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A FACTORIZATION THEOREM AND ITS APPLICATION TO EXTREMALLY DISCONNECTED RESOLUTIONS

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The aim of this paper is to show a factorization theorem for skeletal maps in the sense of Mioduszewski and Rudolf [9], i.e., pseudo-open maps in the sense of Herrlich and Strecker [6] (see also this author [1]), and to apply this theorem to the construction of the greatest extremally disconnected resolution for any Hausdorff space; an *extremally disconnected resolution* is a (continuous) irreducible map of an extremally disconnected space onto a given one. It will be shown that each skeletal map $f: X \to Y$ onto, where X is an extremally disconnected Hausdorff space, has a factorization

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

such that the factor $h: Z \to Y$ is irreducible and Z is an extremally disconnected Hausdorff space.

In contrast to the compact case (see Gleason [4]), there are many extremally disconnected resolutions for any Hausdorff space. The greatest one coincides with the Iliadis resolution [7] modified in [9]. It is also known from [9] that in the category of skeletal maps of Hausdorff spaces the modified Iliadis resolution leads to a functor adjoint to the full embedding of the category of extremally disconnected spaces into the category of Hausdorff spaces. By the use of our factorization theorem the construction of the greatest extremally disconnected resolution falls under a general categorial schema for the construction of adjoints given by Freyd [3].

In fact, our factorization theorem is even more general than we said, and its full statement is the main theorem (Theorem 1) of this paper.

1. Preliminaries. All maps are assumed to be continuous. A map $f: X \to Y$ is *skeletal* if

(1) $\quad \operatorname{Int} f^{-1}(\operatorname{cl} U) = \operatorname{Int} \operatorname{cl} f^{-1}(U) \quad$ for each U open in Y,

or, equivalently, if

(1′) $\quad \mathrm{cl}\, f^{-1}(\mathrm{Int}\, F) = \mathrm{cl}\,\mathrm{Int}\, f^{-1}(F)$ for each F closed in Y.

A map $h: Z \to Y$ onto is said to be *irreducible* iff

(2) $\quad \mathrm{cl}\, h(F) \neq Y$ whenever F is closed and $F \neq Z$;

this notion (see this author [2]) is somewhat different from the usual one: $h(F) \neq Y$ whenever F is closed and $F \neq Z$. If h is closed, the difference vanishes, e.g. in the compact case. Irreducible maps are always skeletal (see [9], p. 27). A map $f: Z \to Y$ is said to be *r.o.-minimal* (see [9], p. 30) if the topology in Z is generated by sets $f^{-1}(U) \cap V$, where U is open in Y and V is regularly open in Z (*regularly open*, shortly, *r.o.*, means that $V = \mathrm{Int}\,\mathrm{cl}\, V$).

A space is said to be *extremally disconnected*, shortly, *e.d.* (see Stone [10]) if the closure of each open subset of it is open.

A map $g: X \to Z$ is said to be *e.d.-preserving* if

(3) $\quad \mathrm{cl}\, g^{-1}(G) = g^{-1}(\mathrm{cl}\, G)$ for each G open in Z;

clearly, an e.d.-preserving map is skeletal.

The following lemmas are obvious:

LEMMA 1. *If $g: X \to Z$ is onto and e.d.-preserving, X is e.d. and Hausdorff, and Z is Hausdorff, then Z is e.d.*

LEMMA 2. *If the composition $X \xrightarrow{g} Z \xrightarrow{h} T$ is e.d.-preserving, and the factor $g: X \to Z$ is onto, then the factor $h: Z \to T$ is e.d.-preserving.*

LEMMA 3. *If a map $h: Z \to Y$ is onto, e.d.-preserving, irreducible and r.o.-minimal, and Z is a T_0-space, then h is a homeomorphism.*

Proof. It is known (see [9], p. 27) that if $h: Z \to Y$ is irreducible, and G is a non-empty r.o. subset in Z, then $G = \mathrm{Int}\,\mathrm{cl}\, h^{-1}(U)$ for a certain U r.o. in Y. To prove Lemma 3 note that if $W = \mathrm{Int}\,\mathrm{cl}\, h^{-1}(U) \cap h^{-1}(V)$, where U is r.o., and V is open in Y, then $W = h^{-1}(U \cap V)$.

