COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 1

IDENTITIES FOR GLOBALS (COMPLEX ALGEBRAS) OF ALGEBRAS

BY

G. GRÄTZER AND H. LAKSER (WINNIPEG, MANITOBA)

The *global* $\mathfrak{P}(\mathfrak{A})$ of a universal algebra $\mathfrak{A} = \langle A, F \rangle$ is the family of all nonvoid subsets ("complexes") of A with operations given by

$$f(A_1, \ldots, A_n) = \{f(a_1, \ldots, a_n) \mid a_i \in A_i\}$$

whenever f is an n-ary operation in F, and $A_1, \ldots, A_n$ are complexes of A. Note that if f is nullary, then its realization in $\mathfrak{P}(\mathfrak{A})$ is the singleton $\{f^{\mathfrak{A}}\}$ subset of A determined by the realization of f in $\mathfrak{A}$. We denote by $\mathfrak{P}_0(\mathfrak{A})$ the analogous algebraic structure on the set of *all* subsets of A, including the empty set.

There is unfortunately no standard terminology or notation for the above concepts. We follow essentially that in [3] and the references therein. In [7], $\mathfrak{P}_0(\mathfrak{A})$ is called the *power algebra* of $\mathfrak{A}$, and in [2], the *complex algebra*. In [8], [9] and [1], $\mathfrak{P}$ is called the complex algebra. In [5], both $\mathfrak{P}$ and $\mathfrak{P}_0$ are considered; $\mathfrak{P}$ is denoted by Com^+ and $\mathfrak{P}_0$ is denoted by Com. These works give necessary and sufficient conditions on the identities of a variety in order that it be closed under the operations of $\mathfrak{P}$ and $\mathfrak{P}_0$ (see [7] and [5] for a discussion of the differences between the various results). In many ways the results in [5] are the most general, permitting infinitary operations and also infinitary relations.

In this paper* we consider an arbitrary variety V of finitary algebras and we determine the identities satisfied by the classes $\{\mathfrak{P}(\mathfrak{A}) \mid \mathfrak{A} \in V\}$ and $\{\mathfrak{P}_0(\mathfrak{A}) \mid \mathfrak{A} \in V\}$ in terms of the identities of V. We then incidentally obtain the above-mentioned results as corollaries in the case of finitary algebras. We apply our results to determine all the varieties determined by the globals of varieties of lattices and of groups.

1. Globals of general algebras. Let V be a variety of algebras of type F, where F is a set of operation symbols. Given any algebra $\mathfrak{A}$ in V, identifying

* The research of both authors was supported by the Natural Sciences and Engineering Research Council of Canada.

the element a with the singleton $\{a\}$ embeds $\mathfrak{A}$ as a subalgebra of $\mathfrak{P}(\mathfrak{A})$ and of $\mathfrak{P}_0(\mathfrak{A})$. Thus $V \subseteq \mathfrak{P}(V)$ and $V \subseteq \mathfrak{P}_0(V)$.

In the rest of this paper we restrict ourselves to the usual case where all the operations in F are *finitary*, unless the contrary is explicitly stated. We consider terms p (*polynomial symbols* in the terminology of [4]) in the variable symbols $v_0, \ldots, v_i, \ldots$ and operation symbols of F. A term p is said to be *linear* if no variable symbol occurs more than once in p. An identity $p \equiv q$ is said to be *linear* if both terms p and q are linear; it is said to be *regular* if the set of variable symbols occurring in p equals the set of variable symbols occurring in q. We use the notation $p(x_1, \ldots, x_n)$ to signify that no variable symbols occur in p other than $x_1, \ldots, x_n$; however, this does not signify that any particular one of the x_i actually does occur in p. If $x_1, \ldots, x_n$ are distinct variable symbols, the term that results from $p(x_1, \ldots, x_n)$ by the (simultaneous) substitution of the term r_i for the variable symbol x_i, $i = 1, \ldots, n$, is denoted by $p(r_1, \ldots, r_n)$. In our work we will need only substitute variable symbols for variable symbols where, in general, the same variable symbol may be substituted for different variable symbols. It is convenient to use the following formalism.

Let $1 \leqslant m \leqslant n$ and let φ: $\{1, \ldots, n\} \to \{1, \ldots, m\}$. Then from $p(x_1, \ldots, x_n)$, with $x_1, \ldots, x_n$ distinct variable symbols, we get the term $p(x_{\varphi(1)}, \ldots, x_{\varphi(n)})$ in the variable symbols $x_1, \ldots, x_m$ by substituting $x_{\varphi(i)}$ for x_i.

