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UNICOHERENCE IN MEANS

BY

PHILIP BACON (GAINESVILLE, FLORIDA)

1. Suppose that X is a set and m is a function from $X \times X$ into X. We say that m is *commutative* if $m(x, y) = m(y, x)$ whenever each of x and y is in X; we say that m is *idempotent* if $m(x, x) = x$ whenever x is in X. A topological space X is said to admit a *mean* if there is a continuous idempotent commutative function m from $X \times X$ into X. Kermit Sigmon has recently shown (in [4]) that a Peano space that admits a mean is necessarily unicoherent. The algebraic and topological hypotheses of Sigmon's theorem can be weakened, as is shown by the following theorem:

(1.1) THEOREM. *Suppose X is a compact connected Hausdorff space and m: $X \times X \to X$ is continuous and idempotent. If X contains a point p such that $m(p, x) = m(x, p)$ for all x in X, then X is unicoherent.*

The present note proves this and other generalizations of Sigmon's result and concludes with an example of a contractible Peano space that admits no mean.

2. All our results derive from the following lemma:

(2.1) *Suppose X is a Hausdorff space, $E = \{E_1, E_2, E_3, E_4\}$ is a collection of open sets such that $E_1 \cap E_3 = \emptyset = E_2 \cap E_4$, and H_1 and K_1 are compact connected sets such that $H_1 \subset E_1 \cup E_2 \cup E_3$, $K_1 \subset E_3 \cup E_4 \cup E_1$, and $H_1 \cap K_1 \cap [E_1 - (E_2 \cup E_4)] \neq \emptyset \neq H_1 \cap K_1 \cap [E_3 - (E_2 \cup E_4)]$. Let p be a point of K_1 and let Y denote $H_1 \cup K_1$. Then there cannot exist a continuous function*

$$m\colon\ Y \times Y \to E_1 \cup E_3 \cup E_2 \cup E_4$$

such that, for each y in Y, $m(y, y) = y$ and $m(y, p) = m(p, y)$.

Proof. Suppose the contrary. Let a be a point of $H_1 \cap K_1 \cap [E_1 - (E_2 \cup E_4)]$ and let b be a point of $H_1 \cap K_1 \cap [E_3 - (E_2 \cup E_4)]$. Since Y is compact, there is a finite collection C of sets relatively open in Y that covers Y and is such that $m(U \times V)$ is entirely a subset of some member of E, whenever $U, V \in C$.

For each x in $E_1 \cup E_2 \cup E_3 \cup E_4$ let T_x be a member of E that contains x. For each y in Y let P_y and V_y be point sets open in Y containing p and y, respectively, such that

$$m(P_y \times V_y) \cup m(V_y \times P_y) \subset T_{m(p,y)}.$$

Since Y is compact, there is a finite subset F of Y that contains p and is such that $\{V_y\colon y \in F\}$ covers Y. Let H be a finite collection of sets relatively open in Y that covers H_1, that refines each of $(V_y\colon y \in F\}$ and C, and that has the property that, if $U \in H$, $m(U \times U)$ is a subset of one of E_1, E_2 and E_3. Let K be a finite collection of sets open in Y that covers K_1, that refines each of $\{V_y\colon y \in F\}$ and C, that has the property that, if $U \in K$, $m(U \times U)$ is a subset of one of E_3, E_4 and E_1, and that has as one of its members an open neighborhood Q of p that is a subset of $V_p \cap \bigcap \{P_y\colon y \in F\}$.

