COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

COMPLEMENTATION IN THE LATTICE OF BOREL STRUCTURES

BY

B. ANISZCZYK (KATOWICE)

1. **Summary.** We will prove that if $L_{\mathscr{B}}$ (the lattice of all Borel substructures of a Borel structure $\mathscr{B}$) is complemented, then it is also antiatomic. We will show also that the converse is not true.

2. **Preliminaries.** If $\mathscr{B}$ is a σ-algebra of subsets of a set X we will say that $\mathscr{B}$ is a *Borel structure* on X and the pair $(X, \mathscr{B})$ is a *Borel space.* If $\mathscr{F}$ is any family of subsets of X, $\sigma(\mathscr{F})$ denotes the Borel structure generated by $\mathscr{F}$ (i.e. the least one containing $\mathscr{F}$). For a Borel space $(X, \mathscr{B})$ let $L_{\mathscr{B}}$ denote the lattice of all Borel structures on X contained in $\mathscr{B}$, with the inclusion as an order relation. Thus if $\mathscr{A}, \mathscr{C} \in L_{\mathscr{B}}$ then $\mathscr{A} \wedge \mathscr{C}$ – the infimum of them – is their intersection and the supremum $\mathscr{A} \vee \mathscr{C} = \sigma(\mathscr{A} \cup \mathscr{C})$ (for details see [2], the following definitions are also there). We say that $L_{\mathscr{B}}$ is *complemented* if for any $\mathscr{A} \in L_{\mathscr{B}}$ there exists $\mathscr{C} \in L_{\mathscr{B}}$ such that $\mathscr{A} \wedge \mathscr{C} = \{\emptyset, X\}$ (the least element in $L_{\mathscr{B}}$) and $\mathscr{A} \vee \mathscr{C} = \mathscr{B}$, we call any such $\mathscr{C}$ the *complement* of $\mathscr{A}$ in $L_{\mathscr{B}}$. $\mathscr{A} \in L_{\mathscr{B}}$, $\mathscr{A} \neq \mathscr{B}$, is an *antiatom* if $\mathscr{B}$ is the only element in $L_{\mathscr{B}}$ strictly greater than $\mathscr{A}$. $L_{\mathscr{B}}$ is called *antiatomic* if every non-unit element of $L_{\mathscr{B}}$ is an infimum of antiatoms.

In [2] Proposition 46(ii) says that if $L_{\mathscr{B}}$ is complemented, then there is no substructure of $\mathscr{B}$, which is countably generated and has uncountably many atoms. The question was raised ([2, P 13]) whether the converse is true. The negative answer to this question was given by the author in 1980, but it has not been published yet. We give this example here. Later on K. P. S. Bhaskara-Rao and B. V. Rao ([3]) have constructed another example of such a Borel space. Fortunately it is much worse than the first one – it is not antiatomic, while the ours is.

3. **Complementation implies antiatomicity.** Let $(X, \mathscr{B})$ be a Borel space. The σ-*ideal* relative to $\mathscr{B}$ is a family $\mathscr{Z} \subseteq \mathscr{B}$ such that if $C \subseteq B \in \mathscr{Z}$ and $C \in \mathscr{B}$ then $C \in \mathscr{Z}$ and $\mathscr{Z}$ contains countable unions of its elements.

LEMMA 1 ([2, Proposition 45]). *Let $\mathscr{B}$ be a Borel structure and $\mathscr{Z} \subseteq \mathscr{B}$*

be a σ-ideal relative to $\mathscr{B}$. Let $\mathscr{A} \in L_{\mathscr{B}}$. Then $\sigma(\mathscr{Z}) \vee \mathscr{A} = \{B \triangle A$: $B \in \mathscr{Z}, A \in \mathscr{A}\}$ ($\triangle$ denotes symmetric difference).

