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$\forall_n$-*THEORIES OF BOOLEAN ALGEBRAS*

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In this paper we give a characterization of complete $\forall_n$-theories of Boolean algebras in terms of the classification of complete theories of Boolean algebras given by Tarski [4] (see also Eršov [1]).

0. Let $\mathscr{L}$ be any first-order language with the variables denoted by x, y, z (with sub- and superscripts). By $\bar{x}, \bar{y}, \bar{z}$ we denote finite sequences of variables. The length of $\bar{x}$ is denoted by $l(\bar{x})$, and the i-th element of $\bar{x}$ by x_i. We write $\forall\bar{x}$ ($\exists\bar{x}$) instead of $\forall x_0 \ldots \forall x_k$ ($\exists x_0 \ldots \exists x_k$).

A formula Φ of $\mathscr{L}$ is called an $\forall_n$-*formula* ($\exists_n$-*formula*) if it is of the form $\forall\bar{x}^{n-1}\exists\bar{x}^{n-2}\ldots\Psi$ ($\exists\bar{x}^{n-1}\forall\bar{x}^{n-2}\ldots\Psi$), where Ψ is open. The set of all $\forall_n$-formulas of $\mathscr{L}$ is denoted by $\forall_n(\mathscr{L})$, and the set of all $\exists_n$-formulas of $\mathscr{L}$ by $\exists_n(\mathscr{L})$. If $\mathscr{T}$ and $\mathscr{T}'$ are theories in $\mathscr{L}$, then we write $\mathscr{T} =_n \mathscr{T}'$ if

$$\mathscr{T} \cap \big(\forall_n(\mathscr{L}) \cup \exists_n(\mathscr{L})\big) = \mathscr{T}' \cap \big(\forall_n(\mathscr{L}) \cup \exists_n(\mathscr{L})\big).$$

For similar structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv_n \mathfrak{B}$ denotes $\mathrm{Th}(\mathfrak{A}) =_n \mathrm{Th}(\mathfrak{B})$.

If $\mathscr{T}$ and $\mathscr{T}'$ are complete, then

a. $\forall_n(\mathscr{L}) \cap \mathscr{T} \subseteq \forall_n(\mathscr{L}) \cap \mathscr{T}'$ *if and only if* $\exists_n(\mathscr{L}) \cap \mathscr{T}' \subseteq \exists_n(\mathscr{L}) \cap T$;

b. $\mathscr{T} =_n \mathscr{T}'$ *if and only if* $\forall_n(\mathscr{L}) \cap \mathscr{T} = \forall_n(\mathscr{L}) \cap \mathscr{T}'$ *or* $\exists_n(\mathscr{L}) \cap \mathscr{T} = \exists_n(\mathscr{L}) \cap \mathscr{T}'$.

By $\mathscr{L}_B$ we denote the language of Boolean algebras with non-logical symbols $-, \cup, \cap, \subseteq, 0, 1$. For any term t of $\mathscr{L}_B$ we put $0t = -t$ and $1t = t$. If $l(\bar{x}) = n$ and $\bar{\varepsilon} \in 2^n$, then $\bar{\varepsilon}\bar{x} = \varepsilon_0 x_0 \cap \ldots \cap \varepsilon_{n-1} x_{n-1}$ and $A(\bar{x}) = \{\bar{\varepsilon}\bar{x}\colon \bar{\varepsilon} \in 2^n\}$.

Now we briefly recall the classification of complete theories of Boolean algebras ([1], [4]). Let $\mathscr{B}$ be a Boolean algebra. By $\mathscr{I}(\mathscr{B})$ we denote the ideal of all elements x of $\mathscr{B}$ such that $x = y \cup z$, where y is atomic and z is atomless. Let $\mathscr{B}_0 = \mathscr{B}$, and $\mathscr{B}_{n+1} = \mathscr{B}_n / \mathscr{I}(\mathscr{B}_n)$. With $\mathscr{B}$ we can connect the triple of ordinals $\langle p, q, r\rangle$, where $p \leqslant \omega$, $q \leqslant \omega$, $r \leqslant 1$, and

$$p = \begin{cases} \min\{n\colon \mathscr{B}_{n+1} \text{ is trivial}\} & \text{if it exists,} \\ \omega & \text{otherwise,} \end{cases}$$

LEMMA 3. *Every $\exists_{n+1}$-sentence of $\mathcal{L}_B$ is equivalent in the theory of Boolean algebras to a sentence of the form*

$$\exists\bar{z}\Big(PART(\bar{z})\wedge \bigwedge_{i\in I}\ \bigvee_{j<l(\bar{z})} \Phi_{i,j}(z_j)\Big),$$

where every $\Phi_{i,j}$ is an $\forall_n$-formula with all quantifiers relativized to z_j.

