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ON THE DERIVATIVE OF TYPE a

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In paper [2] Marcus considered the functions whose set of points of continuity is dense and is a boundary set. He called them *functions of type a*. The derivative of a continuous function which is a function of type a is called a *derivative of type a*. The investigation of derivatives of type a is connected with the investigation of Pompeiu's functions and derivatives (see [3], p. 2), since — as is easily seen — each Pompeiu's derivative is of type a. A lot of articles (see, e.g., [1], [3] and [4]) have been devoted to constructing and investigating Pompeiu's functions. In this paper we construct some class of Pompeiu's derivatives such that their derivatives exist and are equal to 0 on a dense set. This gives a positive solution to the following Marcus' problem ([2], p. 13, Problem 7):

Does there exist a derivative of type a such that the set of its points of differentiability is everywhere dense?

Let I be the closed interval $\langle 0, 1\rangle$ and let, for a function f mapping I into the set R of real numbers, the symbols C_f, D_f and Δ_f denote the sets of points of continuity, discontinuity and differentiability, respectively, of the function f. Further, let $\Delta_f(a)$ stand for a set of all those points $x \in I$ for which there exists a derivative $f'(x) = a$. We shall write $A \Subset B$ if $A \subset B$ and each point of the set A is a point of density of the set B.

THEOREM 1. *Let D and E be disjoint F_σ-sets, contained in the interval I, and let $D \Subset D$. Then there exists an approximatively continuous function $f: I \to I$ such that*

$$D \supset D_f \supset D \cap \overline{I \setminus D} \quad \text{and} \quad \Delta_f(0) \supset E, \tag{1}$$

where $\bar{A}$ denotes the closure of a set A.

In particular, if D is a boundary set in I, then $D_f = D$.

Proof. Let

$$D = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad E = \bigcup_{n=1}^{\infty} E_n,$$

where (A_n) and (E_n) are non-decreasing sequences of closed sets. We now define a sequence of closed sets $B_1, B_2, \ldots$ such that

$$D = \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad B_n \Subset B_{n+1}.$$

Let $B_1 = A_1$ and suppose the sets $B_1, \ldots, B_n$ are defined so that $A_k \subset B_k \Subset B_{k+1} \subset D$. We define the set B_{n+1}. Since $A_{n+1} \cup B_n \Subset D$, it follows from the Luzin-Menchoff theorem that there exists a closed set B_{n+1} such that

$$A_{n+1} \cup B_n \Subset B_{n+1} \subset D.$$

From the relation $A_n \subset B_n \subset D$ we obtain the equality

$$D = \bigcup_{n=1}^{\infty} B_n.$$

Of course, we also have $B_n \Subset B_{n+1}$.

Consider any sequence $a_0, a_1, \ldots$ satisfying the conditions

$$1 \geqslant a_0 > a_1 > \ldots, \quad \lim a_n = 0 \quad \text{and} \quad a_n \leqslant \varrho^2(B_{n+1}, E_{n+1}),$$

where $\varrho(A, B) = \inf\{|x-y|\colon x \in A, y \in B\}$. Put

$$D_{a_0} = \emptyset \quad \text{and} \quad D_{a_n} = B_n \text{ for } n = 1, 2, \ldots$$

Since $D_{a_n} \Subset D_{a_{n+1}}$ for $n = 0, 1, \ldots$, we infer from the Luzin-Menchoff theorem that there exist closed sets $D_{(a_n+a_{n+1})/2}$ such that

$$D_{a_n} \Subset D_{(a_n+a_{n+1})/2} \Subset D_{a_{n+1}}.$$

Analogously we find a countable set A, contained and dense in $(0, a_0\rangle$, containing the numbers $a_0, a_1, \ldots$, and such that with each $a \in A$ we associate a closed set D_a. Moreover, we have

$$D = \bigcup_{a \in A} D_a \quad \text{and} \quad D_b \Subset D_a \text{ when } a, b \in A \text{ and } a < b.$$

For each number $t \in (0, a_0\rangle$ we put

$$D_t = \bigcap_{a \in A \cap (0,t\rangle} D_a.$$

It is easily seen that the closed sets D_t satisfy the conditions

$$D = \bigcup_{t \in (0,a_0\rangle} D_t \quad \text{and} \quad D_u \Subset D_t \text{ when } 0 < t < u \leqslant a_0.$$

Let us put

$$f(x) = \begin{cases} 0 & \text{for } x \in I \setminus D, \\ \sup\{t\colon x \in D_t\} & \text{for } x \in D. \end{cases}$$

