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ON CHARACTERISTIC SETS OF A SYSTEM OF EQUIVALENCE RELATIONS

BY

MIROSLAV KRATKO (KIEV)

The *characteristic set* of an n-tuple of equivalence relations $R_1, \ldots, R_n$ in a set A is defined as a set of zero-one n-sequences (sequences of length n consisting of zeroes and ones) in the following way:

$$h(R_1, \ldots, R_n) = \{(\chi_{R_1}(a, b), \ldots, \chi_{R_n}(a, b)) | \ (a, b) \in A^2\} \subset \{0, 1\}^n,$$

where χ_{R_i} denotes the characteristic function of R_i in A^2. We call such sets of n-sequences *E-sets* and denote n-sequences as follows: $\mathit{1} = (1, \ldots, 1)$ – the sequence of 1's only, $\mathit{0} = (0, \ldots, 0)$ – the sequence of 0's only, $p = (p_1, \ldots, p_n) \in \{0, 1\}^n$. A set A together with an n-tuple of equivalence relations $R_1, \ldots, R_n$ is a *relation system*

$$\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle.$$

A relation system $\mathfrak{A}$ is *normal* iff it does not have two elements $a_1, a_2 \in A$, $a_1 \neq a_2$, such that

$$(\chi_{R_1}(a_1, a_2), \ldots, \chi_{R_n}(a_1, a_2)) = \mathit{1}.$$

It is easy to see that if $C \subset \{0, 1\}^n$ is an E-set, then it is the characteristic set of some normal relation system. All relation systems considered below are normal.

Obviously, not every set $C \subset \{0, 1\}^n$ is an E-set. J. Łoś[1] investigates which sets of zero-one n-sequences are E-sets. He formulated the following necessary condition:

(*) for every $p, q \in C$ there exist $r, s, t, u \in C$ such that for every i, j ($i, j = 1, \ldots, n$) if $p_i = 1 = q_j$ and $p_j = 0 = q_i$, then

$$r_i = r_j = 0 \quad \text{or} \quad s_i = s_j = 0 \quad \text{or} \quad t_i = t_j = 0 \quad \text{or} \quad u_i = u_j = 0$$

(1) J. Łoś, *Characteristic sets of a system of equivalence relations*, Colloq. Math. 42 (1979), pp. 291–293.

and asked if (*) is a sufficient condition for $C \subset \{0, 1\}^n$, to which $\mathit{1}$ belongs, to be an E-set.

First of all, we note that if $C \subset \{0, 1\}^n$ is an E-set and consists of k elements, then it is a characteristic set of a relation system $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ such that A does not have more than $2k$ elements. Now the following example shows that the answer to the Łoś problem (P 1163) is negative:

$$T = \{(1, 1, 1, 1), (1, 1, 0, 0), (1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}.$$

In fact, if T is an E-set, then one of the following cases occurs:

(1) there exist a, b, c such that

$$(\chi_{R_1}(a, b), \chi_{R_2}(a, b), \chi_{R_3}(a, b), \chi_{R_4}(a, b)) = (1, 1, 0, 0),$$
$$(\chi_{R_1}(a, c), \chi_{R_2}(a, c), \chi_{R_3}(a, c), \chi_{R_4}(a, c)) = (0, 0, 1, 1);$$

(2) there exist a, b, c, d such that

$$(\chi_{R_1}(a, b), \chi_{R_2}(a, b), \chi_{R_3}(a, b), \chi_{R_4}(a, b)) = (1, 1, 0, 0),$$
$$(\chi_{R_1}(c, d), \chi_{R_2}(c, d), \chi_{R_3}(c, d), \chi_{R_4}(c, d)) = (0, 0, 1, 1).$$

In both cases, for arbitrary R_1, R_2, R_3, R_4: if (1) holds, then

$$(\chi_{R_1}(b, c), \chi_{R_2}(b, c), \chi_{R_3}(b, c), \chi_{R_4}(b, c)) = \mathit{0} \notin T;$$

if (2) holds, then

$$\{(\chi_{R_1}(a, c), \chi_{R_2}(a, c), \chi_{R_3}(a, c), \chi_{R_4}(a, c)), (\chi_{R_1}(a, d), \chi_{R_2}(a, d), \chi_{R_3}(a, d), \chi_{R_4}(a, d)),$$
$$(\chi_{R_1}(b, c), \chi_{R_2}(b, c), \chi_{R_3}(b, c), \chi_{R_4}(b, c)), (\chi_{R_1}(b, d), \chi_{R_2}(b, d), \chi_{R_3}(b, d), \chi_{R_4}(b, d))\}$$

is not a subset of T.