PROPOSITION 1. *If $L_{\mathscr{B}}$ is complemented, then it is antiatomic.*

Proof. By Corollary 4 of [1] it suffices to prove that if μ is a 0–1-measure on $\mathscr{A} \in L_{\mathscr{B}}$, then it extends to a 0–1 measure on $\mathscr{B}$. Let $\mathscr{Z} = \{A \in \mathscr{A}$: $\mu(A) = 0\}$, $\mathscr{Z}$ is a σ-ideal relative to $\mathscr{A}$ and $\sigma(\mathscr{Z}) = \mathscr{A}$. We can obviously extend $\mathscr{Z}$ to $\mathscr{Z}_0$, a σ-ideal relative to $\mathscr{B}$. By assumption $\sigma(\mathscr{Z}_0)$ has a complement $\mathscr{C}$ in $L_{\mathscr{B}}$. By Lemma 1 $\mathscr{B} = \sigma(\mathscr{Z}_0) \vee \mathscr{C} = \{A \triangle C$: $A \in \mathscr{Z}_0, C \in \mathscr{C}\}$. Thus for any $B \in \mathscr{B}$ there exist $C(B) \in \mathscr{C}$ and $A(B) \in \mathscr{Z}_0$ such that $B = A(B) \triangle C(B)$. Such $C(B)$ is unique because if $B = A_1(B) \triangle C_1(B) = A_2(B) \triangle C_2(B)$, then $A_1(B) \triangle A_2(B) = C_1(B) \triangle C_2(B) \in \mathscr{C} \cap \mathscr{Z}_0$, that is $C_1(B) = C_2(B)$. Let ν be any 0–1 measure on $\mathscr{C}$ (for instance concentrated at a point) and define $\tilde{\mu}(B) = \nu(C(B))$. $\tilde{\mu}$ takes only 0 and 1 as its values. If $\{B_i$: $i = 1, 2, \ldots\}$ are pairwise disjoint, then the corresponding $\{C(B_i)$: $i = 1, 2, \ldots\}$ are also pairwise disjoint, whence

$$\tilde{\mu}\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} C(B_i)\Big) = \sum_{i=1}^{\infty} \nu(C(B_i)) = \sum_{i=1}^{\infty} \tilde{\mu}(B_i).$$

This shows that $\tilde{\mu}$ is a 0–1 measure. For $A \in \mathscr{Z}$, $C(A) = \emptyset$. Thus $\tilde{\mu}$ extends μ. The proof is complete.

This proof is very similar to that of Proposition 39 in [2] and it is not incidental.

4. Complementation and σ-homomorphisms. If $\mathscr{A}$ and $\mathscr{B}$ are Borel structures (not necessarily on the same set) then a function h: $\mathscr{A} \to \mathscr{B}$ is called *σ-homomorphism* if it preserves countable unions and complementations. If such h is one-to-one and onto we call it *isomorphism.*

LEMMA 2. *If h: $\mathscr{A} \to \mathscr{B}$ is a σ-homomorphism, then for any family $\mathscr{F} \subseteq \mathscr{A}$ we have $\sigma(h(\mathscr{F})) = h(\sigma(\mathscr{F}))$.*

The routine proof of this lemma is left to the reader.

The term "complemented" is in some sense justified by the following

LEMMA 3. *If $L_{\mathscr{A}}$ is complemented and h: $\mathscr{A} \to \mathscr{B}$ is a σ-homomorphism, then there is $\mathscr{C} \in L_{\mathscr{A}}$ such that h restricted to $\mathscr{C}$ is an isomorphism of $\mathscr{C}$ and $h(\mathscr{A})$ the image of $\mathscr{A}$.*

Proof. Let $\mathscr{Z}$ denote the σ-ideal relative to $\mathscr{A} - h^{-1}(\emptyset)$ – and $\mathscr{D} = \sigma(\mathscr{Z})$. Because $L_{\mathscr{A}}$ is complemented there exists $\mathscr{C} \in L_{\mathscr{A}}$, a complement of $\mathscr{D}$. It suffices to check that the restriction of h to $\mathscr{C}$ is one-to-one and onto $h(\mathscr{A})$. If $h(C_1) = h(C_2)$, $C_1, C_2 \in \mathscr{C}$, then $h(C_1 \triangle C_2) = h(C_1) \triangle h(C_2) = \emptyset$. This implies $C_1 \triangle C_2 \in \mathscr{Z} \cap \mathscr{C}$, thus $C_1 = C_2$. If we put in Lemma 2 $\mathscr{F} = \mathscr{C} \cup \mathscr{D}$, we obtain $h(\mathscr{C}) = h(\mathscr{A})$.

