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CONDITIONS IMPLYING THE SUMMABILITY OF APPROXIMATE DERIVATIVES

BY

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1. Introduction. Let F: $[0,1] \to R$ be approximately differentiable with finite approximate derivative F'_{ap}. If $DF = \{x\colon F'(x) \text{ exists}\}$ and ΔF denotes the interior of DF, then it is known [2] that ΔF is a dense open subset of [0, 1]. Two recent papers [5] and [3] have investigated the behavior of F'_{ap} over DF and ΔF. The more general result is contained in [5], where it is shown that if $[0, 1]\setminus\Delta F \neq \emptyset$ (i.e., F is not everywhere differentiable in the ordinary sense) and M is any fixed positive integer, then there is a component of ΔF on which F' takes on both M and $-M$. Thus F'_{ap} takes on "much" of its variation over ΔF. From this observation one might expect that F'_{ap} cannot be "well-behaved" over ΔF without being also "well-behaved" over $[0,1]$. For example, if F'_{ap} is unisigned or bounded over ΔF, then $\Delta F = [0,1]$. However, in this paper* it is shown that this is not the case with regard to the summability (Lebesgue integrability) of F'_{ap}. Examples will be given which show that, even under additional restrictions on F, the summability of F'_{ap} over ΔF does not imply its summability over $[0,1]$.

In the positive direction, in Section 3, a theorem is proved to which we have a corollary that F'_{ap} is summable over $[0,1]$ if and only if it is summable over DF. In addition, it is shown that for this type of problem the appropriate set on which to study the behavior of F'_{ap} differs from DF and ΔF. The natural set is $\Delta^* F$ which is the union of all open intervals $I = (a, b)$ satisfying:

(1) F is continuous for all x in (a, b).

(2) F is differentiable for almost all x in (a, b).

It is established that F'_{ap} is summable over $[0,1]$ if and only if it is summable over $\Delta^* F$.

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2. Summability of F'_{ap} over ΔF. It is assumed that the reader has some familiarity with the concepts and notation contained in [7]. The notation $W(F, I)$ will be used to denote the total variation of the function F over the interval I. Further in this section we will make reference to the theorem of Neugebauer [1] that F'_{ap} is summable over $[0, 1]$ if and only if F is absolutely continuous. The theorems in Section 3 improve this result but the proofs are not dependent on it.

The examples mentioned in Section 1 require the following lemma:

LEMMA. *Given an interval $[a, b]$, a nowhere dense, perfect subset E of (a, b), and $M > 0$, there exists a function $F(x)$ on $[a, b]$ such that*

(i) *$F'_{\text{ap}}(x)$ exists on $[a, b]$;*

(ii) *$E = \text{cl}\{x \mid F'(x)$ does not exist$\} = [a, b]\setminus \Delta F$;*

(iii) *$M(b-a) \leqslant W(F, [a, b]) \leqslant 3M(b-a)$, and $W(F, \Delta F) \leqslant 3M|\Delta F|$.*

Proof. Let a' and b' denote the lower and upper bounds of E, respectively, and let $\{C_n = [a_n, b_n]\}$ denote the sequence of intervals contiguous to E in $[a', b']$.

In each interval C_n, choose a sequence of closed intervals $J_{n,k}$, $k = 1, 2, \ldots$, such that

(1) $\quad J_{n,k} \to a_n$ as $k \to \infty$;

(2) $\quad$ each $x \in E$ is a point of dispersion of $\bigcup_{n,k} J_{n,k}$.

On E set $F(x) = Mx - Ma$, and on $[a_n, b_n]$ define $F(x)$ as follows:

Let $L_1(x)$ and $L_2(x)$ denote the lines through the point $(a_n, F(a_n))$ with slopes M and $M+1$, respectively. Let $p_{n,1}$ be the abscissa of the intersection of $L_2(x)$ with the line $y = Mb_n - Ma$. Choose $\hat{J}_{n,1}$ to be the first interval of the sequence $\{J_{n,k}\}$ which lies to the left of $p_{n,1}$.

After choosing $\hat{J}_{n,q}$ for $q = 1, \ldots, m-1$, let r denote the left-hand endpoint of $\hat{J}_{n,m-1}$. Let $p_{n,m}$ be the abscissa of the point of intersection of the lines $L_2(x)$ and $y = Mr - Ma$. Choose $\hat{J}_{m,n}$ to be the first interval of $\{J_{n,k}\}$ which lies to the left of $p_{n,m}$. We note that (1) and (2) imply that

(3) $\quad \hat{J}_{n,m} \to a_n$;

(4) $\quad$ each $x \in E$ is a point of dispersion of $\bigcup_{n,m} \hat{J}_{n,m}$.

