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SOME REMARKS ON THE RELATION OF CLASSICAL SET-VALUED MAPPINGS TO THE BAIRE CLASSIFICATION

BY

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Several applications of the notions considered in this paper are to be found in the joint paper by Marczewski and myself published in Fundamenta Mathematicae 18, as well as in many other papers by Marczewski.

1. Introduction. Let us recall some theorems on set-valued B-measurable mappings (see, e.g., [5]).

Let X and Y be metric spaces, and let X be compact.

Let the topology of the hyperspace 2^X (of all closed subsets of X) be defined (as usually) by assuming that the collection of all sets which are either of the form $\{K: K \subset G\}$ or $\{K: K \cap G \neq \emptyset\}$, where G is open, is an open subbase of 2^X.

A (closed-set-valued mapping) $F: Y \to 2^X$ is said to be of *class* a^+ $(0 \leqslant a < \omega_1)$ if

(1) the set $\{y: F(y) \subset G\}$ is of additive class a for open G

or, equivalently, if

(2) the set $\{y: F(y) \cap K \neq \emptyset\}$ is of multiplicative class a for closed K.

F is said to be of *class* a_- if

(3) the set $\{y: F(y) \cap G \neq \emptyset\}$ is of additive class a for open G

or, equivalently, if

(4) the set $\{y: F(y) \subset K\}$ is of multiplicative class a for closed K.

Recall that the open sets form the additive class 0, the F_σ-sets form the additive class 1, etc. The multiplicative classes are: the class of closed sets, of G_δ-sets, $F_{\sigma\delta}$-sets, etc.

Of course, the functions F of class 0^+ are the *upper semi-continuous functions*, and the functions of class 0_- are the *lower semi-continuous functions* in the usual sense.

2. Some fundamental properties of the classes a^+ and a_-. (For the proofs see [5] and [6].)

THEOREM 1. *The mapping $F: Y \to 2^X$ is of class a, i.e., the set $\{y: F(y) \in G\}$ is of additive class a in Y for each G open in 2^X, iff F is simultaneously of classes a^+ and a_-.*

The proof is founded on the fact that the sets

$$(K: K \subset G) \cap (K: K \cap G_1 \neq \emptyset) \cap \ldots \cap (K: K \cap G_n \neq \emptyset)$$

form an open base of the space 2^X (where K is closed, and G and $G_1, \ldots, G_n$ are open).

THEOREM 2. *If F is either of class a^+ or a_-, then F is of class $a+1$.*

If F is of class a^+, the theorem follows easily from the formula $K = G_1 \cap G_2 \cap \ldots$, where G_n is open.

If F is of class a_-, put $G = K_1 \cup K_2 \cup \ldots$, where K_n is closed and $K_n \subset \operatorname{Int}(K_{n+1})$. Then the formula $F(y) \subset G$ implies, for each y (by the compactness of $F(y)$), the existence of n such that $F(y) \subset K_n$. Since $\{y: F(y) \subset K_n\}$ is by (4) of multiplicative class a, it is of additive class $a+1$.

COROLLARY. *If F is semi-continuous (i.e., F is either of class 0^+ or 0_-), then the set of its points of discontinuity is of the first category.*

Hence, if the space Y is complete, then the set of points of continuity of F is dense in Y.

This holds since these properties belong to any mapping of class 1 (see [4], p. 71, and [1]).

THEOREM 3. *If F_0 and F_1 are both a^+ or both a_-, then their union $F = F_0 \cup F_1$ is a^+ or a_-, respectively.*

Hence the union $K \cup L$ is a continuous mapping $2^X \times 2^X \to 2^X$.

More generally: if F_n is a_- for $n = 0, 1, 2, \ldots$, then so is $\overline{F_0 \cup F_1 \cup \ldots}$

THEOREM 4. *If F_0 and F_1 are both of class a^+, then so is $F_0 \cap F_1$.*

Hence the intersection $K \cap L$ is an upper semi-continuous mapping $2^X \times 2^X \to 2^X$.

More generally: if F_n is a^+ for $n = 0, 1, 2, \ldots$, then so is $F_0 \cap F_1 \cap \ldots$

THEOREM 5. *If F_0 is a_- and F_1 is a^+, then $F = \overline{F_0 - F_1}$ is of class a_-.*

Hence $\overline{K - L}$ is a lower semi-continuous mapping $2^X \times 2^X \to 2^X$.

THEOREM 6. *Let $Z = \{y: F(y) = \emptyset\}$. If F is of class a^+, then Z is a set of additive class a. If F is of class a_-, then Z is of multiplicative class a.*

THEOREM 7. *Let $F: Y \to 2^X$ be of class a and let $H: 2^X \to 2^Z$ be of class β^+ (respectively, β_-). Then the composed mapping $L = H \circ F: Y \to 2^Z$ is of class $(a+\beta)^+$ (respectively, $(a+\beta)_-$).*

3. Applications.

1. *The boundary, i.e., the mapping* $K \cap \overline{X-K}\colon 2^X \to 2^X$, *is of class* 1^+.

Proof. By Theorem 5 the mapping $\overline{X-K}$ is of class 0_-, hence of class 1 (by Theorem 2), and it follows from Theorem 4 that the intersection $K \cap \overline{X-K}$ is of class 1^+.

