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TOPOLOGIES ASSOCIATED TO SOME DECISION STATISTICAL FUNCTIONS

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1. Introduction. Wald proved most of the results of [4] under the hypothesis that the set of decision statistical functions at statistician's disposal is weakly compact in the intrinsic sense. In [3] Le Cam establishes certain theorems of complete class supposing that the class of decision statistical functions at the statistician's disposal is compact in a certain sense and in the same paper he mentions that the compactness of the space H may be replaced by the more general property (W). Kudo retakes the property (W) in [1] giving a precise description of this property and providing some conditions which imply it, partially sketched by Le Cam in [3].

In the present paper we try to give a topological description of these concepts and to establish a tight relation between them and the existence of minimax and Bayes solutions.

Let us consider a statistical decision function problem (Ω, H, r), where Ω is the space of the parameter ω, H the class of decision functions δ to which the choice of a statistician is restricted, and $r(\omega, \delta)$ the risk function imposed on him when δ is chosen and ω is the true value of a parameter. Let us suppose $r(\omega, \delta) \geqslant 0$ for any $\omega \in \Omega$ and $\delta \in H$. Let $\mathscr{H}$ be a class of probability measures given a priori on $(\Omega, \sigma(\Omega))$, where $\sigma(\Omega)$ is a σ-algebra of sets in Ω such that all countable sets belong to $\sigma(\Omega)$. We suppose that $\mathscr{H}$ contains at least the set of probability measures degenerated on Ω, which will be expressed by $\mathscr{H} \supseteq \Omega$. By r we denote also the appropriate extension of the risk function to $\mathscr{H} \times H$. Now we denote by $P = (\mathscr{H}, H, r)$ an extension of the statistical decision problem (Ω, H, r) [4].

In Section 2 a topology $\mathscr{T}_1$ is introduced on H and it is proved that $(H, \mathscr{T}_1)$ is a topological compact space if and only if there is a minimax solution for the problem $P = (\mathscr{H}, H, r)$.

In Section 3 it is shown that a topology $\mathscr{T}_2$ may be built on H so that H is weakly compact in the intrinsic sense if and only if $(H, \mathscr{T}_2)$

is a topological compact space. If $\mu_0 \in \mathscr{H}$ is an a priori distribution on $(\Omega, \sigma(\Omega))$, then a topology $\mathscr{T}_{\mu_0}$ (Section 4) can be associated to H so that $(H, \mathscr{T}_{\mu_0})$ can be a topological compact space if and only if there is a Bayes solution relative to μ_0.

In Section 5 the property $(\tilde{\mathrm{W}})$ is introduced and in Section 6 the property (W) is redefined from [1]. Also the relation between the properties (W) and $(\tilde{\mathrm{W}})$ and the topologies defined in Sections 2-4 is established.

The most important result in Section 6 establishes that, in the case where $\mathscr{H}$ is a countable set, the properties (W), $(\tilde{\mathrm{W}})$ and the compactness in the intrinsic sense are coincident. The problem for the general case remains open.

2. Topology $\mathscr{T}_1$. We say that δ' of H is a *minimax solution* for $P = (\mathscr{H}, H, r)$ if

$$\sup_{\mu\in\mathscr{H}} r(\mu, \delta') = \inf_{\delta\in H}\sup_{\mu\in\mathscr{H}} r(\mu, \delta).$$

Now, we assign to each element $\delta_0 \in H$ a family of neighborhoods $\mathscr{V}(\delta_0, \varepsilon)$ consisting of all elements δ of H such that

$$\sup_{\mu} r(\mu, \delta_0) - \sup_{\mu} r(\mu, \delta) < \varepsilon \quad \text{if } \sup_{\mu} r(\mu, \delta_0) < \infty,$$

$$\sup_{\mu} r(\mu, \delta) > \frac{1}{\varepsilon} \quad \text{if } \sup_{\mu} r(\mu, \delta_0) = \infty,$$

where $\varepsilon > 0$. Such a system of neighborhoods of every $\delta_0 \in H$ defines a topology $\mathscr{T}_1$ in H. Relative to this topology we have

THEOREM 1. *The following statements are equivalent*:

(i) $(H, \mathscr{T}_1)$ *is a compact space.*

(ii) *For any sequence* $(\delta_i)_i$ *of elements of* H, *there exist an element* δ^* *of* H *and a subsequence* $(\delta_{i_j})_j$ *of the sequence* $(\delta_i)_i$ *such that*

$$\underline{\lim}_{j\to\infty} \sup_{\mu} r(\mu, \delta_{i_j}) \geqslant \sup_{\mu} r(\mu, \delta^*). \tag{2.0}$$

(iii) *There exists* δ' *in* H *such that*

$$\sup_{\mu} r(\mu, \delta') = \inf_{\delta}\sup_{\mu} r(\mu, \delta),$$

i.e., there exists a minimax solution for $P = (\mathscr{H}, H, r)$.

