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OPERATORS ON CONTINUOUS FUNCTION SPACES AND L_1-SPACES

BY

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1. Introduction. Let $C(Y)$ denote the Banach space of all bounded continuous complex-valued functions on a topological space Y with the usual supremum norm, and let $C_0(Y)$ denote the subspace of $C(Y)$ consisting of all functions in $C(Y)$ that vanish at infinity. We recall that Y is compact if and only if $C_0(Y) = C(Y)$, and that $C(Y)$ and $C_0(Y)$ are also commutative Banach algebras under the usual pointwise multiplication. A function $f \in C(Y)$ is said to be ***invertible*** if there exists a function $g \in C(Y)$ such that $fg = 1$ on Y. The well-known Banach-Stone theorem (see Dunford and Schwartz [11], Theorem V.8.8) states that if X and Y are compact Hausdorff spaces and T is a linear isometry from $C(X)$ onto $C(Y)$, then there exist a function $r \in C(Y)$ with $|r| = 1$ on Y and a homeomorphism φ from Y onto X such that

$$Tf(y) = r(y)f(\varphi(y)) \tag{1}$$

for all $f \in C(X)$ and all $y \in Y$. Many authors have studied generalizations of this theorem and obtained related results. Among others, Holsztyński [22] proved that if X and Y are compact Hausdorff spaces and T is a linear isometry from $C(X)$ into $C(Y)$, then there exist a closed subset Y_0 of Y, a function $r \in C(Y)$ with $|r| = 1$ on Y_0, and a continuous mapping φ from Y_0 onto X such that (1) holds for all $f \in C(X)$ and all $y \in Y_0$. Related results have been given by Gęba and Semadeni [15], Semadeni [32] (especially, p. 388-395), Williams [33], Amir and Arbel [2], and McDonald [27] (see also Figiel [12] and Holsztyński [23]).

Recently, the author [30] proved that if T is a positive linear operator from $C(X)$ into $C(Y)$, where X and Y are compact Hausdorff spaces, such that $\|T\| \leqslant 1$, then the set $\{f \in C(X)\colon |f| = 1 = |Tf|\}$ is total in $C(X)$, i.e., the linear manifold generated by $\{f \in C(X)\colon |f| = 1 = |Tf|\}$ is dense in $C(X)$, if and only if there exists a continuous mapping φ from Y into X such that $Tf(y) = f(\varphi(y))$ for all $f \in C(X)$ and all $y \in Y$.

On the other hand, Cambern (see [4] and [5]) proved that if X and Y are locally compact Hausdorff spaces and there exists a one-to-one bounded linear operator T from $C_0(X)$ onto $C_0(Y)$ such that $\|T\|\|T^{-1}\| < 2$, then X and Y are homeomorphic (see also Amir [1], Cambern [6] and [7], Gordon [18], and Cengiz [9]). Another treatment of this problem was done by Hewitt [20] (see also Gillman and Jerison [16], especially Chapters 8 and 10).

The main purpose of this paper is to prove similar results in more general settings of topological spaces X and Y and linear operators T from $C(X)$ into $C(Y)$. This will be done in the next section using techniques given in [30]. In particular, it will be proved there that if T is a bounded linear operator from $C(X)$ into $C(Y)$, where X is a completely regular Hausdorff space and Y any topological space, such that

$$Y = \bigcup_{f \in C_0(X)} \{y \in Y: |Tf(y)| > 0\}$$

and also such that $|Tf| > 0$ on Y for all invertible $f \in C(X)$, then there exist a function $r \in C(Y)$ with $|r| > 0$ on Y and a continuous mapping φ from Y into X such that (1) holds for all $f \in C(X)$ and all $y \in Y$.