If we express Lemma 2 in semantical terms, we obtain the following characterization of $\exists_n$-sentences in a given Boolean algebra $\mathcal{B}$:

PROPOSITION 1. *For every sentence $\Phi\in\exists_{n+1}(\mathcal{L}_B)$ there is a finite set $\{\Phi_{ij}\colon i\in I,\ j\in J\}$ of $\forall_n$-sentences such that $\mathcal{B}\models\Phi$ if and only if, for some Boolean algebras $\mathcal{B}_j$ $(j\in J)$,*

$$\mathcal{B}=\prod_{j\in J}\mathcal{B}_j,$$

and for every $i\in I$ there is a $j\in J$ for which $B_j\models\Phi_{ij}$.

PROPOSITION 2. *Let $\mathcal{T}$, $\mathcal{T}'$ be complete theories of Boolean algebras. Then*

$$\exists_{n+1}(\mathcal{L}_B)\cap\mathcal{T}\subseteq\exists_{n+1}(\mathcal{L}_B)\cap\mathcal{T}'$$

if and only if for any complete theories $\mathcal{T}_0,\ldots,\mathcal{T}_k$ such that

$$\mathcal{T}=\mathcal{T}_0\times\ldots\times\mathcal{T}_k$$

there are complete theories $\mathcal{T}_0',\ldots,\mathcal{T}_k'$ such that

$$\mathcal{T}'=\mathcal{T}_0'\times\ldots\times\mathcal{T}_k'$$

and, for every $i\leqslant k$, $\mathcal{T}_i=_n\mathcal{T}_i'$.

Proof. Sufficiency follows immediately from Proposition 1, necessity from the compactness theorem.

3. The proof of the Theorem is by induction. Let us assume its validity for all natural numbers less than n.

Let $\mathcal{T}\in\mathcal{E}_n$, $\mathcal{T}'=\langle p',q',r'\rangle$ and $\mathcal{T}=_n\mathcal{T}'$. By the induction hypothesis,

$$\mathcal{T}'\notin\bigcup_{k<n}\mathcal{E}_k. \tag{3.1}$$

According to the definition of $\mathcal{E}_n$, we should consider four cases. Let us take one of them, say, $n=4k+1$. Proofs of other cases are similar.

If $n=4k+1$, then $\mathcal{T}=\langle k,q,0\rangle$ for some positive integer q, and from (3.1) it follows that $p'\geqslant k$.

If $p'>k$ or $r'=1$, then $\mathcal{T}'=\mathcal{T}'\times\langle k,0,1\rangle^q$.

Since no factor of this product belongs to $\bigcup_{k<n}\mathcal{E}_k$, we have, by Proposition 2 and the induction hypothesis,

$$\mathcal{T}=\prod_{i\leqslant q}\langle p_i,q_i,r_i\rangle$$

and $p_i \geqslant k$ for every $i \leqslant q$. Hence, by Lemma 1, for every $i \leqslant q$, $p_i = k$, $r_i = 0$, $q_i \neq 0$, and $\sum_{i \leqslant q} q_i = q$, which is impossible. So $p' = k$, $r' = 0$.

Analogously, one can show that $q' = q$, which gives the equality $\mathcal{T} = \mathcal{T}'$.

In order to prove the second part of the Theorem it is sufficient to show that if $\mathcal{T} \notin \bigcup_{k \leqslant n} \mathcal{E}_k$, then $\mathcal{T} =_n \langle \omega, 0, 0 \rangle$.

We give only the proof of the inclusion

$$\exists_n(\mathcal{L}_B) \cap \mathcal{T} \subseteq \exists_n(\mathcal{L}_B) \cap \langle \omega, 0, 0 \rangle. \tag{3.2}$$

Proof of the converse inclusion is similar but slightly more labourous.

We start with the easy observation that if $\mathcal{T} \notin \bigcup_{k \leqslant n} \mathcal{E}_k$ and $\mathcal{T} = \prod_{i \leqslant m} \mathcal{T}_i$, then, for at least one $i \leqslant m$, $\mathcal{T}_i \notin \bigcup_{k \leqslant n} \mathcal{E}_k$.

If $\mathcal{T}$, m and $\mathcal{T}_i$ are as above, then we put

$$\mathcal{T}'_i = \begin{cases} \mathcal{T}_i & \text{if } \mathcal{T}_i \in \bigcup_{k \leqslant n} \mathcal{E}_k, \\ \langle \omega, 0, 0 \rangle & \text{otherwise.} \end{cases}$$

But then

$$\prod_{i \leqslant m} \mathcal{T}'_i = \langle \omega, 0, 0 \rangle$$

and, by induction hypothesis, $\mathcal{T}_i =_{n-1} \mathcal{T}'_i$. Hence, by Proposition 2, (3.2) holds.

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