Proof. The equivalence of (i) and (ii) is obvious since a sequence $(\delta_i)_i$ of elements of H converges in the topology $\mathscr{T}_1$ to a decision function δ^* of H if and only if

$$\underline{\lim}_{i\to\infty} \sup_{\mu} r(\mu, \delta_i) \geqslant \sup_{\mu} r(\mu, \delta^*).$$

We now prove that (ii) $\Rightarrow$ (iii). Obviously, there exists a sequence $(\delta_i)_i$ of decision functions of H such that

$$\lim_{i\to\infty}\sup_{\mu} r(\mu, \delta_i) = \inf_{\delta}\sup_{\mu} r(\mu, \delta). \tag{2.1}$$

By (ii) there exists a subsequence $(\delta_{i_j})_j$ of $(\delta_i)_i$ satisfying (2.0), so we have

$$\lim_{j\to\infty}\sup_{\mu} r(\mu, \delta_{i_j}) = \inf_{\delta}\sup_{\mu} r(\mu, \delta) \tag{2.2}$$

and, consequently,

$$\inf_{\delta}\sup_{\mu} r(\mu, \delta) \geqslant \sup_{\mu} r(\mu, \delta^*)$$

which is equivalent to relation (iii) for $\delta' = \delta^*$.

We shall now complete the proof of the theorem by showing that (iii) $\Rightarrow$ (ii). Suppose H does not satisfy (ii). Then there exists a sequence $(\delta_i)_i$ of H such that for any subsequence $(\delta_{i_j})_j$ of the sequence $(\delta_i)_i$ there is no δ'' of H such that

$$\varlimsup_{j\to\infty}\sup_{\mu} r(\mu, \delta_{i_j}) \geqslant \sup_{\mu} r(\mu, \delta''). \tag{2.3}$$

It follows from (2.3) that

$$\varlimsup_{j\to\infty}\sup_{\mu} r(\mu, \delta_{i_j}) < \sup_{\mu} r(\mu, \delta) \quad \text{for any } \delta \in H. \tag{2.4}$$

On the other hand, it follows from (iii) that there exists $\delta^* \in H$ such that

$$\sup_{\mu} r(\mu, \delta^*) = \inf_{\delta}\sup_{\mu} r(\mu, \delta). \tag{2.5}$$

By (2.4) for $\delta = \delta^*$ and (2.5) we have

$$\varlimsup_{j\to\infty}\sup_{\mu} r(\mu, \delta_{i_j}) < \sup_{\mu} r(\mu, \delta^*) = \inf_{\delta}\sup_{\mu} r(\mu, \delta) \leqslant \sup_{\mu} r(\mu, \delta) \tag{2.6}$$

for any $\delta \in H$. Hence we obtain

$$\varlimsup_{j\to\infty}\sup_{\mu} r(\mu, \delta_{i_j}) < \sup_{\mu} r(\mu, \delta^*) \leqslant \sup_{\mu} r(\mu, \delta_{i_j}) \quad \text{for any } j \geqslant 1.$$

Now letting $j \to \infty$ we obtain a contradiction with (2.3), and the theorem is proved.

From properties (i) and (ii) of Theorem 1 it follows that a necessary and sufficient condition for the existence of minimax solutions for the decision problem $P = (\mathscr{H}, H, r)$ is that the space H of decision functions at the statistician's disposal is a topological compact space related to the topology $\mathscr{T}_1$ defined above.

3. Topology $\mathscr{T}_2$. We consider a topology on the space H of the decision functions at the statistician's disposal, which generates the weak compactness in the intrinsic sense defined by Wald in [4].