We say that $C' \subset C \subset \{0, 1\}^n$ can be *enlarged* to an E-set if there exists $C'' \subset C$ such that $C' \subset C''$ and C''' is an E-set. Obviously, if C is an E-set, then each of its subsets can be enlarged to an E-set. The next problem will be called the *generalized Łoś problem*:

Is C an E-set if every two-element subset of C can be enlarged to an E-set?

The answer to this problem is given by the following

THEOREM. *For an arbitrary integer k there exists $C \subset \{0, 1\}^n$ such that C is not an E-set, and every subset of C consisting of at most k elements can be enlarged to an E-set.*

Proof. The product of n-sequences p and q is an n-sequence

$$pq = (p_1 q_1, \ldots, p_n q_n).$$

A set $C \subset \{0, 1\}^n$ is orthogonal iff $pq = 0$ for arbitrary $p, q \in C$, $p \neq q$.

LEMMA. *The set $C \cup \{1\}$ is an E-set for an arbitrary orthogonal $C \subset \{0, 1\}^n$ consisting of more than 2 elements. If C consists of k elements, then cardinality of the normal relation system such that C is its characteristic set is at most k^{k^2}.*

Proof of the Lemma. Let an orthogonal set C consist of $k > 2$ elements $C = p^0, \ldots, p^{k-1}$ and $A = \{a_0, \ldots, a_{k-1}\}$. Define $R_1, \ldots, R_n$ in A as follows:

$$(\chi_{R_1}(a_i, a_j), \ldots, \chi_{R_n}(a_i, a_j)) = p^0 \quad \text{for all } i \neq 0,\ j \neq 0,$$

$$(\chi_{R_1}(a_0, a_i), \ldots, \chi_{R_n}(a_0, a_i)) = p^i \quad \text{for all } i = 1, \ldots, k-1.$$

The relation system $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ has the characteristic set equal to C, so C is an E-set.

Assume that a relation system $\mathfrak{A}_1 = \langle A_1, R_1, \ldots, R_n \rangle$ has its characteristic set equal to C and that A_1 has more than k^{k^2} elements. According to the Ramsey theorem there is a maximal $A' \subset A_1$, $|A'| \geqslant k$, such that for all $a, b \in A'$ the n-sequences $(\chi_{R_1}(a, b), \ldots, \chi_{R_n}(a, b))$ are equal to each other and equal, say to p^0. Choose $\alpha \in A_1/A'$ and consider n-sequences $(\chi_{R_1}(\alpha, a), \ldots, \chi_{R_n}(\alpha, a))$ for all $a \in A'$. None of these sequences equals p^0, because A' is maximal. Therefore, for arbitrary $a_1, a_2 \in A$ we have

$$(\chi_{R_1}(\alpha, a_1), \ldots, \chi_{R_n}(\alpha, a_1)) \neq (\chi_{R_1}(\alpha, a_2), \ldots, \chi_{R_n}(\alpha, a_2)).$$

On the other hand, this is impossible because C has only k elements. The Lemma is thus proved.

Two n-sequences $p, q \in C \subset \{0, 1\}^n$ are parallel in C if there is no $r \in C$ such that $pq = pr = qr$. If p and q are parallel in C, then in any relation system $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ with characteristic set C there are no $a, b, c \in A$ such that

$$(\chi_{R_1}(a, b), \ldots, \chi_{R_n}(a, b)) = p, \quad (\chi_{R_1}(a, c), \ldots, \chi_{R_n}(a, c)) = q.$$

Thus, if C has k parallel elements $q^1, \ldots, q^k$, then the set A has at least $2k$ elements. We denote by (α_i, β_i) a pair of elements of A such that

$$(\chi_{R_1}(\alpha_i, \beta_i), \ldots, \chi_{R_n}(\alpha_i, \beta_i)) = q^i \quad \text{for all } i = 1, \ldots, k.$$