Define $F(x)$ on $J_{n,m}$ as follows:

(5) $\quad$ The endpoints of the graph lie on $L_1(x)$.

(6) $\quad$ Some point of the graph lies on $L_2(x)$.

(7) $\quad$ For each $x \in J_{n,m}$, $L_1(x) \leqslant F(x) \leqslant L_2(x)$.

(8) The derivative $F'(x)$ exists on $\hat{J}_{n,m}$ and the one-sided derivatives exist and equal M at the endpoints.

(9) $F(x)$ assumes no value more than 3 times on $\hat{J}_{n,m}$.

(10) As on E, we set $F(x) = Mx - Ma$ at the remaining points of (a_n, b_n).

On $[a, a')$ and $(b', b]$, let $F(x)$ be a strictly increasing, differentiable function such that

(11) $F(a), F(a'), F(b')$, and $F(b)$ lie on $y = Mx - Ma$, $F'_+(a) = F'_-(b) = 0$ and $F'_-(a') = F'_+(b') = M$.

Condition (i) of the Lemma follows from (4), (8), (10) and (11). We infer from (3) and (6) that $F'(a_n)$ does not exist, and since $\{a_n\}_{n=1}^{\infty}$ is dense in E, condition (ii) is established. The inequality $M(b-a) \leqslant W(F, [a, b])$ follows easily from (11). By (9) and the manner in which the intervals $\hat{J}_{n,m}$ were selected, it follows from (10) and (11) that $F(x)$ assumes no value more than 3 times on $[a, b]$. Thus

$$W(F, [a, b]) \leqslant 3M(b-a) \quad \text{and} \quad W(F, \Delta F) \leqslant 3M|\Delta F|$$

and the proof of the Lemma is complete.

Definition. Given an interval $I = [a, c]$, $b = (a+c)/2$, E a nowhere dense, perfect subset of (a, b), and $M > 0$, we say that a function F on I is of *type* (I, E, M) if F is defined on $[a, b]$ as in the Lemma and on $[b, c]$ by reflecting its graph in the line $x = b$.

From (11) one easily sees that if F is of type (I, E, M), then

$$F(a) = F(c) = F'(a) = F'(b) = F'(c) = 0 \quad \text{and} \quad F(b) = M(b-a).$$

It follows from the Lemma that $F'_{\mathrm{ap}}(x)$ exists on I, that $\mathrm{cl}\{x | F'(x)$ does not exist$\}$ is E along with its reflection $\hat{E}$ in $x = b$, and

$$M|I| \leqslant W(F, I) \leqslant 3M|I|, \quad W(F, \Delta F) \leqslant 3M|\Delta F|.$$

Since approximately differentiable functions satisfy condition (N) (see Section 3) and since F is a continuous function of bounded variation on I, F is absolutely continuous on I.

Example 1. There exists an unbounded, approximately differentiable function F such that $F'_{\mathrm{ap}}(x)$ is summable over ΔF.

Construction. Let $\{I_n = [a_n, c_n]\}_{n=1}^{\infty}$ be a sequence of intervals such that $I_n \to 0$ as $n \to \infty$ and such that 0 is a point of dispersion of $\bigcup_n I_n$. Set

$$b_n = (a_n + c_n)/2.$$

Let $M_n = n/|I_n|$ and let E_n be a nowhere dense, perfect subset of (a_n, b_n) such that if $Q_n = I_n \setminus (E_n \cup \hat{E}_n)$, then $|Q_n| \leqslant 1/3M_n \cdot 2^n$.

On I_n let $F(x)$ be a function of type (I_n, E_n, M_n) and set

$$F(x) = 0 \quad \text{for } x \in Q_0 = [0,1] \setminus \bigcup_n I_n.$$

Since $F(b_n) = M_n(b_n - a_n) = n/2$, F is unbounded on $[0,1]$.

The approximate differentiability of F on $(0,1]$ follows from the Lemma. At $x = 0$ it follows from the fact that 0 is a point of dispersion of $\bigcup I_n$.

On any component of Q_0, the variation of F equals 0. On Q_n,

$$W(F, Q_n) \leqslant 3M_n|Q_n| \leqslant 1/2^n.$$

Since

$$\Delta F = \bigcup_{n=0}^{\infty} Q_n,$$

$F'_{\mathrm{ap}}(x)$ is summable over ΔF.

