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ON ALGEBRAIC OPERATIONS OF A LATTICE ORDERED GROUP

BY

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Let $\mathfrak{A} = (M, F)$ be an algebra with the underlying set M and with the set F of fundamental operations. The set of all algebraic operations of the algebra $\mathfrak{A}$ will be denoted by $a(\mathfrak{A})$ (for the definitions, cf. Section 1).

Higman and Neumann [7] proposed the following problem:

Does there exist a pair of groups $\mathfrak{G}_1 = (G; \circ_1)$ and $\mathfrak{G}_2 = (G; \circ_2)$ with the same underlying set G and with the properties:

(i) $\circ_2 \in a(\mathfrak{G}_1)$ and $\circ_1 \in a(\mathfrak{G}_2)$;

(ii) the operation $\circ_1$ on G coincides neither with $\circ_2$ nor with the operation $\circ_3$ on G defined by $a \circ_3 b = b \circ_2 a$ for each $a, b \in G$?

Hulanicki and Świerczkowski [8] have proved that the answer to this problem is positive.

In the terminology introduced by Marczewski and Goetz (see [6] and [11]) condition (i) can be expressed as follows: the identity mapping on G is a weak isomorphism of $\mathfrak{G}_1$ onto $\mathfrak{G}_2$.

Now let $\mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1)$ be a pair of lattice ordered groups with the same underlying set G. For each $g \in G$ we define binary operations $+(g)$ and $+_1(g)$ on G by putting

$$a+(g)+b = a-g+b \quad \text{and} \quad a+_1(g)+b = a-_1g+_1b$$

for each $a, b \in G$. Obviously,

$$\mathfrak{G}(g) = \big(G; +(g), \wedge, \vee\big) \quad \text{and} \quad \mathfrak{G}_1(g) = \big(G; +_1(g), \wedge_1, \vee_1\big)$$

are lattice ordered groups.

Let us consider the following conditions for $\mathfrak{G}$ and $\mathfrak{G}_1$:

(i_1) $\wedge_1, \vee_1 \in a(\mathfrak{G})$ and $\wedge, \vee \in a(\mathfrak{G}_1)$.

(i_2) For each $g \in G$,

$$\wedge_1, \vee_1 \in a\big(\mathfrak{G}(g)\big) \quad \text{and} \quad \wedge, \vee \in a\big(\mathfrak{G}_1(g)\big).$$

The partial order in $\mathfrak{G}$ and $\mathfrak{G}_1$ will be denoted by $\leqslant$ and $\leqslant_1$, respectively. The positive cone and the negative cone of $\mathfrak{G}$ are the sets

$$G^+ = \{g \in G : g \geqslant 0\} \quad \text{and} \quad G^- = \{g \in G : g \leqslant 0\}$$

(0 denoting the neutral element of $\mathfrak{G}$). Let G_1^+ and G_1^- be defined analogously (with respect to $\mathfrak{G}_1$).

In this paper the following question is investigated: which relations between the partial orders $\leqslant$ and $\leqslant_1$ are consequences of conditions (i_1) and (i_2), respectively?

The following results will be proved:

(A) *If* (i_1) *holds and if the neutral element of* $\mathfrak{G}$ *coincides with the neutral element of* $\mathfrak{G}_1$, *then either*

$$G^+ = G_1^+ \quad \textit{and} \quad G^- = G_1^-$$

or

$$G^+ = G_1^- \quad \textit{and} \quad G^- = G_1^+.$$

(B) *If* (i_2) *holds, then* $\leqslant$ *coincides either with* $\leqslant_1$ *or with* $\geqslant_1$.

Since the partial order in a lattice ordered group is uniquely determined by the group operation and by the corresponding positive cone, we obtain from (A) as a corollary:

(C) *If* $\mathfrak{G} = (G; +, \wedge, \vee)$ *and* $\mathfrak{G}_1 = (G; +, \wedge_1, \vee_1)$ *are lattice ordered groups with the same underlying set and the same group operation and if* (i_1) *holds, then* $\leqslant$ *coincides either with* $\leqslant_1$ *or with* $\geqslant_1$.