We say that H is *weakly compact in the intrinsic sense relative to* $(\mathscr{H}, H, r)$ if for any sequence $(\delta_i)_i$ of H there is a subsequence $(\delta_{i_j})_j$ and there is $\delta^* \in H$ such that

$$\varliminf_{j\to\infty} r(\mu, \delta_{i_j}) \geqslant r(\mu, \delta^*) \quad \text{for any } \mu \in \mathscr{H}.$$

Now, we build the following topology on H. We assign to each element $\delta_0 \in H$ a family of neighborhoods $\mathscr{V}(\delta_0; \mu_1, \ldots, \mu_k, \varepsilon)$ consisting of all elements δ of H such that

$$r(\mu_i, \delta_0) - r(\mu_i, \delta) < \varepsilon \quad \text{if } r(\mu_i, \delta_0) < \infty,$$

$$r(\mu_i, \delta) > \frac{1}{\varepsilon} \quad \text{if } r(\mu_i, \delta_0) = \infty,$$

where k is an arbitrary positive integer, $\mu_1, \ldots, \mu_k$ a finite subset of $\mathscr{H}$ and $\varepsilon > 0$. It is easy to verify that the family of sets obtained by varying $k, \mu_1, \ldots, \mu_k, \varepsilon$ can be used as an open set basis to define a topology. Let us denote this topology by $\mathscr{T}_2$. It is then clear that a sequence $(\delta_i)_i$ of H converges in the topology $\mathscr{T}_2$ to an element δ' of H if and only if

$$\varliminf_{i\to\infty} r(\mu, \delta_i) \geqslant r(\mu, \delta') \quad \text{for any } \mu \in \mathscr{H}.$$

Thus, relatively to the Wald weak compactness in the intrinsic sense, we have

THEOREM 2. *H is weakly compact in the intrinsic sense relative to $(\mathscr{H}, H, r)$ if and only if $(H, \mathscr{T}_2)$ is a compact space.*

Proof. The proof is immediate if we consider the definition of the convergence in the topology $\mathscr{T}_2$ and the definition of the compactness of a space.

It is easy to prove that the topology $\mathscr{T}_2$ is finer than the topology $\mathscr{T}_1$. Hence, because of Theorems 1 and 2, we obtain

COROLLARY 1. *If $(H, \mathscr{T}_2)$ is a compact topological space, then $(H, \mathscr{T}_1)$ is a compact topological space.*

4. Topology $\mathscr{T}_{\mu_0}$. Let μ_0 be an a priori distribution in $(\Omega, \sigma(\Omega))$ and $\delta_0 \in H$.

We say that δ_0 is a *Bayes solution relative to* μ_0 if

$$r(\mu_0, \delta_0) = \inf_{\delta} r(\mu_0, \delta). \tag{4.1}$$

We now topologize the space H by defining a base of open neighborhoods for any point δ' of H. Consider the family of sets of the form $\mathscr{V}_{\mu_0}(\delta', \varepsilon)$ consisting of all elements δ of H such that

$$r(\mu_0, \delta') - r(\mu_0, \delta) < \varepsilon \quad \text{if } r(\mu_0, \delta') < \infty,$$

$$r(\mu_0, \delta) > \frac{1}{\varepsilon} \quad \text{if } r(\mu_0, \delta') = \infty,$$

where $\varepsilon > 0$. Let us denote by $\mathcal{T}_{\mu_0}$ the topology generated by this family of sets obtained by varying ε. It is clear that a sequence $(\delta_i)_i$ of elements of H converges in the topology $\mathcal{T}_{\mu_0}$ to an element δ^* of H if and only if

$$\varliminf_{i\to\infty} r(\mu_0, \delta_i) \geqslant r(\mu_0, \delta^*).$$

As to the compactness of the space H in the topology $\mathcal{T}_{\mu_0}$ we have

THEOREM 3. *$(H, \mathcal{T}_{\mu_0})$ is a compact topological space if and only if there exists δ_0 of H such that δ_0 is a Bayes solution relative to μ_0.*

Proof. Suppose that $(H, \mathcal{T}_{\mu_0})$ is a compact space. It is clear that there exists a sequence $(\delta_i)_i$ of elements of H such that

$$\lim_{i\to\infty} r(\mu_0, \delta_i) = \inf_{\delta} r(\mu_0, \delta). \tag{4.2}$$

Since $(H, \mathcal{T}_{\mu_0})$ is a compact space, there exist a subsequence $(\delta_{i_j})_j$ of the sequence $(\delta_i)_i$ and a decision function $\delta_0 \in H$ such that

$$\varliminf_{j\to\infty} r(\mu_0, \delta_{i_j}) \geqslant r(\mu_0, \delta_0). \tag{4.3}$$

On the other hand,

$$\varliminf_{j\to\infty} r(\mu_0, \delta_{i_j}) = \lim_{i\to\infty} r(\mu_0, \delta_i). \tag{4.4}$$

It follows from (4.2)-(4.4) that

$$\inf_{\delta} r(\mu_0, \delta) \geqslant r(\mu_0, \delta_0).$$

Obviously, the equality must hold in this relation, and δ_0 is a Bayes solution relative to μ_0.