2. Factorization theorems.

THEOREM 1. *If a map $f: X \to Y$ is onto and skeletal, and Y is a T_0-space, then there exists a unique factorization $f: X \xrightarrow{g} Z \xrightarrow{h} Y$ such that $g: X \to Z$ is onto and e.d.-preserving, Z is a T_0-space, and $h: Z \to Y$ is onto, irreducible and r.o.-minimal.*

The factor h is a homeomorphism iff f is e.d.-preserving.

The factor g is a homeomorphism iff f is irreducible and r.o.-minimal.

Proof. Consider an equivalence on X assuming $x \sim y$ whenever

(4) for each U r.o. and V open in Y, there is $x \in \mathrm{Int}\,\mathrm{cl}\, f^{-1}(U) \cap f^{-1}(V)$ iff $y \in \mathrm{Int}\,\mathrm{cl}\, f^{-1}(U) \cap f^{-1}(V)$.

Let $[x]$ denote the equivalence class of x. Consider the topology $\mathcal{T}'$ in X generated by

$$\mathfrak{B} = \{f^{-1}(V) \cap \operatorname{Int}\operatorname{cl} f^{-1}(U) \colon V \text{ open and } U \text{ r.o. in } Y\}.$$

We have $\mathcal{T}' \subset \mathcal{T}$, where $\mathcal{T}$ is the given topology in X. Since $\mathfrak{B}$ is closed with respect to finite intersections, $\mathfrak{B}$ is a base of $\mathcal{T}'$. Let

$$q\colon X' \to X'/\!\sim\; = Z$$

be the quotient map, where X' is X with the topology $\mathcal{T}'$. Let

$$g = q \circ c\colon X \to X' \to Z,$$

where $c\colon X \to X'$ is a contraction. To show that Z is a T_0-space note that

(5) $[x] \cap \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$ implies $[x] \subset \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)$

for each U r.o. and V open in Y, a fact which follows obviously from (4). This means that each member of $\mathfrak{B}$ is a union of equivalence classes of $\sim$. Thus the family

$$g(\mathfrak{B}) = \{g(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)) \colon U \text{ is r.o. and } V \text{ is open in } Y\}$$

is a base of a topology in Z. Let a and b be two different points of Z. Let x and y be such that $a = g(x)$ and $b = g(y)$. There is $[x] \neq [y]$ and, by (4), there exists a U r.o. and a V open in Y such that

$$x \in \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V) \quad \text{and} \quad y \notin \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V),$$

or conversely. Then, by (5),

$$g(x) \in g(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)) \quad \text{and} \quad g(y) \notin g(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)),$$

or conversely; this means that Z is a T_0-space.

To see that g is e.d.-preserving it suffices to show that $g^{-1}(\operatorname{cl} H) \subset \operatorname{cl} g^{-1}(H)$ for an arbitrary H open in Z. Clearly,

$$H = \bigcup \{g(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)) \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V}\},$$

where $\mathcal{U}$ is a family of r.o. sets in Y, and $\mathcal{V}$ is a family of open sets in Y. Suppose that $x \notin \operatorname{cl} g^{-1}(H)$. Then there exists an open neighbourhood W of x such that

$$W \cap \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V) = \emptyset \quad \text{for each } U \in \mathcal{U} \text{ and } V \in \mathcal{V}.$$

Hence

$$\operatorname{Int}\operatorname{cl} f(W) \cap U \cap V = \emptyset \quad \text{for each } U \in \mathcal{U} \text{ and } V \in \mathcal{V}.$$

Since f is skeletal, hence, by (1′), $W \subset \operatorname{Int}\operatorname{cl} f^{-1}(\operatorname{Int}\operatorname{cl} f(W))$. Clearly,

$$\operatorname{Int}\operatorname{cl} f^{-1}(\operatorname{Int}\operatorname{cl} f(W)) \cap \operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V) = \emptyset$$
$$\text{for each } U \in \mathcal{U} \text{ and } V \in \mathcal{V}.$$

Hence $g\big(\operatorname{Int}\operatorname{cl} f^{-1}(\operatorname{Int}\operatorname{cl} f(W))\big)$ is an open neighbourhood of $g(x)$, $x \notin g^{-1}(\operatorname{cl} H)$. Since Y is T_0, we infer that

$$(6) \qquad x \sim y \quad \text{implies} \quad f(x) = f(y).$$

Hence there can be defined a map $h: Z \to Y$ such that

$$(7) \qquad h([x]) = f(x).$$

Clearly, h is continuous and the diagram

$$(8) \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \nearrow_{h} & \\ Z & & \end{array}$$

commutes.