In the third section we shall consider linear contraction operators in L_1-spaces of probability measure spaces. For a probability measure space $(Y, \mathscr{B}, \mu)$, we denote by $L_p(Y, \mathscr{B}, \mu)$, $1 \leqslant p \leqslant \infty$, the (complex) Banach spaces defined as usual with respect to $(Y, \mathscr{B}, \mu)$, and by $(\mathscr{B}(\mu), \mu)$ the measure algebra associated with $(Y, \mathscr{B}, \mu)$ (see Halmos [19], Chapter 8). It will be proved that whenever $(X, \mathscr{A}, \lambda)$ and $(Y, \mathscr{B}, \mu)$ are two probability measure spaces and T is a bounded linear operator from $L_1(X, \mathscr{A}, \lambda)$ into $L_1(Y, \mathscr{B}, \mu)$ such that $\|T\|_1 \leqslant 1$, the set

$$\{f \in L_1(X, \mathscr{A}, \lambda): |f| = 1 = |Tf|\}$$

is total in $L_1(X, \mathscr{A}, \lambda)$ if and only if there exist a function $r \in L_1(Y, \mathscr{B}, \mu)$ with $|r| = 1$ μ-a.e. and a homomorphism φ from $(\mathscr{A}(\lambda), \lambda)$ into $(\mathscr{B}(\mu), \mu)$ such that

$$Tf = r(Uf) \quad \text{for all } f \in L_1(X, \mathscr{A}, \lambda), \tag{2}$$

where U denotes the operator from $L_1(X, \mathscr{A}, \lambda)$ into $L_1(Y, \mathscr{B}, \mu)$ induced by φ. This generalizes the Corollary of [30]. We note that similar results have been given by Banach [3] (Chapter 11) and Lamperty [26] for linear isometries of L_p-spaces, and by Nagasawa [28], deLeeuw et al. [10], Forelli [13] and [14], Cambern [8] and Schneider [31] for linear isometries of H_p-spaces.

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2. Operators on continuous function spaces. Let X and Y be two topological spaces, and let T be a bounded linear operator from $C(X)$ into $C(Y)$. If $y \in Y$, then we set

$$b(y) = \sup\{|Tf(y)|: f \in C(X) \text{ and } \|f\| \leqslant 1\},$$

and

$$C(X)_y = \{f \in C(X): \|f\| \leqslant 1 \text{ and } |Tf(y)| = b(y)\}.$$

We also set

$$Y_+ = \bigcup_{f \in C_0(X)} \{y \in Y: |Tf(y)| > 0\},$$

$$Y_t = \{y \in Y: C(X)_y \text{ is total in } C(X)\} \quad \text{and} \quad Y_1 = Y_+ \cap Y_t.$$

Then we have the following theorem:

THEOREM 1. *Let X be a completely regular Hausdorff space and Y any topological space. Let T be a bounded linear operator from $C(X)$ into $C(Y)$. Then there exist a function $r \in C(Y)$ with $|r| > 0$ on Y_1 and a continuous mapping φ from Y_1 into X such that* (1) *holds for all $f \in C(X)$ and all $y \in Y_1$.*

For the proof of Theorem 1 we need the following lemma:

LEMMA 1. *Let Y be a topological space, and let Φ be a bounded linear functional on $C(Y)$ such that $\|\Phi\| = 1$. Then the set*

$$\{f \in C(Y): \|f\| \leqslant 1 \text{ and } |\Phi f| = 1\}$$

is total in $C(Y)$ if and only if

(3) $\quad |\Phi 1| = 1 \quad$ *and* $\quad \Phi(fg) = (\Phi f)(\Phi g)(\Phi 1)^{-1} \quad$ *for all $f, g \in C(Y)$.*

Proof. By the Gelfand-Naimark theorem (see, for example, Hewitt and Ross [21], Theorem (C.28)), we may and will assume without loss of generality that Y is a compact Hausdorff space.