The following two lemmas follow quite trivially by induction on the complexity of the terms (see [5] for example).

LEMMA 1. *Given two terms $p(x_1, \ldots, x_m)$ and $q(x_1, \ldots, x_m)$, with $x_1, \ldots, x_m$ distinct variable symbols, there are an integer $n \geqslant m$, a surjection*

$$\varphi\colon \{1, \ldots, n\} \to \{1, \ldots, m\},$$

and linear terms $p'(x_1, \ldots, x_n)$, $q'(x_1, \ldots, x_n)$ such that

$$p(x_1, \ldots, x_m) = p'(x_{\varphi(1)}, \ldots, x_{\varphi(n)})$$

and

$$q(x_1, \ldots, x_m) = q'(x_{\varphi(1)}, \ldots, x_{\varphi(n)}).$$

LEMMA 2. *Given a linear term $p(x_1, \ldots, x_n)$ with $x_1, \ldots, x_m$ distinct variable symbols, an algebra $\mathfrak{A} = \langle A, F\rangle$ and subsets $A_1, \ldots, A_n$ of A, we have*

$$p(A_1, \ldots, A_n) = \{p(a_1, \ldots, a_n) \mid a_i \in A_i\}.$$

The last two occurrences of p in Lemma 2 denote, by abuse of notation, the *term functions* on $\mathfrak{P}(\mathfrak{A})$ (or $\mathfrak{P}_0(\mathfrak{A})$) and on $\mathfrak{A}$ determined by the term p. It should also be noted that the linearity of p is essential in Lemma 2. For example, let $\mathfrak{A}$ be a groupoid and let the term $p(v_0)$ be $v_0 v_0$. Then, for a subset A_1 with more than one element, the subset $p(A_1)$ is $\{ab \mid a, b \in A_1\}$ and not $\{p(a) \mid a \in A_1\}$.

For any variety V we denote by $\mathfrak{P}(V)$ (respectively, by $\mathfrak{P}_0(V)$) the variety determined by the class $\{\mathfrak{P}(\mathfrak{A}) \mid \mathfrak{A} \in V\}$ (respectively, by $\{\mathfrak{P}_0(\mathfrak{A}) \mid \mathfrak{A} \in V\}$). The major result in this paper is the following proposition:

PROPOSITION 1. *Let $x_1, \ldots, x_n$ be distinct variable symbols, let $m \leqslant n$, let $\varphi\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ be a surjection, and let $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$ be linear terms. If the variety $\mathfrak{P}(V)$ satisfies the identity*

$$p(x_{\varphi(1)}, \ldots, x_{\varphi(n)}) \equiv q(x_{\varphi(1)}, \ldots, x_{\varphi(n)}),$$

then there is a permutation π on $\{1, \ldots, n\}$ with $\varphi\pi = \varphi$ such that the variety V satisfies the identity

$$p(x_{\pi(1)}, \ldots, x_{\pi(n)}) \equiv q(x_1, \ldots, x_n).$$

Proof. Let $\mathfrak{F}$ be the free algebra in V on the distinct free generators $a_1, \ldots, a_n$ and for each $j = 1, \ldots, m$ let the subset A_j of $\mathfrak{F}$ be determined by setting

$$A_j = \{a_i \mid \varphi(i) = j\}.$$

Then in $\mathfrak{P}(\mathfrak{F})$ we have

$$p(A_{\varphi(1)}, \ldots, A_{\varphi(n)}) = q(A_{\varphi(1)}, \ldots, A_{\varphi(n)}).$$

Clearly, $p(a_1, \ldots, a_n) \in p(A_{\varphi(1)}, \ldots, A_{\varphi(n)})$. By the linearity of q, Lemma 1 yields elements $a_{\alpha(1)}, \ldots, a_{\alpha(n)}$ with $a_{\alpha(i)} \in A_{\varphi(i)}$ such that

$$p(a_1, \ldots, a_n) = q(a_{\alpha(1)}, \ldots, a_{\alpha(n)}).$$

That is, in view of the freeness of $\mathfrak{F}$, there is a mapping

$$\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, n\} \quad \text{with } \varphi\alpha = \varphi$$

such that V satisfies the identity

$$p(x_1, \ldots, x_n) \equiv q(x_{\alpha(1)}, \ldots, x_{\alpha(n)}). \tag{1}$$

Similarly, there is a mapping

$$\beta\colon \{1, \ldots, n\} \to \{1, \ldots, n\} \quad \text{with } \varphi\beta = \varphi$$

such that V satisfies the identity

$$q(x_1, \ldots, x_n) \equiv p(x_{\beta(1)}, \ldots, x_{\beta(n)}). \tag{2}$$

Substituting $x_{\alpha(i)}$ for x_i in (2) and using (1) we conclude that V satisfies

$$p(x_1, \ldots, x_n) \equiv p(x_{\alpha\beta(1)}, \ldots, x_{\alpha\beta(n)}). \tag{3}$$