The question whether (i_1) is equivalent to (i_2) remains open. (**P 1065**)

1. Preliminaries. We use the standard terminology for lattices and lattice ordered groups (cf. [1], [2], and [4]). Let us recall the notion of weak isomorphism of algebras.

Let $\mathfrak{A} = (M; F)$ be an algebra with the underlying set M and with the set F of fundamental operations. The operations $e_j^{(n)}$ of the form

$$e_j^{(n)}(x_1, x_2, \ldots, x_n) = x_j$$

are called *trivial*. The smallest family of operations on M containing all trivial and fundamental operations and closed with respect to compositions is called the *family of algebraic operations* and will be denoted by $a(\mathfrak{A})$.

Let $\mathfrak{A}_1 = (M_1, F_1)$ and $\mathfrak{A}_2 = (M_2, F_2)$ be two given algebras, and let φ be a one-to-one mapping of the set M_1 onto M_2. For each n-ary operation $f \in a(\mathfrak{A}_1)$ we define an n-ary operation f^* on the set M_2 by putting

$$f^*(c_1, c_2, \ldots, c_n) = \varphi\big(f(\varphi^{-1}(c_1), \varphi^{-1}(c_2), \ldots, \varphi^{-1}(c_n))\big)$$

for each n-tuple $(c_1, c_2, \ldots, c_n)$ of elements of M_2. Analogously, for each n-ary operation $g \in \alpha(\mathfrak{A}_2)$ we put

$$g^*(d_1, d_2, \ldots, d_n) = \varphi^{-1}\Big(g\big(\varphi(d_1), \varphi(d_2), \ldots, \varphi(d_n)\big)\Big)$$

for each n-tuple $(d_1, d_2, \ldots, d_n)$ of elements of M_1.

The mapping φ is called a *weak isomorphism* of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ if for each $f \in F_1$ and each $g \in F_2$ we have $f^* \in \alpha(\mathfrak{A}_2)$ and $g^* \in \alpha(\mathfrak{A}_1)$.

The notion of the weak isomorphism of algebras was introduced by Goetz and Marczewski (see [6], [10], and [11]). Weak isomorphisms and weak automorphisms of universal algebras and of special types of algebraic structures were investigated by several authors (see, e.g., Dudek and Płonka [3], Traczyk [14], Senft [12], Sichler [13], the author [9]). Głazek and Michalski [5] studied weak homomorphisms of algebras.

The notion of weak isomorphism of algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be generalized by supposing that there are given subsets $F_1' \subseteq F_1$ and $F_2' \subseteq F_2$ and by assuming that for each $f \in F_1'$ and each $g \in F_2'$ we have $f^* \in \alpha(\mathfrak{A}_2)$ and $g^* \in \alpha(\mathfrak{A}_1)$.

Let $\mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{G}_1 = (G_1; +_1, -_1, \wedge_1, \vee_1)$ be lattice ordered groups and let φ be a one-to-one mapping of the set G onto G_1. Suppose that the relations

$$\wedge^*, \vee^* \in \alpha(\mathfrak{G}_1) \quad \text{and} \quad \wedge_1^*, \vee_1^* \in \alpha(\mathfrak{G})$$

are valid. Then φ is said to be a *w-isomorphism* of $\mathfrak{G}$ onto $\mathfrak{G}_1$.

Let $\mathfrak{G} = (G; +, \wedge, \vee)$ be a lattice ordered group and suppose that $a, b, c_1, c_2, \ldots, c_n$ belong to G. Then we have

$$a + \bigvee c_i + b = \bigvee (a + c_i + b) \quad (i = 1, 2, \ldots, n),$$

and dually. Hence and from the distributivity of the lattice $(G; \wedge, \vee)$ we infer that each non-trivial binary operation $f \in \alpha(\mathfrak{G})$ can be written as

$$(1) \qquad f(x, y) = A_1 \vee A_2 \vee \ldots \vee A_n,$$

where

$$(2) \qquad A_i = B_{i1} \wedge B_{i2} \wedge \ldots \wedge B_{i,k(i)} \quad (i = 1, 2, \ldots, n),$$

$$(3) \quad B_{ij} = \sum \varepsilon_t^{ij} z_t^{ij} \quad (t = 1, 2, \ldots, m(i, j)), \quad \varepsilon_t^{ij} \in \{+, -\}, z_t^{ij} \in \{x, y\}.$$