Now suppose that the set $\{f \in C(Y): \|f\| \leqslant 1 \text{ and } |\Phi f| = 1\}$ is total in $C(Y)$. The Riesz representation theorem (see [21], Section 14) implies that there exist a positive regular measure μ on the Borel subsets of Y with $\mu(Y) = 1$ and a Borel measurable function ξ on Y with $|\xi| = 1$ μ-a.e. such that

$$\Phi f = \int_Y f\xi d\mu \quad \text{for all } f \in C(Y).$$

Let E denote the support of μ (see [21], Theorem (11.25)). If $f \in C(Y)$ satisfies $\|f\| \leqslant 1$ and $|\Phi f| = 1$, then we have

$$1 = \left|\int_Y f\xi d\mu\right| \leqslant \int_Y |f\xi| d\mu \leqslant \int_Y |\xi| d\mu = 1.$$

Hence $|f\xi| = 1$ μ-a.e., and $|f| = 1$ μ-a.e. Therefore

$$E \subset \{y \in Y: |f(y)| = 1\}.$$

By this, we may and will assume without loss of generality that $E = Y$ and the set $\{f \in C(Y)\colon |f| = 1 \text{ on } Y \text{ and } |\Phi f| = 1\}$ is total in $C(Y)$. Then choose an $h \in C(Y)$ with $|h| = 1$ on Y and $|\Phi h| = 1$, and define a linear functional Ψ on $C(Y)$ by the relation

$$\Psi f = \big(\Phi(fh)\big)(\Phi h)^{-1} \quad \text{for all } f \in C(Y).$$

Let ν be a positive regular measure on the Borel subsets of Y with $\nu(Y) = 1$, and η a Borel measurable function on Y with $|\eta| = 1$ ν-a.e. such that

$$\Psi f = \int_Y f\eta d\nu \quad \text{for all } f \in C(Y).$$

Since

$$1 = \Psi 1 = \int_Y \eta d\nu,$$

it follows that $\eta = 1$ ν-a.e. Therefore, if $f, g \in C(Y)$ satisfy $\|f\| \leqslant 1$, $\|g\| \leqslant 1$ and $|\Psi f| = 1 = |\Psi g|$, then

$$\Psi f = f \ \nu\text{-a.e.} \quad \text{and} \quad \Psi g = g \ \nu\text{-a.e.}$$

Hence we have

$$\Psi(fg) = \int_Y fg d\nu = (\Psi f)(\Psi g).$$

Let M denote the linear manifold generated by the set

$$\{f \in C(Y)\colon \|f\| \leqslant 1 \text{ and } |\Psi f| = 1\}.$$

It follows that $\Psi(fg) = (\Psi f)(\Psi g)$ for all $f, g \in M$. Let $f, g \in C(Y)$. Since M is a dense subset of $C(Y)$, there exist $f_n, g_n \in M$, $n = 1, 2, \ldots$, such that

$$\lim_n \|f_n g_n - fg\| = \lim_n \|f_n - f\| = \lim_n \|g_n - g\| = 0.$$

Hence we have

$$\Psi(fg) = \lim_n \Psi(f_n g_n) = \lim_n (\Psi f_n)(\Psi g_n) = (\Psi f)(\Psi g),$$

i.e., Ψ is a (non-zero) multiplicative linear functional on $C(Y)$. Therefore, there exists a point $y \in Y$ such that $\Psi f = f(y)$ for all $f \in C(Y)$ (see, for example, [21], Corollary (C.32)). In particular, it follows that

$$|\Phi 1| = |(\Phi 1)(\Phi h)^{-1}| = |\Psi(1/h)| = |1/h(y)| = 1.$$

The proof of the converse implication is now clear and omitted.

Proof of Theorem 1. For a fixed point $y \in Y_1$, let Φ_y denote the bounded linear functional on $C(X)$ defined by

$$\Phi_y f = Tf(y) \quad \text{for all } f \in C(X).$$

It follows that $\|\Phi_y\| = b(y)$ $(b(y) > 0)$, and the set

$$\{f \in C(X)\colon \|f\| \leqslant 1 \text{ and } |\Phi_y f| = b(y)\}$$

is total in $C(X)$. Hence, by Lemma 1,

$$|\Phi_y 1| = b(y)$$

and

$$\Phi_y(fg) = (\Phi_y f)(\Phi_y g)(\Phi_y 1)^{-1} \quad \text{for all } f, g \in C(X).$$