The function $f\colon I \to \langle 0, a_0\rangle$ is approximatively continuous in I, continuous at each point of the set $I \setminus D$, and $f(x) > 0$ for $x \in D$. Indeed,

f is upper semicontinuous in I. It is evident when $f(x_0) = a_0$. If, however, $f(x_0) < a_0$, then for each u with $f(x_0) < u < a_0$ we have $x_0 \notin D_u$. So, there exists a $\delta_u > 0$ such that

$$(x_0 - \delta_u, x_0 + \delta_u) \cap D_u = \emptyset.$$

Hence, for $x \in (x_0 - \delta_u, x_0 + \delta_0)$ we have $x \notin D_u$ and, consequently, $x \in I \setminus D$ or $x \in D_t$ with some $t < u$, which gives $f(x) \leqslant u$. Thus we have proved that f is upper semicontinuous at each point $x_0 \in I$. Since $0 \leqslant f(x) \leqslant a_0$, f is lower semicontinuous at each point $x_0 \in I$ such that $f(x_0) = 0$. Thus f is lower semicontinuous at each point of the set $I \setminus D$. Consequently, $D \supset D_f$.

In order to show the approximative continuity of f, it remains to prove its approximative lower semicontinuity at each point of D, which is equal to $\{x: f(x) > 0\}$. To that end, let $x_0 \in D$ and $0 < \varepsilon < f(x_0)$. The point x_0, obviously, is a point of density of the set D_ε (because $x_0 \in D_\eta$, where $\varepsilon < \eta < f(x_0)$). Since $f(x) \geqslant \varepsilon$ for $x \in D_\varepsilon$, the function f is approximatively lower semicontinuous at x_0.

If $x_0 \in D \cap \overline{I \setminus D}$, then $f(x_0) > 0$ and there exists a sequence of points $x_n \in I \setminus D$ such that $x_0 = \lim x_n$. Since $f(x_n) = 0$, we have $\lim f(x_n) \neq f(x_0)$ and $x_0 \in D_f$. Thus the second inclusion of (1) has been proved.

To prove the third inclusion of (1), we consider any point $x \in E$ and select an n such that $x \in E_n$. Then, for h satisfying $0 < |h| < \varrho(B_n, E_n)$ we have

$$x + h \in I \setminus B_n = I \setminus D \cup \bigcup_{m=n}^{\infty} (B_{m+1} \setminus B_m).$$

Hence we have $f(x+h) = 0$ if $x + h \in I \setminus D$ and $f(x+h) \leqslant a_m$ if $x + h \in B_{m+1} \setminus B_m = D_{a_{m+1}} \setminus D_{a_m}$, where $m \geqslant n$. So, we always have the estimate

$$0 \leqslant f(x+h) \leqslant a_m \leqslant \varrho^2(B_{m+1}, E_{m+1}) \leqslant \varrho^2(x+h, x) = h^2$$

$(x + h \in B_{m+1}, x \in E_n \subset E_{m+1})$, whence the inequality

$$\left| \frac{f(x+h) - f(x)}{h} \right| = \frac{f(x+h)}{|h|} \leqslant |h|$$

holds. It is now easily seen that $f'(x) = 0$ for $x \in E$, q.e.d.

Remark 1. The function f from Theorem 1, being approximatively continuous, is of the first class of Baire. The set of points of discontinuity of f is then of the first category and, thereby, a boundary set. Hence it appears that the first two inclusions of condition (1) cannot be replaced by the equality $D_f = D$.

THEOREM 2. *There exists a derivative $f: I \to I$ of type a such that the set Δ_f is dense in I.*

Proof. Let a boundary set $D \subset I$ be F_σ, dense and such that $D \in \mathbf{D}$, and let $E \in F_\sigma$ be a dense subset of I, disjoint with the set D. Then, by Theorem 1, there exists an approximatively continuous function $f\colon I \to I$ such that $D_f = D$ and $\Delta_f \supset E$. The function f, being bounded and approximatively continuous, is the derivative of the function

$$F(x) = \int_0^x f(t)\,dt \qquad \text{for } x \in I.$$

Besides, the function f satisfies the conditions of the theorem.

Remark 2. Theorem 2 contains an affirmative answer to the question included in the Marcus problem which was given in the introductory remarks.

The existence of a set D satisfying the conditions listed in the proof of Theorem 2 is evident. Indeed, it is sufficient to consider a measurable set $A \subset I$ such that $|A \cap P| > 0$ and $|P \setminus A| > 0$ for each non-degenerate interval $P \subset I$, and next to take its subset B composed of all points of density of A, belonging to A. Then, each set $D \in F_\sigma$, satisfying the conditions $D \subset B$ and $|B \setminus D| = 0$, fulfils the requirements above.

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