in P\colon \exists z \in S-P \text{ with } \{x, y, u, z\} \in q\}.$$

If A, $B \in S-P$, $A \neq B$, denote by

$$\begin{aligned} &\{c_{i1}, c_{i2}, c_{i3}, c_{i4}\}, && i = 1, 2, \ldots, k,\\ &\{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}, && i = 1, 2, \ldots, s,\\ &\{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}, && i = 1, 2, \ldots, r,\\ &\{\alpha_i, \beta_i, \tau_i, \varrho_i\}, && i = 1, 2, \ldots, h,\end{aligned}$$

respectively, the quadruples of π containing exactly 1, 2, $\geqslant 3$, 0 elements of M_{AB}. Observe that, since $|S-P| \geqslant 2t+4$, we have $r \geqslant h+1+k/2$. Let

$$\begin{aligned} \{A, B, X_{ij}, a_{ij}\} \in q, &\quad i = 1, 2, \ldots, r,\ j = 1, 2, 3, \text{ or } 4,\\ \{A, B, Y_{ij}, b_{ij}\} \in q, &\quad i = 1, 2, \ldots, s,\ j = 1, 2,\\ \{A, B, Z_{i1}, c_{i1}\} \in q, &\quad i = 1, 2, \ldots, k,\end{aligned}$$

where X_{ij}, Y_{ij}, $Z_{i1} \in S-P$.

We will prove that there exists a partial parallel class of quadruples of q having size $t' > t$. We will suppose that, for every $b \in q$, $b \nsubseteq S-P$ (otherwise $\pi(v) > t$, immediately).

At first, suppose that there exists $\bar{i} \in \{1, 2, \ldots, r\}$ such that

$$\{a_{\bar{i}1}, a_{\bar{i}2}, a_{\bar{i}3}, a_{\bar{i}4}\} \subseteq M_{AB}.$$

Let $\bar{i} = 1$. If $r \geqslant 2$, then consider the blocks $\{X_{11}, X_{21}, X_{ij}, x\} \in q$ for $X_{11} \neq X_{ij} \neq X_{21}$ and $x \in P$. If

$$L = \{c_{11}, c_{21}, \ldots, c_{k1}\} \cup \{\alpha_1, \beta_1, \tau_1, \varrho_1, \ldots, \alpha_h, \beta_h, \tau_h, \varrho_h\},$$

since the collection of the blocks $\{X_{11}, X_{21}, X_{ij}, x\}$ has size

$$m \geqslant 2t-k-2s = 2r+k+2h \geqslant 4h+2+2k$$

and $|L| = 4h+k$, there necessarily exists at least a block of type $\{X_{11}, X_{21}, X_{ij}, x\}$ or $\{X_{12}, X_{21}, X_{ij}, x\}$ such that $x \notin L$. It follows that $\pi(v) > t$.

If $r = 1$, then $h = k = 0$, $t = s+1$, $|S-P| = 2t+4 = 2s+6$, $v = 6s+10$. If $s = 0$, then $v = 10$ and it is well known that $\pi(10) = 2$. If $s \geqslant 1$, let $\{X_{11}, Y_{11}, X_{12}, x\} \in q$, $\{X_{11}, Y_{11}, X_{13}, x\} \in q$. Since $x \neq y$, we can suppose $x \neq y_{12}$. Hence it follows that there exists a block $\{A, B, W, w\} \in q$, where $X = X_{ij}$, $(i, j) \neq (1, 1)$, $(1, 2)$, or $W = Y_{ij}$, $w \neq x$ and w, x with the same first index. If b is the block of π containing w and x and

$$\pi' = (\pi - \{b\}) \cup \{\{A, B, W, w\}, \{X_{11}, Y_{11}, X_{12}, x\}\},$$

then $|\pi'| > t$.

Now, suppose that, for every $i \in \{1, 2, \ldots, r\}$,

$$\{a_{i1}, a_{i2}, a_{i3}\} \subseteq M_{AB} \quad \text{and} \quad a_{i4} \notin M_{AB}.$$

We have $r = 2h+k+2$. Consider the blocks $\{X_{11}, X_{12}, X_{ij}, x\} \in q$ for $X_{ij} \neq X_{11}, X_{12}, X_{13}$ and $x \in P$. If $L' = L \cup \{a_{13}\}$, since the collection of the blocks $\{X_{11}, X_{12}, X_{ij}, x\}$ has size $m = 3(r-1) = 6h+3k+3$, and $|L'| = 4h+k+1$, there necessarily exists at least an X_{ij} such that if

$$\{X_{11}, X_{12}, X_{ij}, x\} \in q,$$

then $x \in L'$. It follows that $\pi(v) > t$.