2. w-isomorphism of lattice ordered groups (special case). In this section we assume that

$$\mathfrak{G} = (G; +, \wedge, \vee) \quad \text{and} \quad \mathfrak{G}_1 = (G_1; +_1, \wedge_1, \vee_1)$$

are lattice ordered groups such that

(i) $G = G_1$;

(ii) the neutral element of $\mathfrak{G}$ coincides with that of $\mathfrak{G}_1$ (this element will be denoted by 0);

(iii) the identity mapping on G is a w-isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$.

The case $\operatorname{card} G = 1$ being trivial, we suppose that G has more than one element. Hence there are elements $g_1, g_2 \in G$ with $0 < g_1$ and $0 <_1 g_2$.

According to (iii), $\vee_1$ and $\vee_1^*$ on G coincide (and analogously for the operations $\wedge_1, \vee, \wedge$). There exists a non-trivial binary operation $f \in \alpha(\mathfrak{G})$ such that

$$a_1 \vee_1 a_2 = f(a_1, a_2)$$

is valid for each pair (a_1, a_2) of elements of G. We can write $f(x, y)$ as in (1)-(3).

If m is an integer and $g \in G$, then the multiplication mg in G will have the obvious meaning; an analogous operation in G_1 will be denoted by $m \circ g$.

2.1. LEMMA. *There exists an integer n_1 such that*

$$a_1 \vee_1 0 = n_1 a_1$$

is valid for each element $a_1 \in G$ with $0 < a_1$.

Proof. Choose $0 < a_1 \in G$. In (1)-(3) we put $x = a_1$ and $y = 0$. Then there are integers n_{ij} such that for each B_{ij} we have

$$B_{ij} = n_{ij} a_1.$$

Put

$$m^{(i)} = \min\{n_{i1}, n_{i2}, \ldots, n_{i,k(i)}\} \quad \text{and} \quad n_1 = \max\{m^{(1)}, m^{(2)}, \ldots, m^{(n)}\}.$$

Hence

$$A_i = m^{(i)} a_1 \quad \text{and} \quad f(a_1, 0) = n_1 a_1.$$

It is obvious that n_1 does not depend on the particular choice of the element $a_1 > 0$.

Analogously we can verify the following assertion:

2.2. LEMMA. *There are integers $n_2, m_1, m_2, n_1', n_2', m_1'$ and m_2' such that*

(a) $a_1 \wedge_1 0 = n_1' a_1$ *is valid for each $a_1 \in G$ with $0 < a_1$;*

(b) $a_2 \vee_1 0 = n_2 a_2$ *and* $a_2 \wedge_1 0 = n_2' a_2$ *are valid for each $a_2 \in G$ with $a_2 < 0$;*

(c) $b_1 \vee 0 = m_1 \circ b_1$ *and* $b_1 \wedge 0 = m_1' \circ b_1$ *are valid for each $b_1 \in G$ with $0 >_1 b_1$;*

(d) $b_2 \vee 0 = m_2 \circ b_2$ *and* $b_2 \wedge 0 = m_2' \circ b_2$ *are valid for each $b_2 \in G$ with $b_2 <_1 0$.*

2.3. LEMMA. $n_1 > 0 \Rightarrow n_1 = 1$.

Proof. Let $n_1 > 0$. Choose $a_1 \in G$ with $0 < a_1$ and put $n_1 a_1 = c$. Then $c > 0$, whence according to 2.1 we have

$$c = c \vee_1 0 = n_1 c,$$

thus $n_1 = 1$.

Similarly we obtain

2.4. LEMMA. *If* $n \in \{n_2, m_1, m_2, n_1', n_2', m_1', m_2'\}$ *and* $n > 0$, *then* $n = 1$.

2.5. LEMMA. $n_1 > 0 \Rightarrow m_1 = 1$.

Proof. Let $n_1 > 0$. According to 2.3, $n_1 = 1$. Choose $0 < a_1 \in G$. Then $a_1 \vee_1 0 = a_1$, and thus $a_1 >_1 0$. We have

$$a_1 = a_1 \vee 0 = m_1 \circ a_1,$$

whence $m_1 = 1$.

