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ON RINGS OF DARBOUX FUNCTIONS

BY

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Problems connected with addition and multiplication of Darboux functions were investigated in many papers (see, e.g., notes [2], [4], [9], [10], and some chapters of monograph [1]). It is known that the maximal additive family for the Darboux functions is the family of constant functions (see [1]) and that if f is a discontinuous Darboux function in Baire class 1, then there exists a Darboux function g in Baire class 1 such that $f+g$ is not a Darboux function.

So, the question whether it is possible to form a ring of Darboux functions, containing all continuous functions and a fixed Darboux function f, seems to be interesting. This question becomes especially interesting when f is not assumed to be a function in Baire class 1.

The present paper is intended to discuss this problem. In particular, from Theorem 1 it can be inferred that there exist nonmeasurable Darboux functions for which the formation of the ring described above is possible. The method for proving Theorem 1 has not been chosen accidentally, which is stressed in Theorem 2. However, not all problems concerning this question are solved in the paper. Some questions the present paper gives no answer to are embraced in the form of a problem following Theorem 2.

Theorem 3 of the paper is connected with the problem of extending the topologies, and the result presented in it refers to those given in papers [5], [6], [8], and [11].

We use the standard notions and notation which were used in the monograph of Engelking [3].

By R we denote the set of all real numbers as well as the space of all real numbers with the natural topology of line. By $(R, \mathscr{T})$ we denote the topological space of all real numbers with the topology $\mathscr{T}$ (different from the natural topology).

Throughout the paper, we consider only real functions, i.e., all considered functions assumed their values in the topological space R.

The symbols (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ (for $a < b$ or $b < a$) denote (respectively) open, left-sided open, right-sided open, and closed intervals in R. We denote by $f(a, b), f^{-1}(a, b)$, $f(a, b], f^{-1}(a, b], \ldots$ the images and inverse images of those intervals to avoid superfluous brackets.

By $m(x, y)$ we denote the middle-point of the interval (x, y).

Let $x \in R$ and $A \subset R$. Then by $x < A$ $(x > A)$ we denote the fact that, for every $a \in A$, $x < a$ $(x > a)$.

The symbols A^{d} and $\bar{A}$ denote (respectively) the derived set and the closure of a set $A \subset R$.

Moreover, we use the following symbols:

C – the class of all continuous functions $f\colon R \to R$;

$C(\mathcal{T})$ – the ring of all continuous functions $f\colon (R, \mathcal{T}) \to R$, where $\mathcal{T}$ is some topology in R;

C_f – the set of all continuity points of f (in the natural topology of line);

D_f – the set of all discontinuity points of f (in the natural topology of line).

For a Darboux function $f\colon R \to R$ let $\mathrm{RD}(f)$ denote a class of all rings K of Darboux functions (with the usual addition and multiplication of functions) such that $f \in K$ and $C \subset K$.

Before discussing the results of this paper, we first give the following

DEFINITION 1. (a) We say that $f\colon R \to R$ is *Young's function* if for every point $x_0 \in R$ there exist sequences $\{x_n^-\}$ and $\{x_n^+\}$ such that

$$x_n^- \nearrow x_0, \quad x_n^+ \searrow x_0, \quad \text{and} \quad \lim_{n \to \infty} f(x_n^-) = \lim_{n \to \infty} f(x_n^+) = f(x_0).$$

(b) We say that $f\colon R \to R$ is *c-Young's function* if for every point $x_0 \in D_f$ and every $\varepsilon > 0$ there exists $\delta_{x_0} > 0$ such that if I is a component of C_f and

$$I \cap (x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) \neq \emptyset,$$

then

$$f(I) \cap \big(f(x_0) - \varepsilon, f(x_0) + \varepsilon\big) \neq \emptyset.$$

THEOREM 1. *If $f\colon R \to R$ is a Darboux and c-Young's function such that D_f is a nowhere dense set, then the class* $\mathrm{RD}(f)$ *contains the ring $C(\mathcal{T})$, where $\mathcal{T}$ is some topology in R.*

Proof. Put $C'_f = R \setminus \bar{D}_f$. Let $\{I_n\}$ be a sequence of all components of C'_f (remark that the components of C'_f are open and disjoint intervals).

Let $x \in \bar{D}_f$.