Since K_1 is connected and contains both p and a, there is an integer $r > 1$ and a finite sequence $S(1), \ldots, S(r)$, each term of which is in K, such that $p \in S(1) = Q$, $a \in S(r)$ and $S(j-1) \cap S(j) \cap K_1 \neq \emptyset$ whenever $1 < j \leqslant r$. Similarly, there is an integer $t > r$ and a finite sequence $S(r+1), \ldots, S(t)$, each term of which is in H, such that $b \in S(t)$ and $S(j-1) \cap S(j) \cap H_1 \neq \emptyset$ whenever $r < j \leqslant t$. Finally, there is an integer $n > t$ and a finite sequence $S(t+1), \ldots, S(n)$, each term of which is in K, such that $S(n) = S(1)$ and $S(j-1) \cap S(j) \cap K_1 \neq \emptyset$ whenever $t < j \leqslant n$. Define

$$M = \big((i,j)\colon 1 \leqslant j \leqslant i \leqslant n\big).$$

M is the vertex set of an abstract complex $N(M)$ to which an abstract simplex $\{(x_0, y_0), \ldots, (x_d, y_d)\}$ belongs if and only if

$$\bigcap_{k=0}^{d} [S(x_k) \times S(y_k)] \neq \emptyset.$$

Let $N(E)$ be the nerve of E. We may define a function $f\colon M \to E$ by letting $f(i,j)$ be some element of E that contains $m[S(i) \times S(j)]$, thereby insuring that f is simplicial. Moreover, our constructions are such that we may require

(1) $$f(1,1) = f(n,n) = f(n,1) \in \{E_3, E_4, E_1\};$$

(2) $$f(j,1) = f(n,j) \quad \text{for all } j \text{ in } \{1, \ldots, n\};$$

(3) $$f(j,j) \in \{E_1, E_2, E_3\} \quad \text{for all } j \text{ in } \{r, \ldots, t\};$$

and

(4) $$f(j,j) \in \{E_3, E_4, E_1\} \text{ for all } j \text{ in } \{1, \ldots, r, t, \ldots, n\}.$$

Define a 2-chain w of $N(M)$ with coefficients in Z_2, the cyclic group of order two, as follows:

$$w = \sum_{i=1}^{n-1} \sum_{j=1}^{i} (i, j)(i+1, j)(i+1, j+1) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (i, j)(i+1, j+1)(i, j+1).$$

Computation shows that

$$\partial w = \sum_{i=1}^{n-1} [(n, i)(n, i+1) + (i, i)(i+1, i+1) + (i, 1)(i+1, 1)].$$

Let $f_\#$ denote the chain map induced by the simplicial map f. Condition (2) and the choice of Z_2 as coefficient group insure that

$$f_\# \sum_{i=1}^{n-1} [(n, i)(n, i+1) + (i, 1)(i+1, 1)] = 0.$$

Hence

$$f_\# \partial w = f_\# \sum_{i=1}^{n-1} (i, i)(i+1, i+1).$$

Let u be the chain $\sum_{i=r}^{t-1} (i, i)(i+1, i+1)$. $\partial u = (t, t) + (r, r)$. Since a is not in E_3, conditions (3) and (4) imply that $f_\#(r, r) = E_1$. Similarly, $f_\#(t, t) = E_3$. So $\partial f_\# u = E_1 + E_3$. Condition (3) insures that $f_\# u$ is a linear combination of E_1E_2 and E_2E_3. The only linear combination of E_1E_2 and E_2E_3 with boundary $E_1 + E_3$ is $E_1E_2 + E_2E_3$. Thus

$$f_\# \sum_{i=r}^{t-1} (i, i)(i+1, i+1) = E_1E_2 + E_2E_3.$$

A similar argument shows that

$$f_\# \left[\sum_{i=1}^{r-1} (i, i)(i+1, i+1) + \sum_{i=t}^{n-1} (i, i)(i+1, i+1)\right] = E_3E_4 + E_4E_1.$$

Then

$$\partial f_\# w = f_\# \partial w = f_\# \sum_{i=1}^{n-1} (i, i)(i+1, i+1) = E_1E_2 + E_2E_3 + E_3E_4 + E_4E_1 \neq 0$$

from which it follows that $f_\# w \neq 0$. This contradicts the fact that $N(E)$ contains no 2-simplex.