Conversely, let $(\delta_i)_i$ be a sequence of elements of H. It is sufficient to prove that there exist a subsequence $(\delta_{i_j})_j$ of the sequence $(\delta_i)_i$ and an element $\delta^* \in H$ such that

$$\varliminf_{j\to\infty} r(\mu_0, \delta_{i_j}) \geqslant r(\mu_0, \delta^*). \tag{4.5}$$

But for any subsequence $(\delta_{i_j})_j$ we have

$$\varliminf_{j\to\infty} r(\mu_0, \delta_{i_j}) \geqslant \inf_{\delta} r(\mu_0, \delta). \tag{4.6}$$

Let δ' be a Bayes solution relative to μ_0. Then

$$\inf_{\delta} r(\mu_0, \delta) = r(\mu_0, \delta'). \tag{4.7}$$

The theorem is an immediate consequence of (4.6) and (4.7).

COROLLARY 2. *$(H, \mathcal{T}_\mu)$ is a compact space for any $\mu \in \mathcal{H}$ if and only if for any $\mu \in \mathcal{H}$ there exists δ_μ of H, a Bayes solution relative to μ.*

Since $\mathcal{T}_2$ is finer than $\mathcal{T}_{\mu_0}$, we have

THEOREM 4. *If $(H, \mathcal{T}_2)$ is a compact space, then $(H, \mathcal{T}_\mu)$ is a compact space for any $\mu \in \mathcal{H}$.*

We establish in the sequel that, in general, if $(H, \mathcal{T}_1)$ is a compact topological space, then $(H, \mathcal{T}_\mu)$ need not be a compact topological space for any $\mu \in \mathcal{H}$. For this we give the following

Example 1. Let us consider the decision problem (Ω, H, r), where $\Omega = \{\omega_1, \omega_2, \ldots\}$, $H = \{\delta_1, \delta_2, \ldots\}$ and r is defined by $r(\omega_i, \delta_j) = i/(i+j)$ for $i, j \geqslant 1$. Suppose that $\mathcal{H} = \Omega$.

Since

$$\inf_\delta \sup_\mu r(\mu, \delta) = \sup_\mu r(\mu, \delta_j) = 1 \quad \text{for any } j \geqslant 1,$$

there exists a minimax solution for the decision problem $(\mathcal{H}, H, r)$. According to Theorem 1, $(H, \mathcal{T}_1)$ is a compact space.

On the other hand, we have

$$\inf_\delta r(\mu_i, \delta) = 0, \quad r(\mu_i, \delta_j) \neq 0 \quad \text{for any } i, j \geqslant 1,$$

where μ_i is an a priori distribution which assigns the probability one to the element ω_i of Ω. Thus, according to Theorem 3 for any $\mu \in \mathcal{H}$, $(H, \mathcal{T}_\mu)$ is not a compact space.

So, this example establishes that the converse of Corollary 1 is not generally true.

Now we build a decision problem $(\mathcal{H}, H, r)$ such that, for any $\mu \in \mathcal{H}$, $(H, \mathcal{T}_\mu)$ is a compact space but $(H, \mathcal{T}_1)$ is not a compact space. Thus, according to Corollary 1, $(H, \mathcal{T}_2)$ is not a compact space.

Example 2. Let (Ω, H, r) be a decision problem where $\Omega = \{\omega_1, \omega_2\}$, $H = \{\delta_i\}_{i>0,}$, and r is defined by

$$r(\omega_1, \delta_0) = 0, \quad r(\omega_2, \delta_0) = a_0 > 0,$$

$$r(\omega_1, \delta_1) = a_1 > 0, \quad r(\omega_2, \delta_1) = 0,$$

$$r(\omega_1, \delta_i) = r(\omega_2, \delta_i) = a_i > 0, \quad i \geqslant 2,$$

where $(a_i)_i$ is a sequence of real numbers decreasing to zero. We suppose $\mathcal{H} = \Omega$.