Let $\hat{X}$ denote the Stone-Čech compactification of X (see, for example, Kelley [25], Theorem 5.24). Let $\hat{f}$ denote the continuous extension of $f \in C(X)$ to $\hat{X}$. Since the Banach algebras $C(X)$ and $C(\hat{X})$ are isometrically isomorphic under the mapping $f \to \hat{f}$, the (non-zero) multiplicative linear functional $f \to (\Phi f)(\Phi 1)^{-1}$ on $C(X)$ may be considered to be a (non-zero) multiplicative linear functional on $C(\hat{X})$. Therefore, there exists a (unique) point $\hat{x} = \varphi(y) \in \hat{X}$ such that

$$Tf(y) = T1(y)\hat{f}(\hat{x}) \quad \text{for all } f \in C(X).$$

Here we note that if $f \in C_0(X)$, then $\hat{f} = 0$ on $\hat{X} - X$. This, together with the fact that $y \in Y_1 \subset Y_+$, implies that $\hat{x} \in X$. Hence we can define a mapping φ from Y_1 into X. Moreover, by letting $r = T1$, we have $|r| > 0$ on Y_1, and $Tf(y) = r(y)f(\varphi(y))$ for all $f \in C(X)$, and all $y \in Y_1$. The continuity of φ follows easily from the fact that $r \in C(Y)$ and $|r| > 0$ on Y_1. The proof is complete.

As an immediate corollary to Theorem 1 we have the following result:

THEOREM 2. *Let X be a completely regular Hausdorff space and Y any topological space. Let T be a bounded linear operator from $C(X)$ into $C(Y)$ such that $\|T\| \leqslant 1$. Assume that $Y_+ = Y$ and the set*

$$\{f \in C(X)\colon \|f\| \leqslant 1 \text{ and } |Tf(y)| = 1\}$$

is total in $C(X)$ for all $y \in Y$. Then there exist a function $r \in C(Y)$ with $|r| = 1$ on Y and a continuous mapping φ from Y into X such that (1) *holds for all $f \in C(X)$ and all $y \in Y$.*

THEOREM 3. *Let X be a completely regular Hausdorff space and Y any topological space. Let T be a bounded linear operator from $C(X)$ into $C(Y)$. Assume that $Y_+ = Y$ and $|Tf| > 0$ on Y for all invertible $f \in C(X)$. Then there exist a function $r \in C(Y)$ with $|r| > 0$ on Y and a continuous mapping φ from Y into X such that* (1) *holds for all $f \in C(X)$ and all $y \in Y$.*

Proof. It follows from the Gleason-Kahane-Żelazko theorem ([17], [24] and [34]; see also Żelazko [35] and Rudin [29], Theorem 10.9) that for each $y \in Y$ the bounded linear functional

$$f \to Tf(y)T1(y)^{-1} \quad \text{on } C(X)$$

is multiplicative. Hence, by Lemma 1, $Y_t = Y$ and $Y_1 = Y_+ \cap Y_t = Y$. Therefore, Theorem 1 completes the proof of Theorem 3.

COROLLARY. *Let X and Y be two completely regular Hausdorff spaces. Assume that there exists a one-to-one bounded linear operator T from $C(X)$ onto $C(Y)$ such that*

(i) $Y_+ = Y$,

(ii) $X = \bigcup_{g \in C_0(Y)} \{x \in X: |T^{-1}g(x)| > 0\}$,

(iii) $|Tf| > 0$ *on* Y *for all invertible* $f \in C(X)$, *and*

(iv) $|T^{-1}g| > 0$ *on* X *for all invertible* $g \in C(Y)$.

Then X and Y are locally compact and homeomorphic.

Proof. By Theorem 3, there exist a function $r \in C(Y)$ with $|r| > 0$ on Y and a continuous mapping φ from Y into X such that

$$Tf(y) = r(y)f(\varphi(y)) \quad \text{for all } f \in C(X) \text{ and all } y \in Y. \tag{4}$$

Similarly, there exist a function $s \in C(X)$ with $|s| > 0$ on X and a continuous mapping ψ from X into Y such that

$$T^{-1}g(x) = s(x)g(\psi(x)) \quad \text{for all } g \in C(Y) \text{ and all } x \in X. \tag{5}$$

Since $TT^{-1} = I = T^{-1}T$, it follows from (4) and (5) that $\varphi(\psi(x)) = x$ for all $x \in X$ and $\psi(\varphi(y)) = y$ for all $y \in Y$. Consequently, we observe that φ is a homeomorphism from Y onto X. Conditions (i) and (ii) now imply that X and Y are locally compact. The proof is complete.