At this point we can say that if $t^* = [(v-4)/6]$, then there exists a partial parallel class containing at least $[(v-4)/6]+1$ blocks. Hence $\pi(v) \geqslant [(v+2)/6]$. The proof is complete.

From Theorem 2.1 we have

$$\begin{array}{llll} \pi(16) \geqslant 3, & \pi(20) \geqslant 3, & \pi(22) \geqslant 4, & \pi(26) \geqslant 4, \\ \pi(28) \geqslant 5, & \pi(32) \geqslant 5, & \pi(34) \geqslant 6, & \pi(38) \geqslant 6, \\ \pi(40) \geqslant 7, & \pi(44) \geqslant 7, & \pi(46) \geqslant 8, & \pi(50) \geqslant 8, \\ \pi(52) \geqslant 9, & \pi(56) \geqslant 9, & \pi(58) \geqslant 10, & \pi(62) \geqslant 10, \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

3. Parallel classes in STS(19). In this section we will denote by $\pi(19)$ the largest number such that every STS(19) has a partial parallel class of size $\pi(19)$.

THEOREM 3.1. *Each* STS(19) *has a partial parallel class containing at least four blocks.*

Proof. It is immediate to prove that any STS(19) (S, T) has a partial parallel class containing at least three blocks. Therefore, suppose

$$\pi' = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\} \subseteq T.$$

Let

$$P' = \{1, 2, 3, \ldots, 9\}, \quad M = \{\{x, y\}: x, y \in S-P', x \neq y\}.$$

Since $|M| = \binom{10}{2}$, there are exactly 45 blocks $b \in T$ such that

$$|b \cap (S-P')| = 2 \quad \text{and} \quad |b \cap P'| = 1.$$

Let

$$M_x = \{\{u, v\} \in M: \{x, u, v\} \in T\} \quad \text{for every } x \in P'.$$

Since $|P'| = 9$ and $|M_x| \geqslant 5$, it follows that $|M_x| = 5$ for every $x \in P'$. If $\{1, 0, A\} \in M_1$, then there exists a pair of distinct elements $x, y \in S-(P' \cup \{0, A\})$ such that $\{2, x, y\} \in T$. Hence $\pi(19) \geqslant 4$, which completes the proof.

THEOREM 3.2. *Each* STS(19) *has a partial parallel class containing at least five blocks, i.e.,* $\pi(19) \geq 5$.

Proof. Let (S, T) be an STS(19) and let π be a partial parallel class of triples of T such that $|\pi| = \pi(19)$. From Theorem 3.1 we have $\pi(19) \geq 4$. Suppose that $\pi(19) = 4$. Let

$$P = \{1, 2, \ldots, 9, 0, A, B\} \quad S-P = \{C, D, E, F, G, H, I\},$$

$$\pi = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{0, A, B\}\},$$

$$M = \{\{x, y\}: x, y \in S-P, \ x \neq y\}.$$

Since

$$|M| = \binom{7}{2},$$

there are exactly 21 blocks $b \in T$ such that

$$|b \cap P| = 1 \quad \text{and} \quad |b \cap (S-P)| = 2.$$

For every $x \in S-P$ let f_x: $(S-P)-\{x\} \to P$ be the mapping such that $f_x(y) = z$ if and only if $\{x, y, z\} \in T$. Further, let $(S, \otimes)$ be the idempotent commutative quasigroup associated with (S, T).

(R) Observe that if

$$\{u', u''\}, \{v', v''\} \in M, \quad \{u', u''\} \cap \{v', v''\} = \emptyset,$$

and

$$u' \otimes u'' \in u, \quad v' \otimes v'' \in v,$$

then $u, v \in \pi$ and $u \neq v$.

Since $|\{f_x(y): y \in (S-P)-\{x\}\}| = 6$ for every $x \in S-P$, we can suppose that

$$\begin{matrix}\{C, D, 1\}, \\ \{C, E, 2\} \in T, \\ \{C, F, 4\}.\end{matrix}$$

The following cases are to be considered:

$$(1)\ \begin{matrix}\{C, G, 5\}, \\ \{C, H, 3\} \in T, \\ \{C, I, 6\};\end{matrix} \qquad (2)\ \begin{matrix}\{C, G, 5\}, \\ \{C, H, 3\} \in T, \\ \{C, I, 7\};\end{matrix} \qquad (3)\ \begin{matrix}\{C, G, 0\}, \\ \{C, H, 3\} \in T, \\ \{C, I, 7\};\end{matrix}$$

$$(4)\ \begin{matrix}\{C, G, 5\}, \\ \{C, H, 8\} \in T, \\ \{C, I, 7\};\end{matrix} \qquad (5)\ \begin{matrix}\{C, G, 5\}, \\ \{C, H, 0\} \in T, \\ \{C, I, 7\}.\end{matrix}$$

Further, for every $x \in S-P$ there exist exactly three blocks $b \in T$ such that $|b \cap P| = 2$.