Successively substituting $x_{\alpha\beta(i)}$ for x_i in (3) and the resulting identities, we conclude that V satisfies the identity

(4) $$p(x_1, \ldots, x_n) \equiv p(x_{\gamma(1)}, \ldots, x_{\gamma(n)})$$

for any power γ of $\alpha\beta$. But since $\{1, \ldots, n\}$ is a finite set, there is a power γ of $\alpha\beta$ which is idempotent, this is, which is the identity mapping on $X = \operatorname{Im}(\gamma)$. Since γ is of the form $\gamma_1\beta$, it follows that $\beta\colon X \to \beta(X)$ is a bijection. Since $\varphi\beta = \varphi$ and since $\{\varphi^{-1}(j) \mid j \in \{1, \ldots, m\}\}$ is a partition of $\{1, \ldots, n\}$, we conclude that

$$\beta\colon X \cap \varphi^{-1}(j) \to \beta(X) \cap \varphi^{-1}(j)$$

is a bijection for each $j = 1, \ldots, m$. Thus the sets $\varphi^{-1}(j) - X$ and $\varphi^{-1}(j) - \beta(X)$ have the same number of elements for each j. Consequently, there is a permutation π on $\{1, \ldots, n\}$ with $\varphi\pi = \pi$ such that $\pi|_X = \beta|_X$, that is, such that $\pi\gamma = \beta\gamma$. Substituting $x_{\pi(i)}$ for x_i in (4) and substituting $x_{\beta(i)}$ for x_i in (4) we conclude that V satisfies the identity

$$p(x_{\pi(1)}, \ldots, x_{\pi(n)}) \equiv p(x_{\beta(1)}, \ldots, x_{\beta(n)}).$$

Combining this result with (2) we conclude that V satisfies

$$p(x_{\pi(1)}, \ldots, x_{\pi(n)}) \equiv q(x_1, \ldots, x_n)$$

proving the proposition.

As corollaries to Proposition 1 we get the following two theorems:

THEOREM 1. *Let V be a variety of finitary algebras. Then the identities satisfied by $\mathfrak{P}(V)$ are precisely those identities resulting through identification of variables from the linear identities true in V.*

Proof. That $\mathfrak{P}(V)$ satisfies all the linear identities satisfied in V follows immediately from Lemma 2. The theorem then follows from Lemma 1 and Proposition 1 by observing that permuting the variables in a linear term results in a linear term.

THEOREM 2. *Let V be a variety of finitary algebras. Then the identities satisfied by $\mathfrak{P}_0(V)$ are precisely those regular identities resulting through identification of variables from the linear identities true in V.*

Proof. We observe for any algebra $\mathfrak{A}$ that $\mathfrak{P}(\mathfrak{A})$ is a subalgebra of $\mathfrak{P}_0(\mathfrak{A})$ and the one element left out, $\emptyset$, is an *absorbing element* of $\mathfrak{P}_0(\mathfrak{A})$, that is, that, for any fundamental operation $f, f(A_1, \ldots, A_n) = \emptyset$ if some A_i is $\emptyset$. It is then immediate (see, e.g., [6]) that the identities satisfied by $\mathfrak{P}_0(\mathfrak{A})$ are precisely the regular identities satisfied by $\mathfrak{P}(\mathfrak{A})$. Thus the identities satisfied in $\mathfrak{P}_0(V)$ are the regular identities satisfied in $\mathfrak{P}(V)$. The theorem then follows from Theorem 1.