THEOREM 4. *Let X and Y be two completely regular Hausdorff spaces, and let T be a linear isometry from $C(X)$ onto $C(Y)$. Assume that $Y_+ = Y$ and*

$$X = \bigcup_{g \in C_0(Y)} \{x \in X: |T^{-1}g(x)| > 0\}.$$

Then there exist a function $r \in C(Y)$ with $|r| = 1$ on Y and a homeomorphism φ from Y onto X such that (1) *holds for all $f \in C(X)$ and all $y \in Y$. In particular, X and Y are locally compact and homeomorphic.*

For the proof of this theorem we need the following lemma:

LEMMA 2. *Let X and Y be two topological spaces, and let T be a one-to-one bounded linear operator from $C(X)$ into $C(Y)$ such that $\|T\| \leqslant 1$. Let $f \in C(X)$ satisfy $\|f\| \leqslant 1$ and $|Tf| = 1$ on Y. Then $|f| = 1$ on X.*

Proof. By the Gelfand-Naimark theorem, we may and will assume without loss of generality that X is a compact Hausdorff space. Set $K = \{x \in X: |f(x)| = 1\}$. Since K is a closed subset of X, if $K \neq X$, then there exists a non-zero function $h \in C(X)$ such that $h = 0$ on K. Since T is one-to-one, Th is a non-zero function in $C(Y)$. Choose a point $y \in Y$ such that $Th(y) \neq 0$, and let μ be a positive regular measure on the Borel subsets of X with $\mu(X) = 1$ and ξ a Borel measurable function on X with $|\xi| = 1$ μ-a.e. such that

$$Tg(y) = \int_X g\xi d\mu \quad \text{for all } g \in C(X).$$

Since

$$\|f\| \leqslant 1 \quad \text{and} \quad \left|\int_X f\xi d\mu\right| = |Tf(y)| = 1,$$

it follows that $E \subset K$, where E denotes the support of μ. But this is impossible, since $h = 0$ on K and

$$\left|\int_X h\xi d\mu\right| = |Th(y)| > 0.$$

Therefore, we must have $K = X$, and this completes the proof.

Proof of Theorem 4. Let $f \in C(X)$. It then follows from Lemma 2 that $|f| = 1$ on X if and only if $|Tf| = 1$ on Y. This, together with the Weierstrass-Stone theorem (see, for example, Semadeni [32], Theorem 7.3.8), implies that $Y_t = Y$. Hence, by Theorem 1, there exist a function $r \in C(Y)$ with $|r| = 1$ on Y and a continuous mapping φ from Y into X such that

$$Tf(y) = r(y)f(\varphi(y)) \quad \text{for all } f \in C(X) \text{ and all } y \in Y.$$

Similarly, there exist a function $s \in C(X)$ with $|s| = 1$ on X and a continuous mapping ψ from X into Y such that

$$T^{-1}g(x) = s(x)g(\psi(x)) \quad \text{for all } g \in C(Y) \text{ and all } x \in X.$$

Therefore, as in the proof of the Corollary, we may observe that φ is a homeomorphism from Y onto X. The proof is complete.

THEOREM 5. *Let X and Y be two completely regular Hausdorff spaces. If there exists a linear isometry T from $C(X)$ onto $C_0(Y)$ such that $Y_+ = Y$, then X and Y are compact and homeomorphic.*

For the proof of this theorem we need the following lemma similar to Lemma 2:

LEMMA 3. *Let X and Y be two topological spaces, and let T be a one-to-one bounded linear operator from $C_0(X)$ into $C(Y)$ such that $\|T\| \leqslant 1$. Let $f \in C_0(X)$ satisfy $\|f\| \leqslant 1$ and $|Tf| = 1$ on Y. Then $|f| = 1$ on X, and hence X is compact.*

Since the proof of this lemma is essentially the same as that of Lemma 2, we may omit the details.