It is easy to see that:

(*) for every $\varepsilon > 0$ there exists $\delta_x > 0$ such that if I is a component of C'_f and $I \cap (x - \delta_x, x + \delta_x) \neq \emptyset$, then

$$f(I) \cap \big(f(x) - \varepsilon, f(x) + \varepsilon\big) \neq \emptyset.$$

Let $\{\delta_n^{(x)}\}_{n=1}^{\infty}$ denote the sequence of positive numbers decreasing to zero such that if I is the component of C_f' and

$$I \cap (x-\delta_n^{(x)},\ x+\delta_n^{(x)}) \neq \emptyset,$$

then

$$f(I) \cap (f(x)-1/n, f(x)+1/n) \neq \emptyset.$$

Now we shall execute auxiliary constructions on the right side of x. Consider two cases.

1° There exists a natural number n_0 such that $I_{n_0} = (x, a)$, where a is some real number or $a = +\infty$.

Then there exists a sequence $\{y_i^{(x+)}\}_{i=2}^{\infty}$ of elements from I_{n_0} such that

$$y_i^{(x+)} \searrow x \quad \text{and} \quad f(y_i^{(x+)}) \in \left(f(x)-\frac{1}{i-1}, f(x)+\frac{1}{i-1}\right).$$

Since, for every $i = 2, 3, \ldots$, f is continuous at $y_i^{(x+)}$, there exists a sequence $\{l_{y_i^{(x+)}}\}_{i=2}^{\infty}$ of natural numbers such that, for every $i = 2, 3, \ldots$,

$$A_i = \left(y_i^{(x+)} - \frac{1}{l_{y_i^{(x+)}}},\ y_i^{(x+)} + \frac{1}{l_{y_i^{(x+)}}}\right) \subset I_{n_0},$$

$$f(A_i) \subset \left(f(x)-\frac{1}{i-1}, f(x)+\frac{1}{i-1}\right),$$

and, moreover,

$$y_{i+1} + \frac{1}{l_{y_{i+1}^{(x+)}}} < y_i - \frac{1}{l_{y_i^{(x+)}}} < y_i + \frac{1}{l_{y_i^{(x+)}}} < y_{i-1} - \frac{1}{l_{y_{i-1}^{(x+)}}}$$

for every $i = 3, 4, \ldots$

2° The element x is not a left-hand endpoint of any component of C_f'. Let $\{k_n^{(x+)}\}$ be a sequence of natural numbers such that

$$x < I_{k_n^{(x+)}} < x + \delta_1^{(x)} \quad \text{for } n = 1, 2, \ldots$$

(it is easy to see that $\{k_n^{(x+)}\}$ is an infinite sequence).

Let

$$t_{k_n}^{(x+)} = \min\{m\colon (x+\delta_m^{(x)},\ x+\delta_{m-1}^{(x)}) \cap I_{k_n^{(x+)}} \neq \emptyset\} \qquad \text{for } n = 1, 2, \ldots$$

From the definitions of $t_{k_n}^{(x+)}$ and $\{\delta_n^{(x)}\}$ we infer that (for every $n = 1, 2, \ldots$) the set

$$I_{k_n^{(x+)}} \cap f^{-1}\left(f(x)-\frac{1}{t_{k_n}^{(x+)}-1}, f(x)+\frac{1}{t_{k_n}^{(x+)}-1}\right)$$

is nonempty. Denote by $y_{k_n}^{(x+)}$ (for $n = 1, 2, \ldots$) an arbitrary element of the above intersection.

Now, for every $n = 1, 2, \ldots$, let $l_{k_n}^{(x+)}$ denote a natural number such that

$$\left(y_{k_n}^{(x+)} - \frac{1}{l_{k_n}^{(x+)}},\ y_{k_n}^{(x+)} + \frac{1}{l_{k_n}^{(x+)}}\right) \subset I_{k_n} \cap f^{-1}\left(f(x) - \frac{1}{t_{k_n}^{(x+)} - 1}, f(x) + \frac{1}{t_{k_n}^{(x+)} - 1}\right).$$

Finally, let $\tau_m^+(x) = \{k_n^{(x+)}: t_{k_n}^{(x+)} = m\}$ for $m = 2, 3, \ldots$ (it is possible that $\tau_m^+(x) = \emptyset$ for some m, but for every natural number k there exists $m > k$ such that $\tau_m^+(x) \neq \emptyset$).