Comparing Theorems 1 and 2, one is led to suspect that Theorem 2 can be strengthened to say that the identities satisfied in $\mathfrak{P}_0(V)$ are the consequences of the regular linear identities satisfied in V. There is, however, a very simple counterexample to this hoped-for theorem. Observe, first of all, that if there are no nullary operations in the type of V, then any consequence of a set of regular linear identities is *balanced*, that is, the number of variable symbols, counting repetitions, occurring on each side is the same (see [7]). A counterexample is then the variety V of semigroups satisfying the identity $v_0 v_1 \equiv v_0$, since $\mathfrak{P}_0(V)$ satisfies the identity $v_0 v_0 \equiv v_0$, which is not balanced.

The following corollaries of Theorems 1 and 2 are immediate:

COROLLARY 1. *For any variety V of finitary algebras,*

$$\mathfrak{P}(\mathfrak{P}(V)) = \mathfrak{P}(V) \quad \text{and} \quad \mathfrak{P}_0(\mathfrak{P}_0(V)) = \mathfrak{P}_0(V).$$

COROLLARY 2 ([1], [8]). *Let V be a variety of finitary algebras. Then $\mathfrak{P}(V) = V$ if and only if V is definable by a set of linear identities, and $\mathfrak{P}_0(V) = V$ if and only if V is definable by a set of linear regular identities.*

Proof. The result for $\mathfrak{P}$ is clear by Theorem 1. As for $\mathfrak{P}_0$, $\mathfrak{P}_0(V) = V$ implies, by Theorem 2, that all the identities of V are regular. The corollary then follows by observing that any nontrivial consequence of a set of regular identities is regular.

The major question left unanswered in this paper is that of finding the analogies of Theorems 1 and 2 for infinitary types. It seems unlikely to us that these theorems, exactly as stated, would apply in the infinitary case; indeed, it is doubtful that $\mathfrak{P}(\mathfrak{P}(V)) = \mathfrak{P}(V)$ or $\mathfrak{P}_0(\mathfrak{P}_0(V)) = \mathfrak{P}_0(V)$ in this case. At the Universal Algebra and Lattice Theory Seminar here at the University of Manitoba, Ervin Fried pointed out, however, that the first part of the proof of Proposition 1 can be adapted to show that in the infinitary case the identities satisfied in $\mathfrak{P}(\mathfrak{P}(V))$ are the consequences of the linear identities of V (and analogously for $\mathfrak{P}_0(\mathfrak{P}_0(V))$). Indeed, let the type of all operations be less than the ordinal γ. Starting with

$$\mathfrak{P}(\mathfrak{P}(V)) \models p(x_{\varphi(1)}, \ldots, x_{\varphi(n)}, \ldots) \equiv q(x_{\varphi(1)}, \ldots, x_{\varphi(n)}, \ldots), \qquad \varphi\colon \gamma \to \gamma,$$

we get

$$\mathfrak{P}(V) \models p(x_1, \ldots, x_n, \ldots) \equiv q(x_{\alpha(1)}, \ldots, x_{\alpha(n)}, \ldots)$$

with $\varphi\alpha = \varphi$, as in the derivation of formula (1). Starting over again with this result, we see that there are linear p', q' and $\psi\colon \gamma \to \gamma$ with

$$p(x_1, \ldots, x_n, \ldots) = p'(x_{\psi(1)}, \ldots, x_{\psi(n)}, \ldots)$$

and

$$q(x_{\alpha(1)}, \ldots, x_{\alpha(n)}, \ldots) = q'(x_{\psi(1)}, \ldots, x_{\psi(n)}, \ldots).$$

We then get $\beta\colon \gamma \to \gamma$ with $\psi\beta = \psi$ and

$$V \models q'(x_1, \ldots, x_n, \ldots) \equiv p'(x_{\beta(1)}, \ldots, x_{\beta(n)}, \ldots).$$

Since $p(x_1, \ldots, x_n, \ldots) = p'(x_{\psi(1)}, \ldots, x_{\psi(n)}, \ldots)$ is linear and $\psi\beta = \psi$, we conclude that $p'(x_{\beta(1)}, \ldots, x_{\beta(n)}, \ldots)$ is linear, establishing the result. Thus, in the infinitary case,

$$\mathfrak{P}(\mathfrak{P}(\mathfrak{P}(V))) = \mathfrak{P}(\mathfrak{P}(V)),$$

and similarly for $\mathfrak{P}_0$.