In the proofs of Proposition 1 and Lemma 3 we have used the existence of complements in $L_{\mathscr{B}}$ only for special Borel structures – those of the form

$\sigma(\mathscr{X})$ for some σ-ideal relative to $\mathscr{B}$. This suggests to introduce the weaker notion of complementation for $L_{\mathscr{B}}$. Namely, we will say that $L_{\mathscr{B}}$ *complements ideals* if for any σ-ideal $\mathscr{X}$ relative to $\mathscr{B}$, $\sigma(\mathscr{X})$ has a complement in $L_{\mathscr{B}}$. Lemma 3 and Proposition 1 remain true if we weaken assumptions in this direction. In fact the remaining part of this paper allows it, too. We do not know whether these notions are realy different.

5. Example. If $\varkappa$ denotes a cardinal number we say that a Borel structure $\mathscr{B}$ satisfies $\varkappa$*-chain condition* (abbreviation $\varkappa$-c.c.) if any subfamily of $\mathscr{B}$ of pairwise disjoint nonempty sets is of cardinality not greater than $\varkappa$.

LEMMA 4. *If $\mathscr{B}$ satisfies $\varkappa$-c.c. and $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{A}$ does it also. Isomorphism preserves $\varkappa$-c.c.*

Recall the example given in [1]. If I is any index set let $X(I)$ be the set of points in $\{0, 1\}^I$ all but finitely many coordinates of which are zeros. For $J \subseteq I$ let p_J denote the canonical projection of $\{0, 1\}^I$ onto $\{0, 1\}^J$. Let $\mathscr{B}$ denote the Borel structure on $\{0, 1\}^I$ generated by the sets of the form $p_i^{-1}(\{0\})$ for all $i \in I$. The trace of $\mathscr{B}$ on $X(I)$, say $\mathscr{B}(I)$, is the structure on $X(I)$ which is antiatomic (see [1]) and as we will show for large I is not complemented.

CLAIM 1. *For any set I, $\mathscr{B}(I)$ satisfies ω_1-c.c.*

Proof. Let $\mathscr{F} \subseteq \mathscr{B}(I)$ be a family of nonempty pairwise disjoint sets. With any set $B \in \mathscr{F}$ we can associate some countable set $J(B) \subseteq I$ and some $A(B) \subseteq X(J(B))$ such that $B = X(I) \cap p_{J(B)}^{-1}(A(B))$. Without lost of generality we can assume that $A(B) = \{x_B\}$ for some $x_B \in X(J(B))$. With these assumptions for two different $B, C \in \mathscr{F}$ there is an index $i \in J(B) \cap J(C)$ such that either $p_i(x_B) = 0$ and $p_i(x_C) = 1$ or conversely $p_i(x_B) = 1$ and $p_i(x_C) = 0$. Inductively for any ordinal α we will define a set $J_\alpha \subseteq I$, a family $\mathscr{F}(\alpha)$ and for any finite set $A \subseteq J_\alpha$ a subfamily $\mathscr{F}(\alpha, A)$ of $\mathscr{F}$ (perhaps empty).

Put $J_0 = \emptyset$, $\mathscr{F}(0) = \mathscr{F}$, $\mathscr{F}(0, \emptyset) = \mathscr{F}$. If we have just defined, for some α, J_α, $\mathscr{F}(\alpha)$, $\mathscr{F}(\alpha, A)$ for any finite $A \subseteq J_\alpha$, then in any nonempty family $\mathscr{F}(\alpha, A)$ choose one set $B(\alpha, A)$. Put $J_{\alpha+1} = \bigcup \{J(B(\alpha, A)): \mathscr{F}(\alpha, A) \neq \emptyset\} \cup J_\alpha$,

$$\mathscr{F}(\alpha+1) = \mathscr{F}(\alpha) \setminus \{B(\alpha, A): \mathscr{F}(\alpha, A) \neq \emptyset\},$$

and for any finite set $A \subseteq J_{\alpha+1}$

$$\mathscr{F}(\alpha+1, A) = \{B \in \mathscr{F}(\alpha+1): A = \{i \in J_{\alpha+1}: p_i(B) = 1\}\}.$$

If β is a limit ordinal and we have defined sets J_α and families $\mathscr{F}(\alpha)$ for any $\alpha < \beta$, then put $J_\beta = \bigcup \{J_\alpha: \alpha < \beta\}$ and $\mathscr{F}(\beta) = \bigcap \{\mathscr{F}(\alpha): \alpha < \beta\}$. For any finite $A \subseteq J_\beta$, $\mathscr{F}(\beta, A)$ is defined as for non-limit ordinals.