In an analogous way we may make the corresponding constructions on the left side of x, i.e., if there exists a natural number n_0 such that $I_{n_0} = (a, x)$, we define (analogously as in case 1°) sequences $\{y_i^{(x-)}\}_{i=2}^{\infty}$ and $\{l_{y_i^{(x-)}}\}_{i=2}^{\infty}$; if $I_n \neq (a, x)$ for every n, we define (analogously as in case 2°) the sequences $\{k_n^{(x-)}\}$ and $t_{k_n}^{(x-)}, y_{k_n}^{(x-)}, l_{k_n}^{(x-)}$ for $n = 1, 2, \ldots$, and $\tau_m^-(x)$ for $m = 2, 3, \ldots$

Moreover, for $x \in I_s \subset C'_f$ let $k^{(x)}$ denote a natural number such that

$$\left(x - \frac{1}{k^{(x)}},\ x + \frac{1}{k^{(x)}}\right) \subset I_s.$$

Now we define (for every $x \in R$) a family $B(x)$ of subsets of R in the following way:

if $x \in C'_f$, then we put

$$B(x) = \{(x - 1/t,\ x + 1/t): t = k^{(x)}, k^{(x)} + 1, \ldots\};$$

if $x \in \bar{D}_f$ and x is not an endpoint of any component of C'_f, then we put

$$B(x) = \Bigg\{U_s(x) = \bigcup_{i=s}^{\infty}\Bigg[\bigcup_{k_n^{(x+)} \in \tau_i^+(x)}\left(y_{k_n}^{(x+)} - \frac{1}{l_{k_n}^{(x+)} + s},\ y_{k_n}^{(x+)} + \frac{1}{l_{k_n}^{(x+)} + s}\right) \cup$$
$$\cup \bigcup_{k_n^{(x-)} \in \tau_i^-(x)}\left(y_{k_n}^{(x-)} - \frac{1}{l_{k_n}^{(x-)} + s},\ y_{k_n}^{(x-)} + \frac{1}{l_{k_n}^{(x-)} + s}\right)\Bigg] \cup \{x\}: s = 2, 3, \ldots\Bigg\};$$

if $x \in \bar{D}_f$ and for some natural numbers n_1 and n_2 we have $I_{n_1} = (a_1, x)$ and $I_{n_2} = (x, a_2)$ (where a_1 and a_2 are real numbers or $a_1 = -\infty$ and $a_2 = +\infty$), then we put

$$B(x) = \Bigg\{U_s(x) = \bigcup_{i=s}^{\infty}\Bigg[\left(y_i^{(x+)} - \frac{1}{l_{y_i^{(x+)}} + s},\ y_i^{(x+)} + \frac{1}{l_{y_i^{(x+)}} + s}\right) \cup$$
$$\cup \left(y_i^{(x-)} - \frac{1}{l_{y_i^{(x-)}} + s},\ y_i^{(x-)} + \frac{1}{l_{y_i^{(x-)}} + s}\right)\Bigg] \cup \{x\}: s = 2, 3, \ldots\Bigg\};$$

if $x \in \bar{D}_f$, $I_{n_0} = (a_0, x)$ for some natural number n_0, and x is not a left-hand endpoint of any component of C'_f, then we put

$$B(x) = \left\{U_s(x) = \bigcup_{i=s}^{\infty}\left[\bigcup_{k_n^{(x+)}\in\tau_i^+(x)}\left(y_{k_n}^{(x+)} - \frac{1}{l_{k_n}^{(x+)}+s},\ y_{k_n}^{(x+)} + \frac{1}{l_{k_n}^{(x+)}+s}\right)\cup\right.\right.$$
$$\left.\left.\cup\left(y_i^{(x-)} - \frac{1}{l_{y_i^{(x-)}}+s},\ y_i^{(x-)} + \frac{1}{l_{y_i^{(x-)}}+s}\right)\right]\cup\{x\}\colon\ s = 2, 3, \ldots\right\};$$

if $x\in\bar{D}_f$, $I_{n_0} = (x, a_0)$ for some natural number n_0, and x is not a right-hand endpoint of any component of C'_f, then we define the family $B(x)$ in a similar way as in the above case.

It is easy to see that the family $\{B(x)\}_{x\in R}$ fulfils the conditions (BP1), (BP2), and (BP3) of [3] (p. 24) and at the same time $\{B(x)\}_{x\in R}$ generates some topology $\mathscr{T}$ (see [3], Theorem 1.2.2, p. 35).

We shall show that $C(\mathscr{T})\in \mathrm{RD}(f)$.

First, we show that $f\in C(\mathscr{T})$. In fact, let $x_0\in R$ and $\varepsilon > 0$.

If $x_0\in C'_f$, then there exists a natural number n such that $n \geqslant k^{(x_0)}$ and

$$(x_0 - 1/n,\ x_0 + 1/n) \subset f^{-1}\big(f(x_0)-\varepsilon, f(x_0)+\varepsilon\big),$$

which means that the function f is continuous at x_0 in the topology $\mathscr{T}$.

Now, we assume that $x_0\in\bar{D}_f$. Then let m_1 be a natural number such that $1/m_1 < \varepsilon$.