2. Globals of lattices. Let V be a nontrivial variety of lattices. Then V satisfies the set of linear identities Σ:

$$v_0 \vee v_1 \equiv v_1 \vee v_0,$$

$$v_0 \wedge v_1 \equiv v_1 \wedge v_0,$$

$$(v_0 \vee v_1) \vee v_2 \equiv v_0 \vee (v_1 \vee v_2),$$

$$(v_0 \wedge v_1) \wedge v_2 \equiv v_0 \wedge (v_1 \wedge v_2),$$

all of which are linear and regular, and so hold in both $\mathfrak{P}(V)$ and in $\mathfrak{P}_0(V)$.

THEOREM 3. *If V is a nontrivial variety of lattices, then Σ is a basis for the identities holding in $\mathfrak{P}(V)$ and in $\mathfrak{P}_0(V)$.*

Before proving this theorem we present a sequence of lemmas.

We let $\mathfrak{F}$ denote the lattice obtained by adding a 0 and a 1 to the V-free lattice generated by the countable set $\{a_i \mid i < \omega\}$.

LEMMA 3. *Let $x_1, \ldots, x_n$ be distinct variable symbols and let $p(x_1, \ldots, x_n)$ be a linear lattice term in which the variable symbol x_1 occurs. Then there are choices $\varepsilon_2, \ldots, \varepsilon_n \in \{0, 1\}$ such that the unary polynomial*

$$f(x) = p(x, \varepsilon_2, \ldots, \varepsilon_n)$$

satisfies the equality $f(x) = x$ on $\mathfrak{F}$.

Proof. We proceed by induction on the complexity of p.

If p is the term x_1, the result is immediate.

If p is $p_1 \vee p_2$ and x_1 occurs in, say, p_1, then, by linearity, x_1 does not occur in p_2. Set all the variables in p_2 equal to 0 and, since none of them occur in p_1, the desired result follows from the truth of the result for p_1.

The dual argument obtains if p is a meet.

LEMMA 4. *Let p be a linear term. If $V \models p \equiv q$, then each variable symbol that occurs in p also occurs in q.*

Proof. If x_1 occurs in $p(x_1, \ldots, x_n)$ and not in $q(x_1, \ldots, x_n)$, then, substituting the $\varepsilon_2, \ldots, \varepsilon_n$ given by Lemma 3, we arrive at the contradiction

$$p(a_1, \varepsilon_2, \ldots, \varepsilon_n) = a_1 = q(a_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0, 1\}.$$

Lemma 4 thus implies that any linear identity holding in V is regular.

LEMMA 5. *Let $p = p_1 \vee p_2$ and $q = q_1 \wedge q_2$ be linear lattice terms. Then the identity $p \equiv q$ does not hold in V.*

Proof. Let the distinct $x_1, \ldots, x_n$, $n \geqslant 2$, be all the variable symbols occurring in p and q, and let x_1 occur in p_1. Let $V \models p \equiv q$. Then, by Lemma 4, we can assume that x_1 occurs in q_1. Denote the set of variable symbols occurring in a term t by $\operatorname{var}(t)$. Put

$$X = \operatorname{var}(p_1) \cap \operatorname{var}(q_1), \quad Y = \operatorname{var}(p_1) - \operatorname{var}(q_1),$$
$$U = \operatorname{var}(q_1) - \operatorname{var}(p_1), \quad W = \operatorname{var}(p_2) \cap \operatorname{var}(q_2).$$

Then, by Lemma 4,

$$(*) \qquad \begin{aligned} \operatorname{var}(p_1) &= X \cup Y, & \operatorname{var}(p_2) &= U \cup W, \\ \operatorname{var}(q_1) &= X \cup U, & \operatorname{var}(q_2) &= Y \cup W; \end{aligned}$$

all these disjoint unions.

By definition, X is not empty. We show that neither are Y, U, nor W.

If $W = \emptyset$, then $Y, U \neq \emptyset$; set all variables in U to 1 and all variables in Y to 0. Then p becomes 1, and q becomes 0, contradicting $V \models p \equiv q$.

If $U = \emptyset$, then $X, W \neq \emptyset$; set all the variables in W to 1 and all the variables in X to 0, yielding the contradiction $p = 1$, $q = 0$.

$W = \emptyset$ is the dual situation.

Thus X, Y, U, W are all nonempty.