The above construction assures:

(i) For any $\alpha < \omega_1$, J_α is countable.

(ii) For any $\alpha < \omega_1$, $\mathscr{F} \setminus \mathscr{F}(\alpha)$ is countable.

(iii) If $\alpha < \beta$, then $\mathscr{F}(\beta, \tilde{A}) \subseteq \mathscr{F}(\alpha, A)$ (if $A = \tilde{A} \cap J_\alpha$) or $\mathscr{F}(\beta, \tilde{A}) \cap \mathscr{F}(\alpha, A) = \emptyset$ (if $A \neq \tilde{A} \cap J_\alpha$).

(iv) If $\alpha < \beta$, $\alpha \neq \beta$ and $B \in \mathscr{F}(\beta, \tilde{A}) \subseteq \mathscr{F}(\alpha, A)$ then $J(B) \cap (J_\beta \setminus J_\alpha) \neq \emptyset$. (To see this observe that $B \neq B(\alpha, A)$ and hence there is $i_0 \in I$ such that $p_{i_0}(B) = 1$ and $p_{i_0}(B(\alpha, A)) = 0$ or conversely. But we have $A = \{i \in J_\alpha \colon p_i(B) = 1\} = \{i \in J_\alpha \colon p_i(B(\alpha, A)) = 1\}$, hence $i_0 \notin J_\alpha$. Obviously $i_0 \in J(B)$ and $i_0 \in J(B(\alpha, A)) \subseteq J_{\alpha+1} \subseteq J_\beta$, so $i_0 \in J(B) \cap J_\beta \setminus J_\alpha$.)

By (ii) $\operatorname{card}(\mathscr{F} \setminus \mathscr{F}(\omega_1)) = \operatorname{card}(\bigcup \{\mathscr{F} \setminus \mathscr{F}(\alpha) \colon \alpha < \omega_1\}) \leqslant \omega_1$ and by (iv) if $B \in \mathscr{F}(\omega_1)$, then $J(B)$ should be uncountable which is impossible, so $\mathscr{F}(\omega_1) = \emptyset$. We have obtained $\operatorname{card} \mathscr{F} \leqslant \omega_1$. The proof is finished.

CLAIM 2. *If* card $I > \omega_1$, *then* $\mathscr{B}(I)$ *is not complemented.*

Proof. Let $Y \subseteq X(I)$ be the set consisting of points all but one coordinates of which are zeros. For any $B \in \mathscr{B}(I)$ let $h(B) = B \cap Y$. It is clear that h is a σ-homomorphism of $\mathscr{B}(I)$ onto the structure $\mathscr{A}$ of subsets countable or co-countable in Y. If $L_{\mathscr{B}(I)}$ is complemented, then by Lemma 3 there is $\mathscr{C} \in L_{\mathscr{B}(I)}$ such that h restricted to $\mathscr{C}$ is an isomorphism of $\mathscr{C}$ and $\mathscr{A}$. By Lemma 4, $\mathscr{C}$ satisfies ω_1-c.c., and then $\mathscr{A}$ also. But obviously one-point subsets of Y form a family of nonempty pairwise disjoint elements in $\mathscr{A}$, and by the assumption its cardinality is greater than ω_1. This contradiction proves the claim.

Remark. For I of cardinality not greater than ω_1, $\mathscr{B}(I)$ complements ideals. The author strongly believes that in fact it is complemented.

REFERENCES

[1] B. Aniszczyk, *Antiatomicity of the lattice of Borel structures and extension of two-valued measures*, Colloquium Mathematicum (to appear).

[2] K. P. S. Bhaskara-Rao and B. V. Rao, *Borel spaces*, Dissertationes Mathematicae 190, 1981.

[3] —, —, *On extension of measures*, Colloquium Mathematicum (to appear).

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
KATOWICE, POLAND

Reçu par la Rédaction le 28. 02. 1984