The set C in the assertion of the Theorem can now be constructed as the union of two disjoint sets C_1 and C_2, where C_1 is defined as the set of k orthogonal n-sequences $C = \{p^1, \ldots, p^k\}$; C is a set of k^{k^2} n-sequences parallel in $C_1 \cup C_2$ and such that $pq = 0$ for all $p \in C_1$, $q \in C_2$. This can be easily done by choosing n sufficiently large, e.g., $n = k^{2k^2}$. The set $C_1 \cup C_2$ is not an E-set. Really, if $C = C_1 \cup C_2$ is an E-set and $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ is a relation system with characteristic set C, consider a relation system

$$\mathfrak{A}' = \langle A', R_1', \ldots, R_n' \rangle,$$

where $A' = \{\alpha_1, \ldots, \alpha_{k^{k^2}}\} \subset A$ and R_i' is the relation R_i restricted to A', $i = 1, \ldots, n$. The characteristic set of $\mathfrak{A}'$ is a subset of C_1. According to the Lemma, there are $A'' \subset A'$ and $p \in C_1$ such that $|A''| \geqslant k$ and, for arbitrary $x, y \in A''$,

$$(\chi_{R_1'}(x, y), \ldots, \chi_{R_n'}(x, y)) = p.$$

Let $\alpha_i \in A''$. Consider a set of pairs $\{(\beta_i, z) | \ z \in A''\}$. For arbitrary (β_i, z) we have

$$(\chi_{R_1'}(\beta_i, z), \ldots, \chi_{R_n'}(\beta_i, z)) \in C_1,$$

and if $z_1 \neq z_2$, then

$$(\chi_{R_1'}(\beta_i, z_1), \ldots, \chi_{R_n'}(\beta_i, z_1)) \neq (\chi_{R_1'}(\beta_i, z_2), \ldots, \chi_{R_n'}(\beta_i, z_2)),$$

which is impossible, because C_1 has only k elements.

If $H \subset C_2$ has k elements, $H = \{h^1, \ldots, h^k\}$, then $C_1 \cup H$ is an E-set. A relation system with characteristic set $C_1 \cup H$ can be constructed in the following way:

$$X = \{x_1, \ldots, x_k\}, \quad Y = \{y_1, \ldots, y_k\}, \quad A = X \cup Y,$$

the relations $R_1, \ldots, R_n$ in A are defined by

(a) $$(\chi_{R_1}(x_i, y_i), \ldots, \chi_{R_n}(x_i, y_i)) = h^i \quad \text{for all } i = 1, \ldots, k,$$

(b) $$(\chi_{R_1}(x_i, x_j), \ldots, \chi_{R_n}(x_i, x_j)) = p^1 \quad \text{for all } i \neq j,$$

(c) $$(\chi_{R_1}(y_i, y_j), \ldots, \chi_{R_n}(y_i, y_j)) = p^1 \quad \text{for all } i \neq j,$$

(d)
$$\{(\chi_{R_1}(x_i, y_j), \ldots, \chi_{R_n}(x_i, y_j)) | \ y_j = Y, \ i \neq j\} = \{p^2, \ldots, p^k\} \quad \text{for each } x_i \in X,$$

(e)
$$\{(\chi_{R_1}(x_j, y_i), \ldots, \chi_{R_n}(x_j, y_i)) | \ x_j \in X, \ i \neq j\} = \{p^2, \ldots, p^k\} \quad \text{for each } y_i \in Y,$$

and for all triples of elements $y_i, y_j, y_k \in A$ or $x_i, x_j, x_k \in A$

(f) $$(\chi_{R_1}(x_i, y_j), \ldots, \chi_{R_n}(x_i, y_j)) \cdot (\chi_{R_1}(x_i, y_k), \ldots, \chi_{R_n}(x_i, y_k)) = 0,$$
$$(\chi_{R_1}(x_i, y_k), \ldots, \chi_{R_n}(x_i, y_k)) \cdot (\chi_{R_1}(x_j, y_k), \ldots, \chi_{R_n}(x_j, y_k)) = 0.$$

Conditions (d), (e) and (f) are fulfilled according to the König–Hall theorem. Therefore $H \cup C_1$ is an E-set. Another subset of C which has less than k parallel elements from C_2 can be enlarged to an E-set. The Theorem is proved.

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