Substituting x for each variable symbol in X, y for each in Y, u for each in U and w for each in W yields binary term functions p_1', p_2', q_1', q_2' on $\mathfrak{F}$ with

$$p_1'(x, y) \vee p_2'(u, w) = q_1'(x, u) \wedge q_2'(y, w).$$

Let $y = w = 1$, $x = a_0$, $u = a_2$; we get

$$q_1'(a_0, a_2) = p_1'(a_0, 1) \vee p_2'(a_2, 1).$$

Now $p_1'(x, 1)$ is x or 1, and $q_1'(a_0, a_2) \neq 1$; thus $p_1'(a_0, 1) = a_0$ and, similarly, $p_2'(a_2, 1) = a_2$. Consequently, $q_1'(a_0, a_2) = a_0 \vee a_2$. Similarly,

$$q_2'(a_1, a_3) = a_1 \vee a_3, \quad p_1'(a_0, a_1) = a_0 \wedge a_1, \quad p_2'(a_2, a_3) = a_2 \wedge a_3.$$

Thus

$$(a_0 \wedge a_1) \vee (a_2 \wedge a_3) = (a_0 \vee a_2) \wedge (a_1 \vee a_3),$$

which does not hold in any nontrivial variety of lattices, is true in $\mathfrak{F}$, where a_0, a_1, a_2, a_3 are free generators. This contradiction proves the lemma.

Lemma 6. *Let $x_1, \ldots, x_n$ be distinct variable symbols, let $p(x_1, \ldots, x_n)$ be a linear lattice term, and let $m \leqslant n$. Then either*

$$p(a_1, \ldots, a_m, 0, \ldots, 0) = 0$$

in $\mathfrak{F}$ or there is a linear term $q(x_1, \ldots, x_m)$ with

$$\operatorname{var}(q) \subseteq \operatorname{var}(p) \cap \{x_1, \ldots, x_m\}$$

and

$$p(a_1, \ldots, a_m, 0, \ldots, 0) = q(a_1, \ldots, a_m)$$

in $\mathfrak{F}$.

Proof. The proof is completely straightforward by induction on the complexity of p. If p is the variable symbol x_i and $i \leqslant m$, then $q = p$; if $i > m$, then

$$p(a_1, \ldots, a_m, 0, \ldots, 0) = 0.$$

If $p = p_1 \vee p_2$ or $p = p_1 \wedge p_2$, then $\operatorname{var}(p_1) \cap \operatorname{var}(p_2) = \emptyset$, and the result is immediate by using the inductive assumption that it is true for p_1 and p_2.

Proof of Theorem 3. Let the linear identity $p \equiv q$ hold in the nontrivial variety V. We show that $\Sigma \models p \equiv q$ by induction on the sum of the complexity of p and of q.

By Lemma 4 the identity $p \equiv q$ is regular. Thus if one of p, q is the variable symbol x_i, then so is the other, and so $\Sigma \models p \equiv q$. Consequently, without loss of generality, we may assume that p is a join-term; by Lemma 5, so is q. Thus, there are linear terms p_1, p_2, q_1, q_2 such that neither p_1 nor q_1 are join-terms, such that

$$\operatorname{var}(p_1) \cap \operatorname{var}(q_1) \neq \emptyset,$$

$$\operatorname{var}(p_1) \cap \operatorname{var}(p_2) = \emptyset, \qquad \operatorname{var}(q_1) \cap \operatorname{var}(q_2) = \emptyset,$$

and such that

$$\Sigma \models p \equiv p_1 \vee p_2, \qquad \Sigma \models q \equiv q_1 \vee q_2.$$

Since $V \models \Sigma$, we have $V \models p_1 \vee p_2 \equiv q_1 \vee q_2$. With the notation like in Lemma 5 we have (*), all unions being disjoint.

In $\mathfrak{F}$ substitute 0 for x_i if $x_i \in U \cup W$ and a_i for x_i otherwise. Then in $\mathfrak{F}$ we have

$$p_1(a_1, \ldots, a_n) = z_1 \vee z_2$$

with $z_1 = 0$ or $q'_1(a_1, \ldots, a_n)$, where q'_1 is a linear term with $\operatorname{var}(q'_1) \subseteq X$ and $z_2 = 0$ or $q'_2(a_1, \ldots, a_n)$, where q'_2 is linear with $\operatorname{var}(q'_2) \subseteq Y$. Now $z_1 = 0$ implies $V \models p \equiv q'_2$, violating the regularity of linear identities since $\emptyset \neq X \subseteq \operatorname{var}(p)$; thus $z_1 = q'_1(a_1, \ldots, a_n)$. On the other hand, $z_2 \neq 0$ implies $V \models p_1 \equiv q'_1 \vee q'_2$, a linear identity that violates Lemma 5 since p_1 is not a join-term. Thus $z_2 = 0$, implying

$$p_1(a_1, \ldots, a_n) = q'_1(a_1, \ldots, a_n),$$

that is, $V \models p_1 \equiv q'_1$. Since $\operatorname{var}(q'_1) \subseteq X$, Lemma 4 implies that $Y = \emptyset$.