Proof of (1.1). Suppose there is an X that satisfies the hypotheses of the theorem but is not unicoherent. There exist closed connected sets H_1 and K_1 such that $p \in K_1$, $H_1 \cup K_1 = X$, and $H_1 \cap K_1$ is the union of disjoint non-empty closed sets A and B. Since X is normal, there exist disjoint open sets E_1 and E_3, E_1 containing A and E_3 containing B. Define $E_2 = X - K_1$, $E_4 = X - H_1$. We have a contradiction of (2.1).

A point set M is said to be *compactly connected* if each two points of M lie in a closed, compact and connected subset of M ([3], p. 76). Let us say, for the moment, that a space X has property H if every connected open subset of X is compactly connected. The next theorem applies to spaces having property H. It is not hard to show that every locally connected, locally compact Hausdorff has property H. A locally connected complete Moore space (that is, a locally connected space satisfying Axiom 1 of [3]) has property H ([3], p. 84, Theorem 1). There is a connected locally connected subspace of Euclidean 3-space that does not have property H ([2], p. 362, 5.3).

(2.2) THEOREM. *Suppose that X is a connected, locally connected normal Hausdorff space such that every connected open subset of X is compactly connected. If $m\colon X\times X\to X$ is an idempotent continuous function for which there is a point p in X such that $m(p, x) = m(x, p)$ for all x in X, then X is unicoherent.*

Proof. Suppose on the contrary that X is the union of closed connected sets H and K such that $p \in K$ and $H \cap K$ is the union of disjoint closed sets A and B, A containing a point a and B containing a point b. Since X is normal there exist disjoint open sets E_1 and E_3, E_1 containing A and E_3 containing B. Define $E_2 = X-K$, $E_4 = X-H$. The component of $E_1 \cup E_2 \cup E_3$ that contains H is open and so contains a compact connected set H_1 that contains both a and b. Similarly, there is a compact connected subset K_1 of $E_4 \cup E_3 \cup E_1$ that contains a, b and p. This contradicts (2.1).

(2.3) Definition. A space X is said to be *locally quasi-unicoherent* at a point p of X if for every closed neighborhood U of p there is a neighborhood V of p contained in U such that if C is a closed connected subset of V containing p, if H and K are closed sets whose union is U and if $H \cap C$ and $K \cap C$ are connected, then $H \cap K \cap C$ is a subset of some component of $H \cap K$.

For a global concept similar to that given above see [5].

(2.4) THEOREM. *If X is a locally compact Hausdorff space, if $m\colon X\times X\to X$ is continuous and idempotent, and if p is a point of X such that $m(p, x) = m(x, p)$ for all x in X, then X is locally quasi-unicoherent at p.*

Proof. Let U be a closed neighborhood of p. Since X is locally compact, there is a compact neighborhood N of p that is contained in the interior of U. There is an open neighborhood V of p such that $m(V\times V) \subset N$. Suppose that C is a closed subset of V that contains p, that H and K are closed sets (K containing p) whose union is U, that $H \cap C$ and $K \cap C$ are connected, but that $H \cap K \cap C$ is not a subset of any component of $H \cap K$. Then $H \cap K \cap N$ is the union of disjoint closed sets A and B, each intersecting C. Since A and B are compact, there exist disjoint open sets E_1 and E_3, E_1 containing A and E_3 containing B. Define

$E_2 = \operatorname{Int} U - K$, $E_4 = \operatorname{Int} U - H$, $H_1 = H \cap C$, $K_1 = K \cap C$. This contradicts (2.1).

Example. For each positive integer n let C_n be the circle in R^2 with center at $(1/n, 0)$ and radius $1/n$. Let $A = \bigcup_{n=1}^{\infty} C_n$ and let X be the cone over A, that is, X is $A \times [0, 1]$ with the points in $A \times \{1\}$ identified. Then X is a contractible Peano space that admits no mean, for it is not locally quasi-unicoherent at $((0, 0), 0)$. This is a counterexample to an assertion in [1] (p. 331).

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