We consider the neighbourhood $U_{m_1+1}(x_0)\in B(x_0)$. From the definition of $U_{m_1+1}(x_0)$ (see the definition of the family $\{B(x)\}_{x\in R}$) we infer that

$$f\big(U_{m_1+1}(x_0)\big) \subset \left(f(x_0) - \frac{1}{m_1}, f(x_0) + \frac{1}{m_1}\right) \subset \big(f(x_0)-\varepsilon, f(x_0)+\varepsilon\big).$$

This proves that in this case f is a continuous function at x_0 in the topology $\mathscr{T}$.

Now, we shall show that if $g\in C(\mathscr{T})$, then g is a Darboux function (in the natural topology).

Assume, to the contrary, that g is not a Darboux function. Then there exist two points $x_1 < x_2$ such that $g(x_1) \neq g(x_2)$ and there exists $\alpha\in\big(g(x_1), g(x_2)\big)$ such that $\alpha\notin g(x_1, x_2)$. For instance, let $g(x_1) < g(x_2)$.

Remark that

$$\{x\in(x_1, x_2]\colon\ g[x_1, x) \subset (-\infty, \alpha)\} \neq \emptyset. \tag{1}$$

To see this we suppose first that there exists a component I_1 of C'_f such that $I_1 \supset (x_1, c)$, where $c > x_1$ is some real number. Of course, $g|(x_1, c)$ is a Darboux function. Observe that $c \leqslant x_2$. Really, since g is continuous at x_1 (in the topology $\mathscr{T}$), there exists s_0 such that $U_{s_0}\in B(x_1)$ and $g(U_{s_0}) \subset (-\infty, \alpha)$. From the definition of $\{B(x)\}_{x\in R}$ we deduce that

$$\emptyset \neq (x_1, x_2)\cap U_{s_0} \subset I_1.$$

Let $x_1' \in (x_1, x_2) \cap U_{s_0}$. Then $g(x_1') \in (-\infty, \alpha)$. To see that $c \leqslant x_2$ assume to the contrary that $c > x_2$. Then from the fact that $g|(x_1, c)$ is a Darboux function we infer that $\alpha \in g(x_1', x_2) \subset g(x_1, x_2)$, which is impossible. Thus $c \leqslant x_2$. Let $x_1'' \in (x_1, c) \cap U_{s_0}$. Then since $g(x_1'') \in (-\infty, \alpha)$ and $g|(x_1, c)$ is a Darboux function, we obtain $g[x_1, c) \subset (-\infty, \alpha)$. Thus we have just proved (1) in this case.

Now we assume that $x_1 \notin C_f'$ and x_1 is not a left-hand endpoint of any component of C_f'. By the continuity of g there exists a natural number s_0 such that $U_{s_0} \in B(x_1)$, $U_{s_0} < x_2$, and $g(U_{s_0}) \subset (-\infty, \alpha)$. Then

$$\begin{aligned} &U_{s_0} \cap [x_1, +\infty) \\ &= \bigcup_{i=s_0}^{\infty} \left(\bigcup_{k_n^{(x_1+)} \in \tau_i^+(x_1)} \left(y_{k_n}^{(x_1+)} - \frac{1}{l_{k_n}^{(x_1+)} + s_0},\ y_{k_n}^{(x_1+)} + \frac{1}{l_{k_n}^{(x_1+)} + s_0} \right) \right) \cup \{x_1\}. \end{aligned}$$

Let s_1 be an arbitrary natural number greater than s_0 such that $\tau_{s_1}^+(x_1) \neq \emptyset$ (see the remark after the definition of $\tau_m^+(x)$). Let

$$k_n^{(x_1+)} \in \bigcup_{i=s_1+1}^{\infty} \tau_i^+(x_1)$$

and assume, for instance, that $k_n^{(x_1+)} \in \tau_{s_0'}^+(x_1)$. Put

$$I_{k_n^{(x_1+)}} = (a, b).$$

Of course, $x_1 < b \leqslant x_2$. We show that $g[x_1, b) \subset (-\infty, \alpha)$. First we remark that $y_{k_n}^{(x_1+)} \in U_{s_0}$, which means that $g(y_{k_n}^{(x_1+)}) \in (-\infty, \alpha)$. Thus, since $g|(a, b)$ is a Darboux function and $\alpha \notin g(a, b)$, we have $g(a, b) \subset (-\infty, \alpha)$. Now, let I denote an arbitrary component of C_f' such that $x_1 < I < a$. Let

$$I = I_{k_w^{(x_1+)}}.$$

It is easy to see that $y_{k_w}^{(x_1+)} \in U_{s_0}$ and, consequently, $g(y_{k_w}^{(x_1+)}) \subset (-\infty, \alpha)$, which means that $g(I) \subset (-\infty, \alpha)$. Since I is arbitrary, we have

$$g\big(C_f' \cap [x_1, b)\big) \subset (-\infty, \alpha).$$

Notice that for every $x \in \bar{D}_f \cap [x_1, b)$ there exists a net $\{x_\sigma\}_{\sigma \in \Sigma}$ of elements of $C_f' \cap [x_1, b)$ such that

$$x \in \lim_{\sigma \in \Sigma} x_\sigma$$

(in the topology $\mathscr{T}$), which means, according to the fact that $g \in C(\mathscr{T})$ and the above considerations, that $g[x_1, b) \subset (-\infty, \alpha)$. Thus we have proved relation (1).