Similarly, $U = \emptyset$. Replacing the variables in W by 0, we get $V \models p_1 \equiv q_1$, and so, by the inductive hypothesis, $\Sigma \models p_1 \equiv q_1$. Similarly, replacing the variables in X by 0, we obtain $\Sigma \models p_2 \equiv q_2$. Consequently, $\Sigma \models p \equiv q$. Thus, in view of Theorems 1 and 2, the proof is complete.

Although there are uncountably many varieties of lattices, we get the following corollary:

COROLLARY. *There is exactly one nontrivial variety of globals of lattices.*

3. Globals of groups. We consider groups as algebras whose type includes the binary operation of multiplication, the unary operation $^{-1}$ – the inverse, and the nullary operation e – the identity.

Let Σ_0 be the following system of identities:

$$(v_0 v_1) v_2 \equiv v_0 (v_1 v_2),$$

$$ev_0 \equiv v_0, \quad v_0 e \equiv v_0,$$

$$(v_0 v_1)^{-1} \equiv v_1^{-1} v_0^{-1},$$

$$(v_0^{-1})^{-1} \equiv v_0, \quad e^{-1} \equiv e.$$

These identities are all linear and regular and hold in any variety of groups.

LEMMA 7. *Let V be a nontrivial variety of groups. If $p \equiv q$ is a linear identity holding in V, then $p \equiv q$ is also regular.*

Proof. Assume, to the contrary, that some variable symbol x occurs in p and not in q. Then setting all the other variable symbols equal to e and, if necessary, taking inverses yields the consequence $V \models x \equiv e$, contradicting the nontriviality of V.

THEOREM 4. *Let V be a nonabelian variety of groups. Then Σ_0 is a basis for the identities of $\mathfrak{P}(V)$ and for the identities of $\mathfrak{P}_0(V)$.*

Proof. By Theorems 1 and 2 and by Lemma 7, and in view of the fact that any consequence of a set of regular identities is regular, we need only show that any linear identity holding in V is a consequence of Σ_0.

Let $V \models p \equiv q$ with $p \equiv q$ linear, and so, by Lemma 7, regular. If either p or q is nullary, then so is the other, and $\Sigma_0 \models p \equiv e$, $\Sigma_0 \models q \equiv e$; so $\Sigma_0 \models p \equiv q$. Otherwise, there are distinct variable symbols $x_1, \ldots, x_n$, a permutation π on $\{1, \ldots, n\}$, and $\varepsilon(i)$, $\eta(i) \in \{1, -1\}$, $i = 1, \ldots, n$, such that

$$\Sigma_0 \models p \equiv x_1^{\varepsilon(1)} x_2^{\varepsilon(2)} \ldots x_n^{\varepsilon(n)}, \tag{5}$$

$$\Sigma_0 \models q \equiv x_{\pi(1)}^{\eta(\pi(1))} x_{\pi(2)}^{\eta(\pi(2))} \ldots x_{\pi(n)}^{\eta(\pi(n))}, \tag{6}$$

where, if one wants to be pedantic, one can define xyz to mean $(xy)z$. Thus

$$V \models x_1^{\varepsilon(1)} x_2^{\varepsilon(2)} \ldots x_n^{\varepsilon(n)} \equiv x_{\pi(1)}^{\eta(\pi(1))} x_{\pi(2)}^{\eta(\pi(2))} \ldots x_{\pi(n)}^{\eta(\pi(n))}. \tag{7}$$

For each i set $x_j = e$ for $j \neq i$, and get

$$V \models x_i^{\varepsilon(i)} \equiv x_i^{\eta(i)}.$$

Since $x_i \equiv x_i^{-1}$ implies commutativity, we have

$$\varepsilon(i) = \eta(i) \quad \text{for all } i = 1, \ldots, n.$$

We next claim that whenever $i < j$ we have $\pi(i) < \pi(j)$, and so that π is the identity permutation, that is, that $x_1^{\varepsilon(1)} x_2^{\varepsilon(2)} \ldots x_n^{\varepsilon(n)}$ and $x_{\pi(1)}^{\eta(\pi(1))} x_{\pi(2)}^{\eta(\pi(2))} \ldots x_{\pi(n)}^{\eta(\pi(n))}$ are the same terms. For if $i < j$ and $\pi(i) > \pi(j)$, then, setting all x_k, $k \neq \pi(i)$, $\pi(j)$, equal to e in (7) yields

$$V \models x_{\pi(j)}^{\varepsilon(\pi(j))} x_{\pi(i)}^{\varepsilon(\pi(i))} \equiv x_{\pi(i)}^{\varepsilon(\pi(i))} x_{\pi(j)}^{\varepsilon(\pi(j))},$$

that is, V is abelian.