Analogously we get

2.6. LEMMA. $m_1 > 0 \Rightarrow n_1 = 1$.

Write

$$G^+ = \{g \in G : g \geqslant 0\} \quad \text{and} \quad G_1^+ = \{g \in G : g \geqslant_1 0\}.$$

The symbols G^- and G_1^- have analogous meanings. From 2.3, 2.5 and 2.6 it follows

2.7. LEMMA. *If* $n_1 > 0$, *then* $G^+ = G_1^+$; *hence* $n_1' = 0$ *and* $m_1' = 0$.

2.8. LEMMA. $n_2 > 0 \Rightarrow m_1 = 0$.

Proof. Let $n_2 > 0$. Choose $a_2 \in G$, $a_2 < 0$. According to 2.4 we have $n_2 = 1$, whence

$$a_2 \vee_1 0 = a_2.$$

Thus $a_2 >_1 0$ and

$$0 = a_2 \vee 0 = m_1 \circ a_2,$$

whence $m_1 = 0$.

2.9. LEMMA. $n_1 = 0 \Rightarrow n_2 \neq 0$, $n_2 = 0 \Rightarrow n_1 \neq 0$.

Proof. Assume that $n_1 = 0 = n_2$. Choose $a_1, a_2 \in G$ with $a_2 < 0 < a_1$. Then

$$a_1 \vee_1 0 = 0 = a_2 \vee_1 0,$$

whence $a_1 <_1 0$ and $a_2 <_1 0$. Thus

$$a_1 \vee 0 = m_2 \circ a_1 \quad \text{and} \quad a_2 \vee 0 = m_2 \circ a_2.$$

Since $a_1 \vee 0 = a_1$, we obtain $m_2 = 1$. From $a_2 \vee 0 = 0$ we get $m_2 = 0$, which is a contradiction.

Analogously we obtain

2.9.1. LEMMA. $m_1 = 0 \Rightarrow m_2 \neq 0$, $m_2 = 0 \Rightarrow m_1 \neq 0$.

2.10. LEMMA. *Let* $n_1 > 0$. *Then* $n_2' \nless 0$.

Proof. Suppose that $n_2' < 0$. Choose a_2, $0 > a_2 \in G$. Then

$$a_2 \wedge_1 0 = n_2' a_2 \neq 0$$

and $n_2' a_2 \in G^+$. Thus, according to 2.7, $n_2' a_2 \in G_1^+$; on the other hand, $a_2 \wedge_1 0 \in G_1^-$, whence $a_2 \wedge_1 0 = 0$, a contradiction.

2.11. LEMMA. *Let* $n_1 > 0$. *Then* $n_2 = 0$.

Proof. According to 2.5 and 2.8 we have $n_2 \leqslant 0$. Assume that $n_2 < 0$. Choose $a_2 \in G$ with $a_2 < 0$. Consider the following possibilities for n_2':

(a) $n_2' > 0$. Then according to 2.4 we have $n_2' = 1$, thus $a_2 \wedge_1 0 = a_2$, whence $a_2 <_1 0$. Therefore $a_2 \vee_1 0 = 0$. Since $a_2 \vee_1 0 = n_2 a_2$ and $n_2 < 0$, we get $n_2 a_2 \neq 0$, which is a contradiction.

(b) $n_2' = 0$. Hence $a_2 \wedge_1 0 = 0$, thus $a_2 >_1 0$. Then

$$a_2 = a_2 \vee_1 0 = n_2 a_2,$$

and so $n_2 = 1$, a contradiction.

(c) $n_2' < 0$. This is impossible according to 2.10.

The proof is complete.