Proof of Theorem 5. It follows from Lemma 3 that Y is compact. Hence Theorem 4 completes the proof of Theorem 5.

THEOREM 6. *Let X and Y be two completely regular Hausdorff spaces, and let T be a one-to-one bounded linear operator from $C(X)$ onto $C_0(Y)$. Assume that $Y_1 = Y$. Then the following hold:*

(i) *X and Y are compact and homeomorphic;*

(ii) *there exist an invertible function $r \in C(Y)$ and a homeomorphism φ from Y onto X such that* (1) *holds for all $f \in C(X)$ and all $y \in Y$.*

Proof. By Theorem 1, there exist a function $r \in C(Y)$ with $|r| > 0$ on Y and a continuous mapping φ from Y into X such that

$$Tf(y) = r(y)f(\varphi(y)) \quad \text{for all } f \in C(X) \text{ and all } y \in Y.$$

Since $T1 = r$, we have $r \in C_0(Y)$. On the other hand, since

$$\sup_{y \in Y}\left|\frac{Tf(y)}{r(y)}\right| = \sup_{y \in Y}|f(\varphi(y))| \leqslant \|T^{-1}\|\,\|Tf\| \quad \text{for all } f \in C(X),$$

it follows that $\sup\{|1/r(y)|:\ y \in Y\} < \infty$, and thus there exists a positive number δ such that $|r| > \delta$ on Y. This implies that Y is compact, and hence $\varphi(Y)$ is a closed subset of X. Here, if $\varphi(Y) \neq X$, then there exists a non--zero function $h \in C(X)$ such that $h = 0$ on $\varphi(Y)$. Then we have

$$Th(y) = r(y)h(\varphi(y)) = 0 \quad \text{for all } y \in Y,$$

which is a contradiction since T is one-to-one. Consequently, we must have $\varphi(Y) = X$. Since T is onto, it follows that φ is one-to-one. This completes the proof.

From now on we assume that T is a bounded linear operator from $C_0(X)$ into $C(Y)$, where X and Y are any topological spaces. If $y \in Y$, then we set

$$b_0(y) = \sup\{|Tf(y)|:\ f \in C_0(X) \text{ and } \|f\| \leqslant 1\}$$

and

$$C_0(X)_y = \{f \in C_0(X):\ \|f\| \leqslant 1 \text{ and } |Tf(y)| = b_0(y)\}.$$

We also set

$$Y_{0+} = \{y \in Y:\ b_0(y) > 0\},$$

$$Y_{0t} = \{y \in Y:\ C_0(X)_y \text{ is total in } C_0(X)\},$$

and

$$Y_{01} = Y_{0+} \cap Y_{0t}.$$

Then we have some results similar to those in the above.

THEOREM 7. *Let X be a locally compact Hausdorff space and Y any topological space. Let T be a bounded linear operator from $C_0(X)$ into $C(Y)$. Then there exist a function $r \in C(Y_{01})$ with $|r| > 0$ on Y_{01} and a continuous mapping φ from Y_{01} into X such that* (1) *holds for all $f \in C_0(X)$ and all $y \in Y_{01}$.*

For the proof of this theorem we need the following lemma similar to Lemma 1:

LEMMA 4. *Let Y be a topological space, and let Φ be a bounded linear functional on $C_0(Y)$ such that $\|\Phi\| = 1$. Then the set*

$$\{f \in C_0(Y): \|f\| \leqslant 1 \text{ and } |\Phi f| = 1\}$$

is total in $C_0(Y)$ if and only if there exists a constant r with $|r| = 1$ such that

$$\Phi(fg) = (\Phi f)(\Phi g)r^{-1} \quad \text{for all } f, g \in C_0(Y). \tag{6}$$

Proof. By the Gelfand-Naimark theorem, we may and will assume without loss of generality that Y is a locally compact Hausdorff space. Then the Riesz representation theorem implies that there exist a positive regular measure μ on the Borel subsets of Y with $\mu(Y) = 1$ and a Borel measurable function ξ on Y with $|\xi| = 1$ μ-a.e. such that

$$\Phi f = \int_Y f\xi d\mu \quad \text{for all } f \in C_0(Y).$$

By this, we may apply the arguments of the proof of Lemma 1 to prove the present lemma, and the details are omitted.