It is easy to see that $x_2 \notin \{x \in (x_1, x_2]\colon g[x_1, x) \subset (-\infty, \alpha)\}$.
Write

$$y_0 = \sup \{x \in (x_1, x_2]\colon g[x_1, x) \subset (-\infty, \alpha)\}.$$

We have $x_1 < y_0 < x_2$ and $g[x_1, y_0) \subset (-\infty, \alpha)$. We show that $y_0 \in \bar{D}_f$. Suppose, on the contrary, that $y_0 \in C'_f$. Let k be a natural number such that

$$x_1 < y_0 - 1/k < y_0 + 1/k < x_2 \quad \text{and} \quad (y_0 - 1/k, y_0 + 1/k) \subset I',$$

where I' denotes the component of C'_f such that $y_0 \in I'$. Then, by the definition of y_0, we have $g(y_0 - 1/k, y_0) \subset (-\infty, \alpha)$ and there exists $z \in (y_0, y_0 + 1/k)$ such that $g(z) > \alpha$. Since $g|(y_0 - 1/k, y_0 + 1/k)$ is continuous in the usual sense, g assumes value α for some point of the interval $(y_0 - 1/k, y_0 + 1/k) \subset (x_1, x_2)$, which contradicts our assumption and proves that $y_0 \in \bar{D}_f$.

From the definition of the topology $\mathscr{T}$ we infer that there exists a net $\{z_a\}_{a \in A}$ of elements of (x_1, y_0) such that

$$y_0 \in \lim_{a \in A} z_a$$

(in the topology $\mathscr{T}$). Since $g \in C(\mathscr{T})$ and $\alpha \notin g(x_1, x_2)$, we have $g(y_0) \in (-\infty, \alpha)$. Moreover, there exists a natural number s_1 such that $U_{s_1}(y_0) \in B(y_0)$ and $g(U_{s_1}(y_0)) \subset (-\infty, \alpha)$.

In a similar way as for (1) (we put y_0 in place of x_1) we can prove that

$$\{x \in (y_0, x_2]\colon g[y_0, x) \subset (-\infty, \alpha)\} \neq \emptyset,$$

but this is impossible because $y_0 = \sup \{x \in (x_1, x_2]\colon g[x_1, x) \subset (-\infty, \alpha)\}$. The obtained contradiction completes the proof of the fact that g is a Darboux function.

Finally, we shall show that if $k \in C$, then $k \in C(\mathscr{T})$.

Let $z \in R$ and ε be an arbitrary positive number. Then there exists $\delta > 0$ such that

$$k(z - \delta, z + \delta) \subset \big(f(z) - \varepsilon, f(z) + \varepsilon\big).$$

If $z \in C'_f$, then, of course, k is also continuous at z in the topology $\mathscr{T}$. Now, we assume that $z \in \bar{D}_f$. It is easy to see that there exists a natural number d such that $U_d(z) \in B(z)$ and $U_d(z) \subset (z - \delta, z + \delta)$. Hence

$$k\big(U_d(z)\big) \subset \big(f(z) - \varepsilon, f(z) + \varepsilon\big).$$

This completes the proof.

Theorem 1 shows that there exist nonmeasurable Darboux functions for which $\mathrm{RD}(f) \neq \emptyset$.

Remark. The ring $\mathscr{R} = C(\mathscr{T}) \in \mathrm{RD}(f)$ defined in the proof of Theorem 1 has the following properties:

(A) if $p \in \mathscr{R}$, then $|p| \in \mathscr{R}$.

If we additionally assume that $D_f = \bar{D}_f$, then

(B) if $p \in \mathscr{R}$, then $C_f \subset C_p$.

The method of proving Theorem 1 consisted in constructing a topology $\mathscr{T}$ such that a ring of continuous functions in $\mathscr{T}$ belongs to $\mathrm{RD}(f)$, while $\mathscr{T}$ depends on the choice of sequences $\{y_{k_n}^{(x-)}\}$ and $\{y_{k_n}^{(x+)}\}$ as well as on the length of neighbourhoods of elements of these sequences. Let us denote by K_f the class of all topologies which can be constructed by means of the method described in the proof of Theorem 1 (i.e., with different choices of sequences $\{y_{k_n}^{(x-)}\}$, $\{y_{k_n}^{(x+)}\}$ and different lengths of intervals corresponding to them).