Analogously we have

$$m_1 > 0 \Rightarrow m_2 = 0.$$

By summarizing, from 2.3, 2.5 and 2.11 we obtain the following assertion:

2.12. PROPOSITION. *Let* $n_1 > 0$. *Then* $n_1 = m_1 = 1$ *and* $n_2 = m_2 = 0$.

2.13. LEMMA. *Let* $n_1 = 0$. *Then* $m_2 = 1$ *and* $G_1^- = G^+$.

Proof. Choose a_1 such that $0 < a_1 \in G$. According to the assumption we have

$$a_1 \vee_1 0 = 0,$$

whence $a_1 <_1 0$. Thus $G^+ \subseteq G_1^-$. Moreover,

$$a_1 = a_1 \vee 0 = m_2 \circ a_1$$

and, therefore, $m_2 = 1$. For each $b_2 \in G$ with $b_2 <_1 0$ the relation

$$b_2 \vee 0 = m_2 \circ b_2 = b_2$$

is valid, thus $G_1^- \subseteq G^+$. Hence $G_1^- = G^+$.

2.14. LEMMA. *Let* $n_1 = 0$. *Then* $n_2 = 1$.

Proof. According to 2.9, $n_2 \neq 0$. By 2.13 we have $G^+ = G_1^-$. Assume that $n_2 < 0$. Choose $a_2 \in G$ with $0 > a_2$. Then

$$a_2 \vee_1 0 = n_2 a_2$$

and $n_2a_2 \in G^+, n_2a_2 \neq 0$. Hence $a_2 \vee_1 0 \in G^+ = G_1^-$. Thus $a_2 \vee_1 0 = 0$, which is a contradiction. Therefore $n_2 > 0$. Hence, according to 2.4, $n_2 = 1$.

2.15. LEMMA. *Let $n_1 = 0$. Then $m_1 = 0$ and $m_2 = 1$.*

Proof. Choose $a_2 \in G$ with $a_2 < 0$. According to 2.14 we have $a_2 \vee_1 0 = a_2$, whence $a_2 >_1 0$. Thus we have

$$0 = a_2 \vee 0 = m_1 \circ a_2;$$

therefore, $m_1 = 0$. Hence and from 2.14 (by taking G_1 instead of G) we get $m_2 = 1$.

By summarizing, from 2.13, 2.14 and 2.15 we obtain

2.16. PROPOSITION. *Let $n_1 = 0$. Then $m_1 = 0$ and $n_2 = m_2 = 1$.*

2.17. LEMMA. *If $n_1 < 0$, then $m_1 < 0$.*

Proof. Assume that $n_1 < 0$. If $m_1 > 0$, then from 2.12 (and by replacing G by G_1) we get $n_1 = 1$, a contradiction. If $m_1 = 0$, then by 2.16 (and taking G_1 instead of G) we obtain $n_1 = 0$, a contradiction. Hence $n_1 < 0$ implies $m_1 < 0$.

2.18. PROPOSITION. *$n_1 < 0$ cannot hold.*

Proof. Assume that $n_1 < 0$. Then by 2.17 we have also $m_1 < 0$. Choose $b_1 \in G$ with $0 <_1 b_1$. Put $c = m_1 \circ b_1$. Then

$$b_1 \vee 0 = m_1 \circ b_1 = c,$$

whence $c \neq 0$, $c \in G^+ \cap G_1^-$. From $c \in G_1^-$ we obtain $c \vee_1 0 = 0$.

On the other hand, since $c \in G^+$ and $n_1 < 0$, we have

$$c \vee_1 0 = n_1 c \neq 0,$$

which is a contradiction.

2.19. THEOREM. *Let $\mathfrak{G}$ and $\mathfrak{G}_1$ be lattice ordered groups fulfilling conditions* (i), (ii) *and* (iii). *Let $G \neq \{0\}$. Then one (and only one) of the following conditions holds:*

(a) $G^+ = G_1^+$ *and* $G^- = G_1^-$;

(b) $G^+ = G_1^-$ *and* $G^- = G_1^+$.

Proof. According to 2.18 we have either $n_1 > 0$ or $n_1 = 0$. If $n_1 > 0$, then from 2.12 we infer that (a) holds. If $n_1 = 0$, then, by 2.16, (b) is valid.