Proof of Theorem 7. Lemma 4 implies that if $y \in Y_{01}$, then there exist a constant $r(y)$ with $|r(y)| > 0$ and a point $\varphi(y) \in X$ such that

$$Tf(y) = r(y)f(\varphi(y)) \quad \text{for all } f \in C_0(X).$$

To prove that φ is continuous on Y_{01}, let $y_0 \in Y_{01}$ and $x_0 = \varphi(y_0)$, and let V be a neighborhood of x_0. Since X is locally compact, there exists a function $h \in C_0(X)$ such that $h(x_0) = 1$ and $h = 0$ on $X - V$. Then $Th(y_0) \neq 0$. Since Th is continuous on Y, there exists a neighborhood W of y_0 such that $|Th| > 0$ on W. It then follows that $\varphi(W) \subset V$. Consequently, φ is continuous on Y_{01}, from which it also follows that r is continuous on Y_{01}. The proof is complete.

The following result is an immediate corollary to Theorem 7 and a generalization of the Proposition of [30]:

THEOREM 8. *Let X be a locally compact Hausdorff space and Y any topological space. Let T be a bounded linear operator from $C_0(X)$ into $C(Y)$ such that $\|T\| \leqslant 1$. Then the set*

$$\{f \in C_0(X): \|f\| \leqslant 1 \text{ and } |Tf(y)| = 1\}$$

is total in $C_0(X)$ for all $y \in Y$ if and only if there exist a function $r \in C(Y)$ with $|r| = 1$ on Y and a continuous mapping φ from Y into X such that (1) *holds for all $f \in C_0(X)$ and all $y \in Y$.*

The following result corresponds to the Corollary. Since the proof is similar to that of the Corollary, we omit the details.

THEOREM 9. *Let X and Y be two locally compact Hausdorff spaces. Assume that there exist a one-to-one bounded linear operator T from $C_0(X)$*

onto $C_0(Y)$ and two complex-valued functions r and s on Y and X, respectively, such that

(i) $|r| > 0$ *on* Y *and* $|s| > 0$ *on* X,

(ii) $Tf(y)/r(y)$ *belongs to* $f(X)$ *for all* $f \in C_0(X)$ *and all* $y \in Y$, *and*

(iii) $T^{-1}g(x)/s(x)$ *belongs to* $g(Y)$ *for all* $g \in C_0(Y)$ *and all* $x \in X$.

Then X and Y are homeomorphic.

THEOREM 10. *Let X and Y be two locally compact Hausdorff spaces, and let T be a linear isometry from $C_0(X)$ into $C_0(Y)$. Let Y_{01} and φ be as in Theorem 7. Then $\varphi(Y_{01}) = X$. In particular, if T transforms $C_0(X)$ onto $C_0(Y)$, then φ is a homeomorphism from Y onto X.*

Proof. We proceed as in Holsztyński [22] (see also Semadeni [32], p. 391-394). For $x \in X$, let

$$Y_x = \{y \in Y\colon f \in C_0(X) \text{ and } \|f\| = 1 = |f(x)| \text{ imply } |Tf(y)| = 1\}.$$

We first show that Y_x is non-empty. To do this, consider any finite set of functions $f_1, \ldots, f_n$ in $C_0(X)$ with $\|f_i\| = 1 = |f_i(x)|$ for all $i = 1, \ldots, n$, and define a function h in $C_0(X)$ by

$$h = \frac{1}{n}\sum_{i=1}^{n} \overline{f_i(x)} f_i.$$

Then $\|Th\| = \|h\| = 1$, $|h(x)| = 1$ and $Th \in C_0(Y)$. It follows that

$$\emptyset \neq \{y \in Y\colon |Th(y)| = 1\} \subset \bigcap_{i=1}^{n} \{y \in Y\colon |Tf_i(y)| = 1\}.$$

Hence the family of all sets $\{y \in Y\colon |Tf(y)| = 1\}$, where $f \in C_0(X)$ and $\|f\| = 1 = |f(x)|$, has the finite intersection property. Since these sets are all compact and Y_x is the intersection of these sets, Y_x is non-empty. It is now easy to check that $Y_x \subset Y_{01}$ and $\varphi(y) = x$ for all $y \in Y_x$. Therefore $\varphi(Y_{01}) = X$.