It turns out that in some cases the method presented in the proof of Theorem 1 is not chosen accidentally, which is stated in the following

THEOREM 2. *Let $f\colon R \to R$ be a Darboux function such that $(D_f)^d = \emptyset$. Then a function g belongs to some ring $\mathscr{R} \in \mathrm{RD}(f)$ which fulfils conditions* (A) *and* (B) *if and only if g is continuous in some topology $\mathscr{T}_g \in K_f$.*

Proof. If $D_f = \emptyset$, then K_f is a singleton because only the natural topology belongs to K_f. In this case the proof of this theorem is obvious.

Thus we assume that $D_f \neq \emptyset$.

Necessity. Let

$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$$

be a sequence of all elements of D_f (it is possible that the set D_f is finite).

We shall show that there exists a sequence $\{y_n^{(0+)}\} \subset (x_0, x_1)$ such that $y_n^{(0+)} \searrow x_0$, and

$$\lim_{n\to\infty} f(y_n^{(0+)}) = f(x_0) \quad \text{and} \quad \lim_{n\to\infty} g(y_n^{(0+)}) = g(x_0). \tag{2}$$

Write $f_1 = f - f(x_0)$ and $g_1 = g - g(x_0)$. Since $\mathscr{R} \in \mathrm{RD}(f)$, we have $f_1, g_1 \in \mathscr{R}$. We put $h = |f_1| + |g_1|$. Then, by condition (A), we infer that $h \in \mathscr{R}$. Of course, h is in Baire class 1 and h is a Darboux function, and so h is Young's function (see [2] and [12]). Then there exists a sequence $\{y_n^{(0+)}\}$ of elements of (x_0, x_1) such that

$$y_n^{(0+)} \searrow x_0 \quad \text{and} \quad \lim_{n\to\infty} h(y_n^{(0+)}) = h(x_0) = 0.$$

Therefore we have (2).

In a similar way we can prove that there exists a sequence $\{y_n^{(0-)}\}$ of elements of (x_{-1}, x_0) such that $y_n^{(0-)} \nearrow x_0$, and

$$\lim_{n\to\infty} f(y_n^{(0-)}) = f(x_0) \quad \text{and} \quad \lim_{n\to\infty} g(y_n^{(0-)}) = g(x_0).$$

In general, we can prove that for every $k = 0, \pm 1, \pm 2, \dots$ there exist sequences $\{y_n^{(k+)}\}$ and $\{y_n^{(k-)}\}$ of points of the intervals (x_k, x_{k+1}) and (x_{k-1}, x_k) (respectively) such that([1]) $y_n^{(k+)} \searrow x_k$, $y_n^{(k-)} \nearrow x_k$, and

$$\lim_{n\to\infty} f(y_n^{(k+)}) = \lim_{n\to\infty} f(y_n^{(k-)}) = f(x_k)$$

and

$$\lim_{n\to\infty} g(y_n^{(k+)}) = \lim_{n\to\infty} g(y_n^{(k-)}) = g(x_k).$$

Since every element of each of the sequences $\{y_n^{(k\pm)}\}_{n=1}^{\infty}$ is a continuity point of the function g in the natural topology (see condition (B)) for every $k = 0, \pm 1, \pm 2, \dots$, the numbers $l_{y_n^{(x_k \pm)}}$ (chosen in the proof of Theorem 1) can be chosen in such a way that they fulfil all the required (in this proof) conditions, and moreover (we assume $y_n^{(k\pm)} = y_n^{(x_k\pm)}$)

$$g\left(y_n^{(k\pm)} - \frac{1}{l_{y_n^{(k\pm)}}},\, y_n^{(k\pm)} + \frac{1}{l_{y_n^{(k\pm)}}}\right)$$
$$\subset \left(g(y_n^{(k\pm)}) - \frac{|g(x_k) - g(y_n^{(k\pm)})|}{2},\, g(y_n^{(k\pm)}) + \frac{|g(x_k) - g(y_n^{(k\pm)})|}{2}\right)$$
$$\text{if } g(y_n^{(k\pm)}) \neq g(x_k)$$

and

$$g\left(y_n^{(k\pm)} - \frac{1}{l_{y_n^{(k\pm)}}},\, y_n^{(k\pm)} + \frac{1}{l_{y_n^{(k\pm)}}}\right)$$
$$\subset \left(g(x_k) - \frac{1}{n},\, g(x_k) + \frac{1}{n}\right) \quad \text{if } g(y_n^{(k\pm)}) = g(x_k).$$

According to our assumption and the proof of Theorem 1, sequences $\{y_n^{(k\pm)}\}_{n=1}^{\infty}$ and $\{l_{y_n^{(k\pm)}}\}$ for $k = 0, \pm 1, \pm 2, \dots$ generate some topology $\mathcal{T}_g \in K_f$.