2.20. COROLLARY. *Let*

$$\mathfrak{G} = (G; +, -, \wedge, \vee) \quad \textit{and} \quad \mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1)$$

be lattice ordered groups such that

$$(\alpha) \qquad \wedge_1, \vee_1 \in a(\mathfrak{G}) \quad \textit{and} \quad \wedge, \vee \in a(\mathfrak{G}_1).$$

Then $\leqslant$ coincides either with $\leqslant_1$ or with $\geqslant_1$.

2.21. Remark. It can be shown by examples that assumption (α) in 2.20 cannot be omitted.

3. w-isomorphism (general case). Now assume that $\mathfrak{G}$ and $\mathfrak{G}_1$ are lattice ordered groups fulfilling condition (i) from Section 2. For each $g \in G$ we put

$$G(g)^+ = \{h \in G : h \geqslant g\}, \quad G(g)^- = \{h \in G : h \leqslant g\},$$

$$G_1(g)^+ = \{h \in G : h \geqslant_1 g\}, \quad G_1(g)^- = \{h \in G : h \leqslant_1 g\}.$$

3.1. Lemma. *Let $\mathfrak{G}$ and $\mathfrak{G}_1$ be lattice ordered groups with the same underlying set G,* $\operatorname{card} G > 1$. *Let $g \in G$. Suppose that the relations*

$$\wedge_1, \vee_1 \in a(\mathfrak{G}(g)) \quad \textit{and} \quad \wedge, \vee \in a(\mathfrak{G}_1(g))$$

are valid. Then one (and only one) of the following conditions holds:

(a$_1$) $G(g)^+ = G_1(g)^+$ *and* $G(g)^- = G_1(g)^-$;

(b$_1$) $G(g)^+ = G_1(g)^-$ *and* $G(g)^- = G_1(g)^+$.

Proof. According to the assumption, the identity mapping on G is a w-isomorphism of $G(g)$ onto $G_1(g)$. Since g is the neutral element in both G_1 and G_2, the assertion follows immediately from 2.19.

3.2. Lemma. *Let $\mathfrak{G}$ and $\mathfrak{G}_1$ be lattice ordered groups with the same underlying set G. Suppose that, for each $h \in G$, the identity mapping on G is a w-isomorphism of $\mathfrak{G}(h)$ onto $\mathfrak{G}_1(h)$. Let $g, a, b \in G$; $g < a < b$. If* (a$_1$) *holds, then $a <_1 b$. If* (b$_1$) *is valid, then $b <_1 a$.*

Proof. Let (a$_2$) and (b$_2$) be the conditions that we obtain from (a$_1$) and (b$_1$) if we replace the element g by the element a. According to 3.1, either (a$_2$) or (b$_2$) is valid. Suppose that (a$_1$) holds. Hence $g <_1 a$, and thus (b$_2$) cannot be valid. Therefore, (a$_2$) is true and this implies that $a <_1 b$ holds. The case where (b$_1$) is valid is analogous.

3.3. Lemma. *Let $\mathfrak{G}$ and $\mathfrak{G}_1$ be as in 3.2. Let $g \in G$,* $\operatorname{card} G \neq 1$. *If* (a$_1$) *holds, then $\leqslant$ coincides with $\leqslant_1$. If* (b$_1$) *is valid, then $\leqslant$ is dual to $\leqslant_1$.*

Proof. Suppose that (a$_1$) holds. Let $a, b \in G$, $a < b$. Choose $u \in G$ with $u < a$ and $u < g$. Let (a$_3$) and (b$_3$) be the conditions that we obtain from (a$_1$) and (b$_1$) if we replace g by the element u. There exists $g_1 \in G$ with $g < g_1$. Hence from 3.2 it follows that (b$_3$) cannot hold; thus, according to 3.1, (a$_3$) is valid. Hence and from 3.2 we infer that $a < b$ is true. The case where (b$_1$) holds can be treated analogously.

3.4. Theorem. *Let*

$$\mathfrak{G} = (G; +, -, \wedge, \vee, \leqslant) \quad \textit{and} \quad \mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1, \leqslant_1)$$

be lattice ordered groups such that, for each $g \in G$,

(a) $\wedge$ *and* $\vee$ *belong to* $a(\mathfrak{G}_1(g))$,

(b) $\wedge_1$ *and* $\vee_1$ *belong to* $a(\mathfrak{G}(g))$.