Since

$$\sup_\mu r(\mu, \delta_i) = a_i > 0,\ i \geqslant 0, \quad \inf_\delta \sup_\mu r(\mu, \delta) = 0,$$

there exists a minimax solution for $(\mathscr{H}, H, r)$. By Theorem 1, $(H, \mathscr{T}_1)$ is not a compact space. But it is clear that

$$\inf_{\delta} r(\mu_1, \delta) = r(\mu_1, \delta_0) \quad \text{and} \quad \inf_{\delta} r(\mu_2, \delta) = r(\mu_2, \delta_1),$$

i.e., δ_1 is a Bayes solution relative to μ_1 and δ_2 is a Bayes solution relative to μ_2. Thus, according to Theorem 3, $(H, \mathscr{T}_\mu)$ is a compact space for any $\mu \in \mathscr{H}$.

5. The property $(\tilde{\mathrm{W}})$. Let us consider

$$\mathscr{F} = \{f \mid f\colon \mathscr{H} \to [0, \infty]\}.$$

We build a topology in $\mathscr{F}$ by defining a base of open neighborhoods for any point $f^0 \in \mathscr{F}$. Consider the family of sets of the form $\mathscr{V}(f^0; \mu_1, \ldots, \mu_k, \varepsilon)$ consisting of all elements g of $\mathscr{F}$ such that

$$|f^0(\mu_i) - g(\mu_i)| < \varepsilon \quad \text{if } f^0(\mu_i) < \infty,$$

$$g(\mu_i) > \frac{1}{\varepsilon} \quad \text{if } f^0(\mu_i) = \infty,$$

where $\mu_1, \ldots, \mu_k$ are elements from $\mathscr{H}$ and ε is a positive number. It is clear that the family of sets obtained by varying $k, \mu_1, \ldots, \mu_k, \varepsilon$ can be used to define a topology $\mathscr{S}$. We shall refer to it as to the pointwise convergence topology in $\mathscr{F}$.

Let us write

$$H(r) = \{f \in \mathscr{F} \mid \underset{\delta \in H}{\exists} \underset{\mu \in \mathscr{H}}{\forall} r(\mu, \delta) = f(\mu)\}.$$

Clearly, $H(r) \subset \mathscr{F}$. Let $\overline{H(r)}$ be the closure of $H(r)$ with respect to $\mathscr{S}$. Let us consider

$$H(r)^{\sim} = \{f \in \overline{H(r)} \mid \underset{(\delta_i)_i \subset H}{\exists} \underset{\mu \in \mathscr{H}}{\forall} \overline{\lim_{i \to \infty}} r(\mu, \delta_i) = f(\mu)\}.$$

The set H is said to have *the property* $(\tilde{\mathrm{W}})$ if for any $f \in H(r)^{\sim}$ there exists $f^* \in H(r)$ such that

$$f^*(\mu) \leqslant f(\mu) \quad \text{for any } \mu \in \mathscr{H}.$$

Relative to this property and topologies defined in Sections 2-4, we have the following

THEOREM 5. *If H has the property $(\tilde{\mathrm{W}})$, then $(H, \mathscr{T}_1)$ is a compact space.*

Proof. Let $(\delta_i)_i$ be a sequence of decision functions such that

$$\lim_{i \to \infty} \sup_{\mu} r(\mu, \delta_i) = \inf_{\delta} \sup_{\mu} r(\mu, \delta). \tag{5.1}$$

It is clear that for any $\mu \in \mathscr{H}$ we have

(5.2) $$\lim_{i\to\infty} \sup_{\mu} r(\mu, \delta_i) \geqslant \overline{\lim_{i\to\infty}}\, r(\mu, \delta_i).$$

Let us consider

$$f(\mu) = \overline{\lim_{i\to\infty}}\, r(\mu, \delta_i) \quad \text{for any } \mu \in \mathscr{H}.$$

Clearly, $f \in H(r)^{\sim}$. Since H has the property $(\tilde{\mathrm{W}})$, there exists $f^* \in H(r)$ such that

(5.3) $$f^*(\mu) \leqslant f(\mu) \quad \text{for any } \mu \in \mathscr{H}.$$

Because of $f^* \in H(r)$ there exists $\delta^* \in H$ such that

(5.4) $$f^*(\mu) = r(\mu, \delta^*) \quad \text{for any } \mu \in \mathscr{H}.$$

From relations (5.1)-(5.4) we obtain

$$\sup r(\mu, \delta^*) \leqslant \inf_{\delta} \sup_{\mu} r(\mu, \delta),$$

i.e., δ^* is a minimax solution to the decision problem $P = (\mathscr{H}, H, r)$. Thus, according to Theorem 1, $(H, \mathscr{T}_1)$ is a compact space and the theorem is proved.