In particular, if T transforms $C_0(X)$ onto $C_0(Y)$, then, clearly, φ is one-to-one and T^{-1} is a linear isometry from $C_0(Y)$ onto $C_0(X)$. Therefore, applying the above-given argument to T^{-1}, we observe that φ^{-1} is continuous on X and $\varphi^{-1}(X) = Y$. This completes the proof.

3. Operators on L_1-spaces. In this section we consider linear contraction operators in L_1-spaces of probability measure spaces. Let $(X, \mathscr{A}, \lambda)$ and $(Y, \mathscr{B}, \mu)$ be two probability measure spaces, and let T be a bounded linear operator from $L_1(X, \mathscr{A}, \lambda)$ into $L_1(Y, \mathscr{B}, \mu)$ such that $\|T\|_1 \leqslant 1$. Write

$$G(T) = \{f \in L_1(X, \mathscr{A}, \lambda)\colon |f| = 1 = |Tf|\}.$$

Then we have the following theorem, which generalizes the Corollary of [30]:

THEOREM 11. *$G(T)$ is total in $L_1(X, \mathscr{A}, \lambda)$ if and only if there exist a function $r \in L_1(Y, \mathscr{B}, \mu)$ with $|r| = 1$ μ-a.e. and a homomorphism φ from the measure algebra $(\mathscr{A}(\lambda), \lambda)$ associated with $(X, \mathscr{A}, \lambda)$ into the measure algebra $(\mathscr{B}(\mu), \mu)$ associated with $(Y, \mathscr{B}, \mu)$ such that $Tf = r(Uf)$ for all $f \in L_1(X, \mathscr{A}, \lambda)$, where U denotes the operator from $L_1(X, \mathscr{A}, \lambda)$ into $L_1(Y, \mathscr{B}, \mu)$ induced by φ.*

Proof. The sufficiency of the condition is easily checked. To prove the necessity of it, fix an $h \in G(T)$ and define a linear operator U from $L_1(X, \mathscr{A}, \lambda)$ into $L_1(Y, \mathscr{B}, \mu)$ by the relation

$$Uf = \overline{Th}T(fh) \quad \text{for all } f \in L_1(X, \mathscr{A}, \lambda).$$

It follows that $\|U\|_1 = \|T\|_1 \leqslant 1$ and $U1 = 1$. We now show that U is positive. To do this, let $0 \leqslant f,\ g \in L_1(X, \mathscr{A}, \lambda)$ and $f+g = 1$. Then it follows that

$$Uf + Ug = U1 = 1 \quad \text{and} \quad 1 \leqslant \|Uf\|_1 + \|Ug\|_1 \leqslant \|f\|_1 + \|g\|_1 = 1.$$

Therefore, an easy argument shows that $0 \leqslant Uf,\ Ug \leqslant 1$. This and an approximation argument imply that if $0 \leqslant f \in L_1(X, \mathscr{A}, \lambda)$, then $Uf \geqslant 0$, i.e., U is positive. From this and from the fact that $G(U) = \{f/h\colon f \in G(T)\}$ is total in $L_1(X, \mathscr{A}, \lambda)$, a slight modification of the Corollary of [30] implies that U is induced by a homomorphism φ from $(\mathscr{A}(\lambda), \lambda)$ into $(\mathscr{B}(\mu), \mu)$. Let us now set $r = Th/Uh$. It then follows that $r \in L_1(Y, \mathscr{B}, \mu)$ and $|r| = 1$ μ-a.e. Moreover, for any $f \in L_1(X, \mathscr{A}, \lambda)$, we have

$$Tf = (Th)\,U(f/h) = (Th/Uh)\,Uf = r(Uf),$$

and hence the proof is complete.

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