We shall show that h is irreducible.

To do this, let $G = g\big(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)\big)$, where U is r.o. and V is open in Y, be an arbitrary open set of the base in Z. We have, f being skeletal,

$$\begin{aligned} \operatorname{cl} h(Z \setminus G) &= \operatorname{cl} h\big(Z \setminus g\big(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)\big)\big) \\ &= \operatorname{cl} h\Big(g\big(X \setminus \big(\operatorname{Int}\operatorname{cl} f^{-1}(U) \cap f^{-1}(V)\big)\big)\Big) \\ &= \operatorname{cl} f\big(X \setminus \operatorname{Int} f^{-1}(\operatorname{cl} U)\big) \cup \operatorname{cl} f\big(f^{-1}(Y \setminus V)\big) \\ &= \operatorname{cl} f\big(\operatorname{cl}\big(X \setminus f^{-1}(\operatorname{cl} U)\big)\big) \cup \operatorname{cl}(Y \setminus V) \\ &= \operatorname{cl} f\big(f^{-1}(Y \setminus \operatorname{cl} U)\big) \cup \operatorname{cl}(Y \setminus V) \\ &= \operatorname{cl}\big(Y \setminus (\operatorname{cl} U \cap V)\big) \subset Y \setminus U \cap V. \end{aligned}$$

Hence $\operatorname{cl} h(Z \setminus G) \neq Y$. Thus h is irreducible.

To prove that h is r.o.-minimal, let H be an r.o. set in Z. There exists U r.o. in Y such that $H = \operatorname{Int}\operatorname{cl} h^{-1}(U)$, h being irreducible (see [9], p. 27). Since g is e.d.-preserving,

$$H = g\big(g^{-1}\big(\operatorname{Int}\operatorname{cl} h^{-1}(U)\big)\big) = g\big(\operatorname{Int}\operatorname{cl} g^{-1}\big(h^{-1}(U)\big)\big) = g\big(\operatorname{Int}\operatorname{cl} f^{-1}(U)\big).$$

On the other hand, we have $h^{-1}(V) = g\big(f^{-1}(V)\big)$ for each V open in Y. Hence the family $\{h^{-1}(V) \cap H: V \text{ is open in } Y \text{ and } H \text{ is r.o. in } Z\}$ coincides with $g(\mathfrak{B})$ which is a base in Z. Thus h is r.o.-minimal.

Now we shall show the uniqueness of our construction. Suppose that f admits another factorization

$$X \xrightarrow{\varphi} T \xrightarrow{\psi} Y$$

such that φ is e.d.-preserving, ψ is irreducible and r.o.-minimal, and T is a T_0-space. Clearly, sets of the form

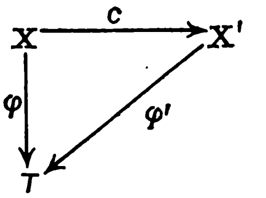
$$W = \operatorname{Int}\operatorname{cl} \psi^{-1}(U) \cap \psi^{-1}(V),$$

where U is r.o. and V is open in Y, form a base in T. Since φ is e.d.-preserving,

(9) $$\varphi^{-1}(W) = \mathrm{Int\,cl} f^{-1}(U) \cap f^{-1}(V).$$

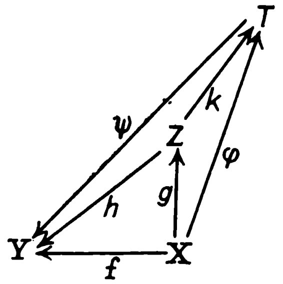
Hence there exists a (continuous) map $\varphi'\colon X' \to T$ filling up the diagram