Then either $\leqslant_1$ coincides with $\leqslant$ or $\leqslant_1$ is dual to $\leqslant$.

This follows from 3.1 and 3.3.

3.4.1. Remark. If $\mathfrak{G}$ and $\mathfrak{G}_1$ are as in 3.4, then the operations $+$ and $+_1$ on G need not coincide; e.g., it can happen that the operation $+$ is commutative while the operation $+_1$ is not commutative.

3.5. THEOREM. *Let*

$$\mathfrak{G} = (G;\, +,\, -,\, \wedge,\, \vee) \quad \text{and} \quad \mathfrak{G}_1 = (G_1;\, +_1,\, -_1,\, \wedge_1,\, \vee_1)$$

be lattice ordered groups. Let φ be a w-isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$ such that $\varphi(0)$ is the neutral element in G_1. Then either

$$\varphi(G^+) = G_1^+ \quad \text{and} \quad \varphi(G^-) = G_1^-$$

or

$$\varphi(G^+) = G_1^- \quad \text{and} \quad \varphi(G^-) = G_1^+.$$

Moreover, if $+_i^*$ coincides with $+$, then φ is either an isomorphism or a dual isomorphism of the lattice $(G;\, \wedge,\, \vee)$ onto the lattice $(G_1;\, \wedge_1,\, \vee_1)$.

This follows immediately from 2.19 and 2.20.

3.6. THEOREM. *Let*

$$\mathfrak{G} = (G;\, +,\, -,\, \wedge,\, \vee) \quad \text{and} \quad \mathfrak{G}_1 = (G_1;\, +_1,\, -_1,\, \wedge_1,\, \vee_1)$$

be lattice ordered groups, and φ a one-to-one mapping of the set G onto G_1. Suppose that, for each $g \in G$, φ is a w-isomorphism of $\mathfrak{G}(g)$ onto $\mathfrak{G}(\varphi(g))$. Then φ is either an isomorphism or a dual isomorphism of the lattice $(G;\, \wedge,\, \vee)$ onto the lattice $(G_1;\, \wedge_1,\, \vee_1)$.

This is a consequence of 3.4.

REFERENCES

[1] G. Birkhoff, *Lattice theory*, 3rd edition, Providence 1967.

[2] P. Conrad, *Lattice ordered groups*, Tulane University 1970.

[3] J. Dudek and E. Płonka, *Weak automorphisms of linear spaces and of some other abstract algebras*, Colloquium Mathematicum 22 (1971), p. 201-208.

[4] L. Fuchs, *Partially ordered algebraic systems*, Oxford 1963.

[5] K. Głazek and J. Michalski, *On weak homomorphisms of general non-indexed algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 22 (1974), p. 651-656.

[6] A. Goetz, *On weak isomorphisms and weak homomorphisms of abstract algebras*, Colloquium Mathematicum 14 (1966), p. 163-167.

[7] G. Higman and B. H. Neumann, *Groups as groupoids with one law*, Publicationes Mathematicae 2 (1952), p. 215-221.

[8] A. Hulanicki and S. Świerczkowski, *On group operations other than xy or yx*, ibidem 9 (1962), p. 142-148.

[9] J. Jakubík, *W-isomorphisms of distributive lattices*, Czechoslovak Mathematical Journal 26 (1976), p. 330-338.

[10] E. Marczewski, *A general scheme of the notion of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 6 (1958), p. 731-736.

[11] — *Independence in abstract algebras. Results and problems*, Colloquium Mathematicum 14 (1966), p. 169-188.

[12] R. Senft, *On weak automorphisms of universal algebras*, Dissertationes Mathematicae 74 (1970).

[13] J. Sichler, *Weak automorphisms of universal algebras*, Algebra Universalis 3 (1973), p. 1-7.

[14] T. Traczyk, *Weak isomorphisms of Boolean and Post algebras*, Colloquium Mathematicum 13 (1965), p. 159-164.

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