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ON SOME PROPERTIES OF A RIEMANNIAN SPACE V_4 WITH CONSTANT SCALAR CURVATURE

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1. Introduction. Let V_n be an n-dimensional Riemannian space of class C^4 with a non-singular fundamental quadratic differential form $ds^2 = g_{ij}(x)dx^i dx^j$.

Denote the curvature tensor, Ricci tensor and scalar curvature of the space by R_{hijk}, R_{ij} and $\varkappa = R_{ij}\cdot g^{ij}/n(n-1)$, respectively.

A tensor $a_{i_1\ldots i_p j_1\ldots j_p l}$, where $2 \leqslant p \leqslant n$, is called a *$1/p\times p$ tensor* (cf. [5]) if

$$a_{i_1\ldots i_p j_1\ldots j_p l} = a_{j_1\ldots j_p i_1\ldots i_p l} \quad \text{and} \quad a_{i_1\ldots i_p [j_1\ldots j_p] l} = a_{i_1\ldots i_p j_1\ldots j_p l}.$$

If $n \geqslant 4$ and $2 \leqslant p \leqslant n-2$, the tensor

$$(1)\qquad {}^0a_{i_1\ldots i_p j_1\ldots j_p l} = \frac{\varrho^2}{(p!)^2} I^{a_1\ldots a_{n-p}}{}_{i_1\ldots i_p} a_{a_1\ldots a_{n-p}\beta_1\ldots\beta_{n-p} l} I^{\beta_1\ldots\beta_{n-p}}{}_{j_1\ldots j_p},$$

where $I_{i_1\ldots i_n}$ is a unit n-vector and $\varrho = (-1)^{p(n-p)/2}$, is called the *dual of the tensor* $a_{i_1\ldots i_{n-p}j_1\ldots j_{n-p}l}$ (cf. [5]).

If $n = 2p$, a $1/p\times p$ tensor $a_{i_1\ldots i_p j_1\ldots j_p l}$ which satisfies the identities

$$\Theta a_{i_1\ldots i_p j_1\ldots j_p l} = {}^0a_{i_1\ldots i_p j_1\ldots j_p l},$$

where $\Theta = \pm 1$, is called a *self-dual $1/p\times p$ tensor*. Moreover, if $\Theta = 1$, the tensor $a_{i_1\ldots i_p j_1\ldots j_p l}$ is called a *self-dual $1/p\times p$ tensor of the first kind*, and if $\Theta = -1$ — *of the second kind* (cf. [5]).

2. A necessary and sufficient condition for a Riemannian space V_4 to have constant scalar curvature. Consider a 4-dimensional Riemannian space of class C^4 with a non-singular metric tensor g_{ij}.

Let B_{hijk} be a tensor field of class C^1 of the form

$$(2)\qquad B_{hijk} \overset{\mathrm{df}}{=} a(R_{hk}g_{ij} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ik}g_{hj}) + bR(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

where $a \neq -2b$ are arbitrary constants.

THEOREM 1. *A Riemannian space V_4 has constant scalar curvature if and only if the covariant derivative of tensor field B_{hijk} is a self-dual $1/2\times 2$ tensor field of the second kind.*

Proof. From (1) and (2) as well as from the formula (cf. [2] and [4])

$$I^{i_1\ldots i_m j_{m+1}\ldots j_n} I_{i_1\ldots i_m k_{m+1}\ldots k_n} = m!(n-m)!A^{[j_{m+1}\ldots j_n]}_{k_{m+1}\ldots k_n}$$

it follows that

$$\begin{aligned}{}^0B_{hijk;\varrho} &= \frac{a}{4}(I_{hi\alpha\beta}I_{jk\gamma\delta}R^{\alpha\delta}{}_{;\varrho}g^{\beta\gamma} - I_{hi\alpha\beta}I_{jk\gamma\delta}R^{\alpha\gamma}{}_{;\varrho}g^{\beta\delta} + \\ &\quad + I_{hi\alpha\beta}I_{jk\gamma\delta}R^{\beta\gamma}{}_{;\varrho}g^{\alpha\delta} - I_{hi\alpha\beta}I_{jk\gamma\delta}R^{\beta\delta}{}_{;\varrho}g^{\alpha\gamma}) + \\ &\quad + \frac{bR;\varrho}{4}(I_{hi\alpha\beta}I_{jk\gamma\delta}g^{\alpha\delta}g^{\beta\gamma} - I_{hi\alpha\beta}I_{jk\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}) \\ &= -a(R_{;\varrho}g_{hj}g_{ik} + R_{hk;\varrho}g_{ij} + R_{ij;\varrho}g_{hk} - R_{ik;\varrho}g_{hj} - \\ &\quad - R_{hj;\varrho}g_{ik} - R_{;\varrho}g_{hk}g_{ij}) - bR_{;\varrho}(g_{hj}g_{ik} - g_{hk}g_{ij}).\end{aligned}$$

Hence

$$\begin{aligned}(3)\qquad {}^0B_{hijk;\varrho} &= -a(R_{hk;\varrho}g_{ij} - R_{hj;\varrho}g_{ik} + R_{ij;\varrho}g_{hk} - \\ &\quad - R_{ik;\varrho}g_{hj}) + (a+b)R_{;\varrho}(g_{hk}g_{ij} - g_{hj}g_{ik}),\end{aligned}$$

where semicolon denotes the covariant derivative.