We note that since $F(x)$ is unbounded, F is not absolutely continuous on $[0,1]$, and thus $F'_{\mathrm{ap}}(x)$ is not summable over $[0,1]$.

Example 2. There exists an approximately differentiable ACG_* function F such that F is summable over ΔF but not over $[0,1]$. Moreover, there is an everywhere differentiable function G such that $F(x) = G(x)$ on $[0,1] \setminus \Delta F$.

Construction. Let $I_n = [a_n, c_n] = [n^{-1/2}, n^{-1/2} + 2^{-n}]$ and set $b_n = (a_n + c_n)/2$. It is easy to verify that 0 is a point of dispersion of $\bigcup_n I_n$.

Let $M_n = 2^n/n$ and let E_n be a perfect subset of (a_n, b_n) such that if $Q_n = I_n \setminus (E_n \cup \hat{E}_n)$, then $|Q_n| \leqslant 1/3M_n \cdot 2^n$.

On I_n let F be a function of type (I_n, E_n, M_n) and set

$$F(x) = 0 \quad \text{on } [0,1] \setminus \bigcup_n I_n.$$

The approximate differentiability of F on $[0,1]$ and its summability over ΔF follows exactly as in Example 1.

Since $F(b_n) = M_n|I_n| = 2/n$, F is not of bounded variation in $[0,1]$, and thus F'_{ap} is not summable over $[0,1]$. However, F is AC over each interval $[(n+1)^{-1/2}, n^{-1/2}]$ and is continuous at 0, since its graph is bounded by the curve $y = x^2$ and the x-axis.

Let a'_n and b'_n denote the lower and upper bounds, respectively, of E_n in (a_n, b_n). On $[a_n, a'_n]$ and $[b'_n, b_n]$ set $G(x) = F(x)$. On $[a'_n, b'_n]$ put $G(x) = M_n(x - a_n)$ and reflect the graph in the line $x = b_n$. Set

$$G(x) = F(x) = 0 \quad \text{on } [0,1] \setminus \bigcup_n I_n.$$

It follows from the Lemma (especially, (11)) that G is differentiable on $(0, 1]$ and that $F(x) = G(x)$ on $[0, 1] \setminus \Delta F$. We have $G'(0) = 0$, since its graph is also bounded by $y = x^2$ and the x-axis, and Example 2 is complete.

3. Some conditions implying the summability of F'_{ap}. Before proceeding to the first theorem it is necessary to mention some auxiliary properties of approximately differentiable functions. Any function, approximately differentiable on $[0, 1]$, has associated with it a sequence of closed sets E_n, whose union is $[0, 1]$, such that on each E_n the function is absolutely continuous [5]. Therefore, any approximately differentiable function is Baire* 1 and Darboux (see [4]) and also satisfies Lusin's condition (N). (A function F: $[0, 1] \to R$ is *Baire** 1 if every closed set has a portion on which the restriction of F is continuous.)

THEOREM 1. *Let F: $[0, 1] \to R$ be Baire* 1 and Darboux. Let*

$$U(F) = \operatorname{int}\{x: F \textit{ is continuous at } x\}.$$

Suppose that F has property (N) *on $U(F)$. Let*

$$P = \{x: F' \textit{ exists at } x \textit{ and } F'(x) > 0\} \cap U(F).$$

Then F is absolutely continuous if and only if the function F' is summable over P.

Proof. The necessity is obvious.

Sufficiency. Let

$$g(x) = \begin{cases} F'(x) & \text{if } x \in P, \\ 0 & \text{if } x \notin P. \end{cases}$$

Let

$$G(x) = \int_0^x g(t)\,dt.$$

Then $G(x)$ is absolutely continuous and non-decreasing over $[0, 1]$. Let $H(x) = G(x) - F(x)$. This new function is Baire* 1 and Darboux. Further, if

$$U(H) = \operatorname{int}\{x: H \text{ is continuous at } x\},$$

then $U(H) = U(F)$. (It should be noted that $U(H)$ is a dense open subset of $[0, 1]$, since F is Baire* 1.) It is claimed that $H(x)$ is non-decreasing. By Theorem 1 of [4], p. 187, it suffices to show that $H(x)$ is non-decreasing over every component of $U(F)$. Let I be such a component and let $[c, d]$ be a closed subinterval of I. Clearly, the function F satisfies the hypothesis of Theorem 7.7 of [7], p. 287. Therefore, F is absolutely continuous on $[c, d]$. In turn, this means that H is absolutely continuous over $[c, d]$. The function F is differentiable almost everywhere in $[c, d]$.