We shall show that g is continuous in the topology $\mathcal{T}_g$. By the fact that $D_f = \bar{D}_f$, condition (B), and the definition of $\mathcal{T}_g$ it is sufficient to prove that g is continuous (in $\mathcal{T}_g$) at every discontinuity point of f. For instance, we show that g is continuous at x_0 (the proof of the fact that g is continuous at $x_k \neq x_0$ is very similar to the above one).

Let $\varepsilon > 0$. Then there exists a natural number N such that $1/N < \varepsilon$ and, for $n \geqslant N$,

$$g(y_n^{(0\pm)}) \in (g(x_0) - \varepsilon/2,\, g(x_0) + \varepsilon/2).$$

([1]) If there exists m such that $x_m \in D_f$ and $D_f \leqslant x_m$ ($x_m \leqslant D_f$), then we put $(x_m, x_{m+1}) = (x_m, +\infty)$ ($(x_{m-1}, x_m) = (-\infty, x_m)$).

Consider $U_N(x_0) \in B(x_0)$ (see the proof of Theorem 1). Then

$$g(U_N(x_0)) \subset (g(x_0)-\varepsilon, g(x_0)+\varepsilon),$$

which means that x_0 is a continuity point of g in the topology $\mathscr{T}_g$.

The sufficiency is a simple consequence of Theorem 1 and the Remark.

In connection with Theorem 2 we pose the following problem:

PROBLEM. (**P 1326**) Assume that, for some Darboux function f, $\mathrm{RD}(f) \neq \emptyset$. Then characterize a function g such that g belongs to some ring $\mathscr{R} \in \mathrm{RD}(f)$.

The partial answer to this question is contained in Theorem 2.

We assume the following notation:

B_1 – the class of all functions $f\colon R \to R$ in Baire class 1;

$B_1(\mathscr{T})$ – the class of all functions $f\colon (R, \mathscr{T}) \to R$ in Baire class 1;

Dbx – the class of all Darboux functions $f\colon R \to R$.

Now, we shall show that Theorem 1 (with the proof) can be applied to solving some problems in connection with the extension of the topology.

In papers [5], [6] and [11] the following problem has been studied (²):

Under what hypotheses on the topology $\mathscr{T}$ stronger than the natural topology of line does the equality $C = C(\mathscr{T})$ take place?

This problem suggests the following question: does there exist topology $\mathscr{T}^*$ stronger than natural topology of line such that

1° $C \subsetneqq C(\mathscr{T}^*)$,

2° $B_1 = B_1(\mathscr{T}^*)$,

3° $C(\mathscr{T}^*) \subset \mathrm{Dbx}$?

The answer to this question is positive (see Theorem 3).

Before we formulate and prove Theorem 3 we assume the following notation: for $x \in R$ let $S(x)$ denote the class of all open local bases in x (in the natural topology of line), and let $\mathscr{B}_{\mathscr{T}^*}(x)$ denote some open local base in x (in the topology $\mathscr{T}^*$).

THEOREM 3. *Let A be an arbitrary nonempty, countable and closed subset of R. Then there exists a topology $\mathscr{T}^*$ stronger than the natural topology of line, which fulfils conditions* 1°–3° *and such that $\mathscr{B}_{\mathscr{T}^*}(x) \in S(x)$ for $x \in R \setminus A$ and $\mathscr{B}_{\mathscr{T}^*}(x) \notin S(x)$ for $x \in A$.*

Proof. Let $\{(a_n, b_n)\}$ be a sequence of all components of $R \setminus A$ (it is possible that, for some m_1, $a_{m_1} = -\infty$ or, for some m_2, $b_{m_2} = +\infty$). Assume that for every n

$$c_n = \begin{cases} m(a_n, b_n) & \text{if } -\infty < a_n < b_n < +\infty, \\ a_n + 1 & \text{if } a_n > -\infty \text{ and } b_n = +\infty, \\ b_n - 1 & \text{if } a_n = -\infty \text{ and } b_n < +\infty. \end{cases}$$

(²) In [8] this problem has been investigated in general topological spaces.