(1) If $\{D, F, x\} \in T$, then $x \in \{7, 8, 9, 0, A, B\}$. Let $x = 7$. If

$$\{G, H, u\}, \{H, I, v\} \in T,$$

then

$$\{u, v\} \cap \{0, A, B\} \neq \emptyset.$$

Since we can suppose $\{C, 7, 0\}, \{C, 8, A\}, \{C, 9, B\} \in T$, we necessarily obtain $\pi(19) \geqslant 5$.

(2) We can suppose $\{C, 0, 6\}$, $\{C, A, 8\}$, $\{C, B, 9\} \in T$. If

$$\{H, I, x\}, \{D, G, y\}, \{E, G, z\}, \{F, H, t\} \in T,$$

then

$$x \in \{8, 9, 0, A, B\}, \{y, z\} \subseteq \{4, 7, 0, A, B\}, \quad t \in \{5, 7, 0, A, B\}.$$

For $x = 8$ [resp. $x = 9$], we have $\{y, z\} = \{A, B\}$, hence $t = B$ [resp. $\{y, z\} = \{4, A\}$, hence $t = A$]. Therefore we have $\pi(19) \geqslant 5$ ($\{D, G, 4\}$ or $\{E, G, 4\}$, $\{C, 0, 6\}$, $\{1, 2, 3\}$, $\{7, 8, 9\}$, $\{F, H, B\}$ [resp. $\{F, H, A\}$]).

For $x = 0$ [resp. $x = A$, $x = B$], we have

$$\{D, G, 4\} \text{ or } \{D, G, x\} \in T.$$

It follows that $\{E, G, x\}$ or $\{E, G, 4\} \in T$, respectively. Necessarily, $\{F, H, 7\} \in T$. At this point we have $\pi(19) \geqslant 5$ with $\{F, H, 7\}$, $\{D, G, x\}$ and $\{C, A, 8\}$ for $x \neq A$ or $\{C, 9, B\}$ for $x = A$ belonging to T.

(3) We can suppose that

$$\{C, 5, 8\}, \{C, 6, A\}, \{C, 9, B\} \in T.$$

If $\{D, I, x\} \in T$, then $x \in \{4, 8, 9, 0\}$. Further, if $y \in E \otimes G$, $z = G \otimes H$, $t = G \otimes F$, $w = F \otimes I$, $u = E \otimes I$, $v = I \otimes H$, then

$$\{y, z\} \subseteq \{4, 7, A, B\}, \quad y \neq z, \quad t \in \{5, 6, 7, A, B\}, \quad w \in \{5, 6, 8, 9, 0\},$$
$$\{u, v\} \subseteq \{4, 8, 9, 0\}, \quad u \neq v.$$

For $x = 8$ it follows that $\{y, z\} = \{4, B\}$, hence $w = 9$. Since

$$\{C, 5, 8\}, \{F, I, 9\}, \{E, G, 4\} \text{ or } \{G, H, 4\} \in T,$$

we have $\pi(19) \geqslant 5$.

For $x = 9$ it follows that $\{y, z\} = \{A, B\}$. If $y = B$, $z = A$ (or $y = A$, $z = B$), we obtain $t = 5$, hence $w = 6$, and finally $u = 8$. At this point, since necessarily $I \otimes H = 4$, we have $\pi(19) \geqslant 5$.

For $x = 4$ [resp. $x = 0$] it follows that $\{y, z\} = \{4, A\}$ [resp. $\{y, z\} = \{4, 7\}$]. Necessarily we have

$$\{G \otimes I, E \otimes I, F \otimes I\} = \{8, 9\} \quad [\text{resp. } E \otimes G = G \otimes H = 7].$$

In the remaining cases we will assume that

$$|b \cap \{f_u(y) \colon y \in (S-P)-\{u\}\}| \leqslant 2$$

for every $u \in S-P$, $b \in \pi$.