(10)



It is an easy consequence of (9), since T is a T_0-space, that $x \sim y$ implies $\varphi'(x) = \varphi'(y)$. Since q is a quotient, there exists a map $k\colon Z \to T$ such that $k \circ q = \varphi'$. Hence, by (10), $k \circ g = \varphi$. Thus, by (8), we have a commutative diagram

(11)



Since g is onto, we have, by (11), $\psi \circ k = h$. Clearly, k is r.o.-minimal as the inner factor of h being r.o.-minimal. The inner factor of an irreducible map is irreducible (see [2]). Hence, by Lemmas 2 and 3, k is a homeomorphism.

To show the next thesis let us suppose that f is e.d.-preserving. Hence, by Lemma 2, h is e.d.-preserving. Thus, by Lemma 3, it is a homeomorphism.

It remains to show that if f is irreducible and r.o.-minimal, then g is a homeomorphism. Note that g is irreducible and r.o.-minimal. Thus, by Lemmas 2 and 3, g is a homeomorphism, which completes our proof.

Note. If a skeletal map $f\colon X \to Y$ is onto and admits a factorization

$$X \xrightarrow{g'} Z' \xrightarrow{h'} Y$$

such that g' is e.d.-preserving, Z' is Hausdorff and h' is irreducible (in general not r.o.-minimal), then there exists an e.d.-preserving contraction $c\colon Z' \to Z$ (Z is the factor space constructed in the proof of Theorem 1).

We shall show that the space Z constructed in Theorem 1 is, in general, not Hausdorff even for skeletal maps from a compact metric space onto a segment.

Example. Let $X = [-1, 0]\times[0, 1]\cup[0, 1]\times[1, 2] \subset R^2$ with the topology induced from the plane and let $f\colon X\to Y = [-1, 1]$ be the projection, i.e., $f(x, y) = x$. Clearly, f is skeletal. Consider three points: $p_1 = (0, 0)$, $p_2 = (0, 1)$ and $p_3 = (0, 2)$. It is easy to see that no two of them are equivalent in the sense of (4) and that each neighbourhood (in the topology of the factor space constructed in the proof of Theorem 1) of the point $[p_2]$ contains both $[p_1]$ and $[p_3]$. Hence Z is not even a T_1-space.

It will be shown in the following theorem that the assumption that X is e.d. deletes the defect:

THEOREM 2. *If a map $f\colon E\to Y$ is onto and skeletal, Y is Hausdorff and E is e.d. Hausdorff space, then there exists a unique factorization*

$$f\colon E\xrightarrow{g} Z\xrightarrow{h} Y$$

such that both g and h are skeletal, Z is e.d. Hausdorff space and h is irreducible r.o.-minimal map.

The factor h is a homeomorphism iff f is e.d.-preserving.

The factor g is a homeomorphism iff f is irreducible and r.o.-minimal.

Proof. It suffices to show, in view of Theorem 1 and Lemma 1, that Z is Hausdorff. Let $[x] \neq [y]$ and let us suppose that $f(x) \neq f(y)$. In this case $[x]$ and $[y]$ are separated by disjoint open neighbourhoods because Y is Hausdorff. If $f(x) = f(y)$, then, by (4), there exists U r.o. in Y such that $x\in \operatorname{Int}\operatorname{cl} f^{-1}(U)$ and $y\notin \operatorname{Int}\operatorname{cl} f^{-1}(U)$, or conversely. Since E is e.d., $\operatorname{cl} f^{-1}(U)$ is open and $x\in\operatorname{cl} f^{-1}(U)$ and $y\notin\operatorname{cl} f^{-1}(U)$, or conversely. Suppose that $x\in\operatorname{cl} f^{-1}(U)$ and $y\in E\setminus\operatorname{cl} f^{-1}(U)$. Consider r.o. set $V = Y\setminus\operatorname{cl} U$. Since f is skeletal and E is e.d.,

$$\operatorname{cl} f^{-1}(V) = E\setminus\operatorname{cl} f^{-1}(U).$$

Clearly, $\operatorname{cl} f^{-1}(U)$ and $\operatorname{cl} f^{-1}(V)$ are open and disjoint neighbourhoods of $[x]$ and $[y]$, respectively, which completes the proof.