Consequently, by (5) and (6), $\Sigma_0 \models p \equiv q$, proving the theorem.

Now let $\Sigma_1 = \Sigma_0 \cup \{v_0 v_1 \equiv v_1 v_0\}$.

THEOREM 5. *Let V be a variety of abelian groups not of exponent 2. Then Σ_1 is a basis for the identities of $\mathfrak{P}(V)$ and for the identities of $\mathfrak{P}_0(V)$.*

Proof. We proceed exactly as in the proof of Theorem 4. If $p \equiv q$ is a linear identity satisfied in V, then it is regular. Then, as in Theorem 4, we need only consider the nonnullary case. Then there are an $n \geqslant 1$ and distinct variable symbols $x_1, \ldots, x_n$ and $\varepsilon(i)$, $\eta(i) \in \{1, -1\}$, $i = 1, \ldots, n$, such that

$$\Sigma_1 \models p \equiv x_1^{\varepsilon(1)} x_2^{\varepsilon(2)} \ldots x_n^{\varepsilon(n)}, \qquad \Sigma_1 \models q \equiv x_1^{\eta(1)} x_2^{\eta(2)} \ldots x_n^{\eta(n)},$$

since the commutative law is in Σ_1. Now the identity $x_i \equiv x_i^{-1}$ does not hold in V; thus we get $\varepsilon(i) = \eta(i)$, and so $\Sigma_1 \models p \equiv q$ exactly as in the proof of Theorem 4, completing the proof.

Finally, let Σ_2 consist of the identities

$$(v_0 v_1) v_2 \equiv v_0 (v_1 v_2), \qquad e v_0 \equiv v_0,$$
$$v_0 v_1 \equiv v_1 v_0, \qquad v_0^{-1} \equiv v_0.$$

THEOREM 6. *Let V be the variety of groups of exponent 2. Then Σ_2 is a basis for the identities of $\mathfrak{P}(V)$ and for the identities of $\mathfrak{P}_0(V)$.*

Proof. Proceeding exactly as in the proofs of the preceding two theorems, we find that if $V \models p \equiv q$, then either $\Sigma_2 \models p \equiv e$, $q \equiv e$ or $\Sigma_2 \models p \equiv x_1 \ldots x_n$, $q \equiv x_1 \ldots x_n$. In either case $\Sigma_2 \models p \equiv q$, proving the theorem.

COROLLARY. *There are exactly three nontrivial varieties of globals of groups.*

REFERENCES

[1] M. N. Bleicher, H. Schneider and R. L. Wilson, *Permanence of identities on algebras*, Algebra Universalis 3 (1973), pp. 72–93.

[2] N. D. Gautam, *The validity of equations of a complex algebra*, Arch. Math. Logik Grundlag. 3 (1957), pp. 117–124.

[3] M. Gould, J. A. Iskra and C. Tsinakas, *Globally determined lattices and semilattices*, Algebra Universalis 19 (1984), pp. 137–141.

[4] G. Grätzer, *Universal Algebra*, Springer-Verlag, New York, N. Y., 1979.

[5] – and S. Whitney, *Infinitary varieties of structures closed under the formation of complex structures*, Colloq. Math. 48 (1984), pp. 1–5.

[6] H. Lakser, R. Padmanabhan and C. R. Platt, *Subdirect decomposition of Płonka sums*, Duke Math. J. 39 (1972), pp. 485–488.

[7] A. Shafaat, *On varieties closed under the construction of power algebras*, Bull. Austral. Math. Soc. 11 (1974), pp. 213–218.

[8] S. Whitney, *Model classes for linear theories* (*Abstract*), Notices Amer. Math. Soc. 21 (1974), A-317, A-500.

[9] – *Théories linéaires*, Ph. D. Thesis, Université Laval, Québec, 1977.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA

Reçu par la Rédaction le 15.9.1986