THEOREM 6. *If H has the property $(\tilde{\mathrm{W}})$, then $(H, \mathscr{T}_{\mu})$ is a compact space for any $\mu \in \mathscr{H}$.*

Proof. Let μ_0 be an element of $\mathscr{H}$. We shall show that if H has the property $(\tilde{\mathrm{W}})$, then $(H, \mathscr{T}_{\mu_0})$ is a compact space. It is clear that there exists a sequence $(\delta_i)_i$ of elements of H such that

(5.5) $$\lim_{i\to\infty} r(\mu_0, \delta_i) = \inf_{\delta} r(\mu_0, \delta).$$

For any $\mu \in \mathscr{H}$ set

$$f(\mu) = \overline{\lim_{i}}\, r(\mu, \delta_i).$$

Since $f \in H(r)^{\sim}$, there exists $f^* \in H(r)$ such that for any $\mu \in \mathscr{H}$

(5.6) $$f^*(\mu) \leqslant f(\mu).$$

Since $f^* \in H(r)$, there exists $\delta^* \in H$ such that, for any $\mu \in \mathscr{H}$, $r(\mu, \delta^*) = f^*(\mu)$. This relation holds for every $\mu \in \mathscr{H}$, so it must hold for μ_0. Thus, because of (5.6), (5.5) and the form of f, we obtain

$$\inf_{\delta} r(\mu_0, \delta) \geqslant r(\mu_0, \delta^*),$$

i.e., δ^* is a Bayes solution relative to μ_0. According to Theorem 3, $(H, \mathscr{T}_{\mu_0})$ is a compact space. Since μ_0 can be chosen arbitrarily, the theorem is proved.

THEOREM 7. *If* $(H, \mathcal{T}_2)$ *is a compact space, then* H *has the property* $(\tilde{W})$.

Proof. Let f be an element of $H(r)^{\sim}$. Therefore, there exists a sequence $(\delta_i)_i$ of elements of H such that

(5.7) $$\overline{\lim_i}\, r(\mu, \delta_i) = f(\mu) \quad \text{for any } \mu \in \mathcal{H}.$$

Since $(H, \mathcal{T}_2)$ is a compact space, there exist a subsequence $(\delta_{i_j})_j$ of the sequence $(\delta_i)_i$ and δ^* of H such that

(5.8) $$\underline{\lim_j}\, r(\mu, \delta_{i_j}) \geqslant r(\mu, \delta^*) \quad \text{for any } \mu \in \mathcal{H}.$$

One can easily verify that

(5.9) $$\overline{\lim_i}\, r(\mu, \delta_i) \geqslant \underline{\lim_j}\, r(\mu, \delta_{i_j}) \quad \text{for any } \mu \in \mathcal{H}.$$

It follows from (5.7)-(5.9) that

$$r(\mu, \delta^*) \leqslant f(\mu) \quad \text{for any } \mu \in \mathcal{H}.$$

The proof of the theorem is an immediate consequence of this relation.

6. The property (W). Let $(\mathcal{F}, \mathcal{I})$ be the topological space defined in Section 5.

Definition [1]. A subset $E \subset \mathcal{F}$ is said to be *half-closed* if for any element $f^* \in \bar{E}$ (where $\bar{E}$ is the closure of E with respect to $\mathcal{I}$) there exists an element $f^0 \in E$ such that

(6.1) $$f^0(\mu) \leqslant f^*(\mu) \quad \text{for any } \mu \in \mathcal{H}.$$

Definition [1]. H has *the property* (W) if $H(r)$ (defined in Section 5) is half-closed.

We have the following relation between the property $(\tilde{W})$ introduced in Section 5 and the property (W) defined in [1]:

THEOREM 8. *If* H *has the property* $(\tilde{W})$, *then* H *has the property* (W).

Proof. We prove that $H(r)$ is a half-closed set. For this purpose, let f^* be an element of $\overline{H(r)}$. But then it follows that $f^* \in H(r)^{\sim}$. Since H has the property $(\tilde{W})$, there exists an f^0 of $H(r)$ such that (6.1) holds.

The following example provides a decision problem for which $(H, \mathcal{T}_1)$ is a compact topological space and H does not have the property (W).