3. Application to the construction of the greatest e.d. resolution. Now the following construction of the greatest e.d. resolution is possible:

Let X be a Hausdorff space. Consider all skeletal maps $f\colon Y\to X$ onto, where Y is e.d. Hausdorff space. These maps do not necessarily form a set. By Theorem 2, for each skeletal map $f\colon Y\to X$ onto, where Y is e.d., there exists a factorization

$$Y\xrightarrow{g} Z\xrightarrow{h} X,$$

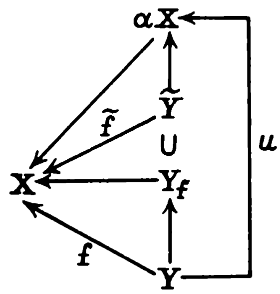
where $h\colon Z\to X$ is irreducible. It was proved in [9], p. 27, that for each Hausdorff space X there exists a *set* of irreducible maps onto X such that each irreducible map onto X is isomorphic to a map from this set. Hence there exists a *set* $S(X)$ of irreducible maps $f\colon Y_f\to X$ from e.d. space Y_f onto X such that each skeletal map $f'\colon Y'\to X$, where Y' is e.d., admits a decomposition

$$Y' \to Y_f \xrightarrow{f} X \quad \text{for some } f \in S(X).$$

Let $\tilde{Y}$ be the disjoint union of all Y_f for $f \in S(X)$ and let $\tilde{f}\colon \tilde{Y} \to X$ be the map induced by maps from $S(X)$. Clearly, $\tilde{Y}$ is e.d. and $\tilde{f}$ is skeletal, $\tilde{f}\,|\,Y_f$ being skeletal. There exists, by Theorem 2, a factorization

$$Y \to aX \xrightarrow{a^X} X,$$

where $a^X\colon aX \to X$ is irreducible and r.o.-minimal, and aX is e.d. For each skeletal map $f\colon Y \to X$ onto, where Y is e.d., there exists a unique map $u\colon Y \to X$ such that $a^X \circ u = f$. This map is a composition of maps given in the following diagram:



Thus the map $a^X\colon aX \to X$ is the greatest e.d. resolution.

This construction of the greatest extremally disconnected resolution depends on the existence of any skeletal map from e.d. space onto X. The existence of such maps can be obtained by the Kuratowski-Zorn Lemma, as was shown by Mioduszewski [8]; in fact, the map constructed there is an e.d. resolution, but not necessarily the greatest one; a similar construction for the compact case which leads to the, unique in that case, e.d. resolution, was given by Hager [5].

REFERENCES

[1] A. Błaszczyk, *On locally H-closed spaces and the Fomin H-closed extension*, Colloquium Mathematicum 25 (1972), p. 241-253.

[2] — *On irreducible maps and extremally disconnected spaces*, Prace Matematyczne, Uniwersytet Śląski, Katowice (to appear).

[3] P. Freyd, *Abelian categories*, New York, Evanston and London 1964.

[4] A. M. Gleason, *Projective topological spaces*, Illinois Journal of Mathematics 2 (1958), p. 482-489.

[5] A. W. Hager, *The projective resolution of a compact space*, Proceedings of the American Mathematical Society 28 (1971), p. 262-266.

[6] H. Herrlich and G. E. Strecker, *H-closed spaces and reflective subcategories*, Mathematische Annalen 177 (1968), p. 302-309.

[7] С. Илиадис, *Абсолюты хаусдорфовых пространств*, Доклады Академии Наук СССР 149 (1963), p. 22-25.

[8] J. Mioduszewski, *On a method which leads to extremally disconnected covers*, Proceedings of the Third Symposium on General Topology and its Relations to Modern Analysis and Algebra, Prague 1971, p. 309-311.

[9] — and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Mathematicae 66 (1969).

[10] M. H. Stone, *Algebraic characterizations of special Boolean rings*, Fundamenta Mathematicae 29 (1937), p. 223-302.

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