Further, $G' = g(x)$ for almost all x in $[c, d]$. Let x be any point at which both F' exists and $G' = g$. Then H' exists at x and $H' = g - F'$. Now, if $x \in P$, then $H' = 0$, and if $x \notin P$, then $F' < 0$ and $g = 0$ so that $H' > 0$. Consequently, H' is non-negative at almost all points where it exists and H is non-decreasing over $[c, d]$ (see [7], p. 286, second paragraph). Since $[c, d]$ was an arbitrary subinterval of I, H is non-decreasing on I. This yields that H is non-decreasing and Darboux on $[0, 1]$. The function H is therefore continuous on $[0, 1]$, i.e., $[0, 1] = U(H) = U(F)$. But this means that Theorem 7.7 of [7] can again be applied to F over $[0, 1]$ to establish that F is absolutely continuous on $[0, 1]$.

COROLLARY 1. *Let F: $[0, 1] \to R$ have a finite approximate derivative F'_{ap} at each point of $[0, 1]$. Then F'_{ap} is summable over $[0, 1]$ if and only if F'_{ap} is summable over $DF = \{x\colon F' \text{ exists at } x\}$.*

Proof. The necessity is obvious.

Sufficiency. Clearly, if F'_{ap} is summable over DF, then F'_{ap} is summable over

$$P = \{x\colon F' \text{ exists and } F' > 0\} \cap U(F).$$

Therefore, F satisfies the hypothesis of Theorem 1 and is absolutely continuous. Thus DF has measure 1 and F'_{ap} is summable over $[0, 1]$.

At this point the behavior of F'_{ap} over a larger set than ΔF is considered and shown to be more representative of the behavior of F'_{ap} over $[0, 1]$.

THEOREM 2. *Let F: $[0, 1] \to R$ have a finite approximate derivative F'_{ap} at each point of $[0, 1]$. Let $\Delta^* F$ be the union of all open intervals I such that*

(1) *F is continuous on I;*

(2) *F is differentiable at almost all x in I.*

Then F'_{ap} is summable over $[0, 1]$ if and only if F'_{ap} is summable over $\Delta^* F$.

Proof. The necessity is obvious.

Sufficiency. Clearly, $\Delta^* F$ is a dense open set, since $\Delta F \subset \Delta^* F$. Also, if I is a component of $\Delta^* F$, then F satisfies conditions (1) and (2) on I. In fact, the summability of F'_{ap} over $\Delta^* F$ implies, by Theorem 1, that F is absolutely continuous over the closure of the component I. Therefore, $\Delta^* F$ cannot have, in this case, any abutting intervals. If $\Delta^* F = [0, 1]$, then the proof is completed. Assume instead that $[0, 1] \setminus \Delta^* F = P \neq \emptyset$. Then, from the above, P is perfect. The function F'_{ap} is Baire 1. Thus there is an open interval (a, b) with $(a, b) \cap P \neq \emptyset$, and F'_{ap} is bounded on $(a, b) \cap P$. However, then F'_{ap} is summable over $(a, b) \cap P$ and also over $(a, b) \setminus P$, i.e., F'_{ap} is summable over (a, b). Again by Theorem 1, F is absolutely continuous on $[a, b]$ and F satisfies (1) and (2) on (a, b). So $(a, b) \subset \Delta^* F$, contradicting $(a, b) \cap P \neq \emptyset$.

The next corollary is rather easy but serves a useful purpose. It illustrates that the type of examples given in Section 2 presents, in a sense, the "only possible" such examples. More precisely, any approximately differentiable function, non-absolutely continuous, but having F'_{ap} summable over ΔF must behave on some subinterval like the function constructed in the Lemma.

COROLLARY 2. *Let $F\colon [0,1] \to R$ have a finite approximate derivative F'_{ap} at all points of $[0,1]$. Suppose that F'_{ap} is summable over ΔF. Then either F is absolutely continuous or there is an open interval (a,b) with*

(i) $|(a,b)\setminus \Delta F| > 0$;

(ii) *F is continuous in (a,b);*

(iii) *F is differentiable almost everywhere in (a,b).*

Proof. If $|\Delta^* F\setminus \Delta F| = 0$, then F'_{ap} is summable over $\Delta^* F$, and hence F is absolutely continuous.

If $|\Delta^* F\setminus \Delta F| > 0$, there is a component (a,b) of $\Delta^* F$ for which $|(a,b)\setminus \Delta F| > 0$. This component clearly satisfies (ii) and (iii).

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