Let us define the function f: $R \to R$ in the following way:

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ \sin \dfrac{1}{x-a_n} & \text{if } x \in (a_n, c_n] \text{ and } a_n > -\infty, \\ \sin \dfrac{1}{x-2|c_n - x| - a_n} & \text{if } x \in [c_n, b_n) \text{ and } b_n < +\infty,\ a_n > -\infty, \\ \sin \dfrac{1}{x-2|c_n - x| - b_n + 2} & \text{if } x \in [c_n, b_n) \text{ and } b_n < +\infty,\ a_n = -\infty, \\ \sin \dfrac{1}{c_n - b_n + 2} & \text{if } x \in (a_n, c_n] \text{ and } b_n < +\infty,\ a_n = -\infty, \\ \sin \dfrac{1}{c_n - a_n} & \text{if } x \in (c_n, b_n] \text{ and } b_n = +\infty. \end{cases}$$

It is easy to see that f is a Darboux and c-Young's function such that $D_f = A$, and so D_f is a closed and boundary set (i.e., D_f is nowhere dense). Let $\mathscr{T}^*$ be some topology of class K_f (defined before Theorem 2). Of course, $\mathscr{T}^*$ is stronger than the natural topology of line. It is easy to see that $\mathscr{B}_{\mathscr{T}^*}(x) \in S(x)$ for $x \in R \setminus A$ and $\mathscr{B}_{\mathscr{T}^*}(x) \notin S(x)$ for $x \in A$. In the proof of Theorem 1 we have shown that condition 3° is fulfilled and $C \subset C(\mathscr{T}^*)$. Moreover, since $f \in C(\mathscr{T}^*) \setminus C$, condition 1° is also fulfilled.

Now, we show that condition 2° is fulfilled. Let $g \in B_1(\mathscr{T}^*)$ and let α be an arbitrary real number. Then there exists a sequence $\{g_k\}$ of continuous functions in the topology $\mathscr{T}^*$ such that

$$\lim_{k \to \infty} g_k = g.$$

Consider an arbitrary component (a_n, b_n) of $R \setminus A$. Let $\{p_m^{(n)}\}$ and $\{q_m^{(n)}\}$ be arbitrary sequences of elements of the intervals (a_n, c_n) and (c_n, b_n) (respectively) such that $p_m^{(n)} \searrow a_n$ and $q_m^{(n)} \nearrow b_n$. Then $g_k | [p_m^{(n)}, q_m^{(n)}]$ is a continuous function in the natural topology of the segment $[p_m^{(n)}, q_m^{(n)}]$. Moreover, it is easy to see that

$$\lim_{k \to \infty} g_k | [p_m^{(n)}, q_m^{(n)}] = g | [p_m^{(n)}, q_m^{(n)}],$$

which means that $g | [p_m^{(n)}, q_m^{(n)}]$ is in Baire class 1 in the natural topology of the segment $[p_m^{(n)}, q_m^{(n)}]$. Hence the sets

$$\{x \in [p_m^{(n)}, q_m^{(n)}]: g(x) > \alpha\} \quad \text{and} \quad \{x \in [p_m^{(n)}, q_m^{(n)}]: g(x) < \alpha\}$$

are of type F_σ (in the natural topology of $[p_m^{(n)}, q_m^{(n)}]$), which implies that

$$\{x \in [p_m^{(n)}, q_m^{(n)}]: g(x) > \alpha\} = \bigcup_{s=1}^{\infty} F_{m,s}^{(n)}$$

and

$$\{x \in [p_m^{(n)}, q_m^{(n)}]:\ g(x) < \alpha\} = \bigcup_{s=1}^{\infty} K_{m,s}^{(n)},$$

where $F_{m,s}^{(n)}$ and $K_{m,s}^{(n)}$ are closed subsets of $[p_m^{(n)}, q_m^{(n)}] \subset R$.

Remark that

$$(a_n, b_n) = \bigcup_{m=1}^{\infty} [p_m^{(n)}, q_m^{(n)}],$$

which means that

$$\{x \in R:\ g(x) > \alpha\} = \{x \in A:\ g(x) > \alpha\} \cup \bigcup_n \bigcup_m \bigcup_s F_{m,s}^{(n)}$$

and

$$\{x \in R:\ g(x) < \alpha\} = \{x \in A:\ g(x) < \alpha\} \cup \bigcup_n \bigcup_m \bigcup_s K_{m,s}^{(n)}.$$

Hence the sets $\{x \in R:\ g(x) > \alpha\}$ and $\{x \in R:\ g(x) < \alpha\}$ are of type F_σ (in the natural topology of line), which implies that $g \in B_1$.

We have shown that $B_1(\mathscr{T}^*) \subset B_1$. The inverse inclusion is obvious. This completes the proof.

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