Thus the boundary is of class 2 (cf. also [4], p. 73). On the other hand, it may fail to be of class 1; such is the case where X denotes the Cantor discontinuum, since then the boundary is discontinuous at each $K \neq \emptyset$ (cf. the Corollary to Theorem 2).

2. *The mapping* $\overline{\mathrm{Int}(K)}$ *is of class* 1_-.

Proof. Let G be an open set. We have to show that (see (3)) the set $\{y\colon \overline{\mathrm{Int}(K)} \cap G \neq \emptyset\}$ is an F_σ-set.

Now let $R_1, R_2, \ldots$ be a countable open base of X. Our conclusion follows from the equivalences

$$\begin{aligned}[\overline{\mathrm{Int}(K)} \cap G \neq \emptyset] &\equiv [\mathrm{Int}(K) \cap G \neq \emptyset] \\ &\equiv \exists n\colon [(R_n \subset K) \cap (R_n \cap G \neq \emptyset)],\end{aligned}$$

since the set $\{K\colon (R_n \subset K)\}$ is closed (see [4], p. 50) as well as the set $\{K\colon (R_n \subset K) \cap (R_n \cap G \neq \emptyset)\}$.

Let us add that the mapping $\overline{\mathrm{Int}(K)}$ is not of the first class if X denotes the Cantor discontinuum (it is discontinuous at each $K \neq \emptyset$).

3. *The derivative* $F(K) = K^d$ *is a mapping of class* 1^+.

We prove first the following lemma:

Lemma. *Let* G *be open; then the set* $T = \{K\colon K \cap G$ *is infinite*$\}$ *is* G_δ.

It is sufficient of course to show that the set

$$T_n = \{K\colon K \cap G \text{ has at least } n \text{ points}\}$$

is open.

Now

$$K \in T_n \equiv \exists G_1, \ldots, G_n\colon \forall m \leqslant n\colon (K \cap G_m \neq \emptyset),$$

where $G_1, \ldots, G_n$ are some open disjoint sets contained in G.

Put $Z(H) = \{K\colon K \cap H \neq \emptyset\}$ for any open H. Hence

$$T_n = \bigcup_{G_1, \ldots, G_n} \bigcap_{m \leqslant n} Z(G_m).$$

Since $Z(G_m)$ is open (see [3], p. 162), so is T_n, and thus the Lemma is shown.

Now let Q be a closed set. We have to show that the set $\{K\colon K^d \cap Q \neq \emptyset\}$ is a G_δ-set in 2^X.

Let $G_1, G_2, \ldots$ be a decreasing sequence of open sets such that $\bar{G}_n \subset G_{n+1}$ and $Q = \bar{G}_1 \cap \bar{G}_2 \cap \ldots$ In view of the Lemma, it is sufficient to prove that

$$[K^d \cap Q \neq \emptyset] \equiv [K \cap G_n \text{ is infinite for each } n].$$

1. Let $K^d \cap Q \neq \emptyset$ and let $p \in K^d \cap Q$. Since $Q \subset G_n$, the set $K \cap G_n$ is infinite for each n.

2. Now let $K^d \cap Q = \emptyset$. Then for each $x \in Q$ there is an open $H(x)$ such that $x \in H(x)$ and $K \cap H(x)$ is finite (or void). Since the family $\{H(x)\}$ is an open cover of the compact set Q, there is a finite subcover

$$H = H(x_1) \cup \ldots \cup H(x_n), \qquad Q \subset H.$$

It follows that $K \cap H$ is finite, and since H is an open set containing Q, we have $G_n \subset H$ for sufficiently large n, and hence $K \cap G_n$ is finite.

Thus our theorem is proved. It follows that the derivative is of class 2 (the last statement has been suggested by Banach; see [7], p. 157).

On the other hand, the derivative may fail to be of class 1 (which is seen by taking for X the Cantor discontinuum and applying the Corollary to Theorem 2).

4. *The mapping $F(K) = K^c$, where K^c is the set of condensation points of K, may fail to be B-measurable* (see [4], p. 73).

Such is the case of X being the Cantor discontinuum. In this case, the family of countable closed sets is (by a theorem of Hurewicz [2]) a non-Borel set. Hence, by Theorem 6, K^c is not a B-measurable mapping.

5. *The following operations do not lead out of the B-measurable mappings*:

(1) $\overline{\bigcup_n K_n}$ *(closure of finite or countably infinite union)*,

(2) $\bigcap_n K_n$ *(finite or countably infinite intersection)*,

(3) $\overline{K - L}$ *(closure of the difference)*,

(4) *the composition $F \circ H$ of mappings F and H*,

(5) K^d *(derivative)*.

In other words, the family B_0 of closed-set-valued mappings, containing all continuous maps and generated by the operations (1)-(5), is contained in the family of B-measurable mappings. This follows from the fact that the family of B-measurable mappings is closed under each of the operations (1)-(5) (by Theorems 3-5 and the proposition 3 in Section 3), and B_0 is the smallest family having these properties.

4. Problem. What is the precise Baire class of the n-th derivative $K^{dd\ldots d}$? **(P 1139)**

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