Now, if the covariant derivative of tensor field B_{hijk} is a self-dual $1/2\times 2$ tensor field of the second kind, i.e., if $B_{hijk;a} = -{}^0B_{hijk;a}$, then we infer from (2) and (3) that

$$R_{;a}(g_{hk}g_{ij} - g_{hj}g_{ik}) = 0.$$

Multiplying the last identity by $g^{hk}g^{ij}$ and summing for the indices i, j, h, k, we obtain $R_{;a} = 0$ or, equivalently, $R = \text{const}$.

Conversely, if $R = \text{const}$, then from (2) and (3) it follows that

$$B_{hijk;a} = a(R_{hk;a}g_{ij} - R_{hj;a}g_{ik} + R_{ij;a}g_{hk} - R_{ik;a}g_{hj})$$

and

$${}^0B_{hijk;a} = -a(R_{hk;a}g_{ij} - R_{hj;a}g_{ik} + R_{ij;a}g_{hk} - R_{ik;a}g_{hj}),$$

hence

$$B_{hijk;a} = -{}^0B_{hijk;a}.$$

Thus $B_{hijk;a}$ is a self-dual $1/2\times 2$ tensor field of the second kind.

3. Some properties of a conformal-symmetric space V_4. As is known, the *conformal curvature tensor* C_{hijk} of a V_n is defined by the formula

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ih}) + \\ + \frac{1}{(n-1)(n-2)}R(g_{ij}g_{hk} - g_{ik}g_{hj}).$$

Chaki and Gupta [1] have called a Riemannian space V_n *conformal-symmetric* if the conformal curvature tensor of this space satisfies identities $C_{hijk;a} = 0$.

Hence, if V_n is a conformal-symmetric space, then there is

$$(4) \qquad R_{hijk;a} = \frac{1}{n-2}(R_{hk;a}g_{ij} - R_{hj;a}g_{ik} + R_{ij;a}g_{hk} - R_{ik;a}g_{hj}) + \\ + \frac{1}{(n-1)(n-2)} R_{;a}(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

By virtue of theorem 1 and formula (4) we have

THEOREM 2. *A conformal-symmetric space V_4 has constant scalar curvature if and only if the covariant derivative of the curvature tensor of this space is a self-dual $1/2 \times 2$ tensor of the second kind.*

If a Riemannian space V_n is a conformal to flat space, i.e., $C_{hijk} = 0$, then V_n is a conformal-symmetric space and, consequently, the curvature tensor of this space satisfies identities (4). In particular, we have

THEOREM 3. *A conformal to flat space V_4 has constant scalar curvature if and only if the convariat derivative of the curvature tensor of this space is a self-dual $1/2 \times 2$ tensor of the second kind.*

Let f^{ij} be an arbitrary self-dual bi-vector field. Then, by the definition (cf. [3]), there is

$$f^{ij} = \Theta \cdot \tfrac{1}{2} I^{ij}_{\alpha\beta} f^{\alpha\beta},$$

where $\Theta = \pm 1$.

It is easy to show (cf. [5]) that for an arbitrary pair of self-dual bi-vector fields of the same kind hold the identities

$$(5) \qquad f^{i(j}h_i^{\ k)} = \tfrac{1}{4} g^{jk} f^{\alpha\beta} h_{\alpha\beta}.$$

If a Riemannian space V_4 is a conformal-symmetric space, then from (4) and (5) it follows that

$$R_{hijk;l} f^{hi} h^{jk} = -\tfrac{1}{6} R_{;l} f^{\alpha\beta} h_{\alpha\beta}$$

or, equivalently,

$$(6) \qquad (R_{hijk;l} + \tfrac{1}{6} R_{;l} g_{[h[j} g_{i]k]}) f^{hi} h^{jk} = 0,$$

where

$$g_{[h[j}g_{i]k]} \stackrel{\mathrm{df}}{=} \tfrac{1}{2}(g_{h[j}g_{|i|k]} - g_{i[j}g_{|h|k]}).$$

Now, if we put

$$(7) \qquad F_{hijk} \stackrel{\mathrm{df}}{=} R_{hijk} + \tfrac{1}{6} R g_{[h[j} g_{i]k]},$$

then, by (6) and (7), we get

$$F_{hijk;l} f^{hi} h^{jk} = 0.$$

This identity implies, as is easy to show (cf. [5]), that

$$F_{hijk;l} = -{}^{0}F_{hijk;l}.$$

Therefore, we have

COROLLARY 1. *If a Riemannian space V_4 is a conformal symmetric space, then the covariant derivative of tensor F_{hijk} defined by formula* (7) *is a self-dual* $1/2 \times 2$ *tensor of the second kind.*

In particular,

COROLLARY 2. *If a Riemannian space V_4 is a conformal to flat space, then the covariant derivative of tensor F_{hijk} defined by formula* (7) *is a self-dual* $1/2 \times 2$ *tensor of the second kind.*

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