Example 3. We consider the decision problem (Ω, H, r) such that $\Omega = \{\omega_i\}_{i \geqslant 1}$, $H = \{\delta_j\}_{j \geqslant 0}$ and

$$r(\omega_i, \delta_0) = a, \quad i \geqslant 1,$$
$$r(\omega_i, \delta_j) = b, \quad i \leqslant j,\ j \geqslant 1,$$
$$r(\omega_i, \delta_j) = c, \quad 1 \leqslant j < i,$$

where $c > a > b > 0$. Suppose that $\mathcal{H} = \Omega$.

Since

$$\inf_{\delta}\sup_{\mu} r(\mu, \delta) = \sup_{\mu} r(\mu, \delta_0) = a,$$

δ_0 is a minimax solution. Thus, by Theorem 1, $(H, \mathcal{T}_1)$ is a compact space. Now, we prove that H does not have the property (W). Let $f_j(\mu_i)$ be defined by

$$f_j(\mu_i) = r(\mu_i, \delta_j) \quad \text{for any } i \geqslant 1,\ j \geqslant 0.$$

Hence $H(r) = \{f_j\}_{j\geqslant 0}$. But $\overline{H(r)} = H(r) \cup \{f^*\}$, where f^* is an element of $\mathcal{F}$ such that $f^*(\mu) = b$ for any $\mu \in \mathcal{H}$. We suppose that H has the property (W). Then there exists $\tilde{f} \in H(r)$ such that for any $\mu \in \mathcal{H}$ we have $\tilde{f}(\mu) \leqslant f^*(\mu)$, i.e., there exists $j \geqslant 0$ such that $f_j(\mu) \leqslant b$ for any $\mu \in \mathcal{H}$. Because of the form of f_j we obtain a contradiction. Hence H does not have the property (W). Now, according to Theorem 8, H does not have the property ($\tilde{\mathrm{W}}$).

The following example states a decision problem such that $(H, \mathcal{T}_\mu)$ is a compact space for any $\mu \in \mathcal{H}$ but H does not have the property (W).

Example 4. Let us consider the decision problem from Example 2. It was shown in Section 4 that H of Example 2 is a compact space relative to $\mathcal{T}_\mu$ for any $\mu \in \mathcal{H}$. Now, we show that H does not have the property (W).

For any $i \geqslant 0$ we write

$$f_i(\mu_1) = r(\mu_1, \delta_i) \quad \text{and} \quad f_i(\mu_2) = r(\mu_2, \delta_i).$$

We obtain $H(r) = \{f_i\}_{i\geqslant 0}$. It easy to show that $\overline{H(r)} = H(r) \cup \{f^*\}$, where $f^* \in \mathcal{F} = \{f \mid f\colon \{\mu_1, \mu_2\} \to [0, \infty]\}$ is defined by

$$f^*(\mu_1) = f^*(\mu_2) = 0.$$

Since there exists no $i_0 \geqslant 0$ such that $f_{i_0}(\mu) \leqslant 0$ for any $\mu \in \mathcal{H}$, H has neither the property (W) nor, according to Theorem 8, the property ($\tilde{\mathrm{W}}$).

It follows from Theorems 7 and 8 that if H is a weak compact space in the intrinsic sense, then H has the property (W). The converse of this statement is partially justified by the following theorem:

THEOREM 9. *Let r be a bounded function and let $\mathcal{H}$ be a denumerable set. If H has the property* (W), *then $(H, \mathcal{T}_2)$ is a compact topological space.*

Proof. Let $(\delta_n)_n$ be a sequence of decision functions of H. For any $\mu \in \mathcal{H}$ we put

$$g(\mu) = \overline{\lim_{n}}\, r(\mu, \delta_n).$$

Clearly, $g \in \mathcal{F}$. Let $\mathcal{H} = \{\mu^1, \mu^2, \ldots\}$. Since $\{r(\mu^1, \delta_n)\}_n$ is a bounded sequence, by the Cesàro lemma there is a convergent subsequence

$$r(\mu^1, \delta_{n_1}), r(\mu^1, \delta_{n_1'}), r(\mu^1, \delta_{n_1''}), \ldots$$

the limit of which is obviously less than or equal to $g(\mu^1)$. But the bounded sequence

$$r(\mu^2, \delta_{n_1}), r(\mu^2, \delta_{n_1'}), r(\mu^2, \delta_{n_1''}), \dots$$

contains a convergent subsequence

$$r(\mu^2, \delta_{n_1}), r(\mu^2, \delta_{n_2}), r(\mu^2, \delta_{n_2'}), r(\mu^2, \delta_{n_2''}), \dots$$

the limit of which is at most $g(\mu^2)$. By the boundedness of the sequence

$$r(\mu^3, \delta_{n_1}), r(\mu^3, \delta_{n_2}), r(\mu^3, \delta_{n_2'}), r(\mu^3, \delta_{n_2''}), \dots$$

we may choose, according to the Cesàro lemma, the convergent subsequence

$$r(\mu^3, \delta_{n_1}), r(\mu^3, \delta_{n_2}), r(\mu^3, \delta_{n_3}), r(\mu^3, \delta_{n_3'}), \dots$$

the limit of which is at most $g(\mu^3)$.