(4) We can suppose that

$$\{C, 0, 3\}, \{C, A, 6\}, \{C, B, 9\} \in T.$$

If $\{D, E, x\}, \{F, G, y\}, \{H, I, z\} \in T$, then

$$x \in \{3, 0, A, B\}, \quad y \in \{6, 0, A, B\}, \quad z \in \{9, 0, A, B\}.$$

We prove that $|\{x, y, z\} \cap \{3, 6, 9\}| \leqslant 1$. In fact, if $x = 3$ and $y = 6$ (or $x = 3$, $z = 9$, or $y = 6$, $z = 9$), then

$$\{D, F, a\}, \{D, G, e\} \in T \quad \text{for } \{a, e\} \subseteq \{0, A, B\}.$$

If $z = 9$, we have

$$\{a, e, D\otimes H, D\otimes I\} = \{0, A, B\};$$

if $z \neq 9$, then

$$H\otimes I \in \{0, A, B\},$$

hence $\pi(19) \geqslant 5$. Therefore, let

$$|\{x, y, z\} \cap \{3, 6, 9\}| = 1.$$

Suppose that $x = 3$. It follows that

$$F\otimes G = H\otimes I \in \{0, A, B\},$$

and further

$$\{E\otimes I, D\otimes I\} \subseteq \{8, 0, A, B\} - \{H\otimes I\}.$$

Now, it is easy to see that there exist two blocks ($\{F, G, F\otimes G\}$ and $b \in \{\{H, I, z\}, \{E, I, E\otimes I\}, \{D, I, D\otimes I\}\}$) satisfying the condition (R).

Necessarily, $x = y = z \in \{0, A, B\}$. Observe that

$$u\otimes v \neq x \quad \text{for every } \{u, v\} \in M - \{\{D, E\}, \{F, G\}, \{H, I\}\};$$

further, we can consider

$$E\otimes F \in \{1, 5\}, \quad D\otimes F \in \{2, 5\}, \quad F\otimes H \in \{5, 7\},$$

$$F\otimes I \in \{5, 8\}, \quad D\otimes H \in \{2, 7\}, \quad D\otimes I \in \{2, 8\}.$$

Since $\{C, 3, 0\} \in T$ and $\{E, F, 1\}$ or $\{D, F, 2\} \in T$, it follows that $\pi(19) \geqslant 5$.

(5) We can suppose that

$$\{C, 8, A\}, \{C, 9, B\}, \{C, 3, 6\} \in T \quad \text{or} \quad \{C, 8, A\}, \{C, 3, 9\}, \{C, 6, B\} \in T.$$

Observe that if $\{D, E, x\}, \{F, G, y\} \in T$, then

$$x \in \{3, 7, 0\}, \qquad y \in \{6, 7, 0\}.$$

If $x = 3$ and $y = 6$, then

$$\{D\otimes F, D\otimes G\} = \{E\otimes F, E\otimes G\} = \{0, 7\};$$

hence

$$\{G\otimes I, F\otimes I, E\otimes I, D\otimes I\} = \{A, B\}.$$

If $x = 3$ and $y = 7$ [resp. $y = 0$], then necessarily

$$D\otimes H = E\otimes H = 0 \quad [\text{resp. } D\otimes I = E\otimes I = 7].$$

If $x = 7$, then $F\otimes H = 5$, $G\otimes H = 4$. Hence $F\otimes G = 7$. It follows that

$$\{F\otimes I, H\otimes I\} = \{A, B\} \quad \text{with } \pi(19) \geqslant 5.$$

Finally, if $x = 0$, then

$$H\otimes I = 8, \quad (F\otimes I, F\otimes G) \in \{(5, 0), (0, 6)\};$$

hence $F\otimes H = 7$. Now we have

$$\{C, 8, A\}, \{D, E, 0\}, \{F, H, 7\} \in T,$$

and so $\pi(19) \geqslant 5$.

This completes the proof of the theorem.

REFERENCES

[1] H. Hanani, *On quadruple systems*, Canad. J. Math. 12 (1960), pp. 145–157.

[2] T. P. Kirkman, *On a problem in combinatorics*, Cambridge and Dublin Mathematical Journal 2 (1847), pp. 191–204.

[3] C. C. Lindner, *On the size of partial parallel classes in Steiner systems*, Istituto di Matematica "Guido Castelnuovo", Seminario di Geometrie Combinatorie, Universitá di Roma, n. 16 maggio 1979.

[4] – and K. T. Phelps, *A note on partial parallel classes in Steiner systems*, Discrete Math. 24 (1978), pp. 109–112.

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