Continuing this procedure, we obtain a double index sequence

$$\begin{array}{l} n_1, n_1', n_1'', n_1''', \dots, \\ n_1, n_2, n_2', n_2'', \dots, \\ n_1, n_2, n_3, n_3', n_3'', \dots, \\ \dots\dots\dots\dots \\ n_1, n_2, n_3, \dots, n_p, n_p', n_p'', \dots, \\ \dots\dots\dots\dots\dots\dots \end{array}$$

so that the sequence of rank p is taken from the sequence of rank $p-1$ and has the first $p-1$ terms in common with it.

Then let us consider the diagonal sequence of indices

$$n_1, n_1, n_3, \dots, n_p, \dots$$

This sequence is a subsequence of all subsequences having the rank greater than or equal to p and, on the other hand, since the sequence of rank p is obtained from the sequence having the rank $p-1$, it follows that n_p also belongs to sequences having the rank less than p. Therefore, the sequence of functions $\{f_{n_p}\}$ is obtained from the initial sequence $\{f_n\}$, $f_n = r(\cdot, \delta_n)$.

From the construction it follows that this sequence converges at all points $\mu \in \mathscr{H}$. Therefore, there is $h \in \mathscr{F}$ such that

$$h(\mu) = \lim_{p\to\infty} r(\mu, \delta_{n_p}) \quad \text{for any } \mu \in \mathscr{H}. \tag{6.2}$$

Moreover, since the sequence $\{r(\cdot, \delta_{n_p})\}_p$ is a subsequence of $\{r(\cdot, \delta_n)\}_n$, we have

$$\lim_{p\to\infty} r(\mu, \delta_{n_p}) \leqslant \overline{\lim_n}\, r(\mu, \delta_n) \quad \text{for any } \mu \in \mathscr{H}.$$

Hence $h(\mu) \leqslant g(\mu)$ for any $\mu \in \mathscr{H}$. By (6.2) we obtain $h \in \overline{H(r)}$. On the other hand, since H has the property (W), there exists $\delta^0 \in H$ such that

$$r(\mu, \delta^0) \leqslant h(\mu) \quad \text{for any } \mu \in \mathscr{H}. \tag{6.3}$$

From (6.2) and (6.3) we get

$$r(\mu, \delta^0) \leqslant h(\mu) = \lim_{p\to\infty} r(\mu, \delta_{n_p}) \quad \text{for any } \mu \in \mathscr{H}. \tag{6.4}$$

Therefore, the sequence $(\delta_n)_n$ contains a subsequence $(\delta_{n_p})_p$ and there exists $\delta^0 \in H$ such that we have (6.4), i.e., H is a weak compact space in the intrinsic sense. Now, according to Theorem 2, $(H, \mathscr{T}_2)$ is a compact topological space and the theorem is proved.

Hence we obtain

COROLLARY 3. *Under the assumptions of Theorem* 9 *the following statements are equivalent*:

(i) H *has the property* (W).

(ii) H *has the property* $(\tilde{\mathrm{W}})$.

(iii) $(H, \mathscr{T}_2)$ *is a compact space.*

COROLLARY 4. *Under the assumptions of Theorem* 9 *we have*

(i) $(H, \mathscr{T}_\mu)$ *is a compact space for any* $\mu \in \mathscr{H}$.

(ii) $(H, \mathscr{T}_1)$ *is a compact space.*

For the general case, we were not able to obtain a counterexample showing that Theorem 9 fails.

REFERENCES

[1] H. Kudo, *On the property* (W) *of the class of statistical decision functions*, Annals of Mathematical Statistics 37 (1966), p. 1631-1642.

[2] C. Kuratowski, *Topologie*, Warszawa 1933.

[3] L. Le Cam, *An extension of Wald's theory of statistical decision functions*, Annals of Mathematical Statistics 26 (1955), p. 69-81.

[4] A. Wald, *Statistical decision functions*, J. Wiley, 1950.

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